Universidad Autónoma de Querétaro
Facultad de Ingeniería
Maestría en Ciencias Ingeniería Matemática

Simulation of a Lévy processes based on statistics of the Mexican Stock Exchange.

Tesis

Que como parte de los requisitos para obtener el grado de
Maestro en Ciencias en Ingeniería Matemática

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Centro Universitario Querétaro, QRO
February 2020
México

To my wife, friends, and the López Jiménez clan.

Acknowledgments

I would like to first thank CONACyT, for the support which allowed me to dedicate the time required to complete the masters program and finish this master's thesis. Second I would like to thank my alma mater the Autonomous University of Querétaro for allowing me to use their facilities and assigning a personal space from which to work and for allowing access to their extensive database of academic texts.

Next I would like to thank Dr. Samuel Estala Arias, my Co-Director Dr. Victor Aguilar, and M.M.A Ivan García Gonzaléz for all their guidance and above all, their patience. I would also like to thank the Coordinator of the Master's program, Dr. Roberto Gomez Loenzo for his guidance through out the program.

Finally I would like to thank all the teacher, tutors, advisers, and faculty who's help was invaluable in the acceptance, support, guidance, and completion of this humble investigation.

Abstract

This work was inspired by the lack of knowledge on how the stock exchange behaves using conventional techniques. So we set off to discover a new way to simulate the volatility but for the case of the Mexican stock exchange, seeing on how this is the second largest stock exchange in Latin America. Our main objective was to simulate the stock exchange using α-stable Lévy processes, we will focus primarily on developing an algorithm that will best imitate a single put option in the Mexican Stock exchange. In this text we will provide theoretical justification along with empirical evidence to support our claim. We obtained all our historical data from a free source yahoo finance we also specify the period with we will be focusing on.

keywords *- Brownian motion, Lévy processes, stochastic differential equations, Financial markets*

Resumen

En este trabajo se presenta se presenta un panorama del mercado financiero mexicano, teniendo como objetivo generar un modelo que aproxime a los datos históricos de varias acciones. A lo largo de la historia predecir el mercado financiero ha sido uno de los grandes objetivos de Economistas y Analistas de datos, y es que con ello se podría obtener información importante para las inversiones en diferentes acciones. Es muy importante estar al día y es por ello que en este trabajo se aborda esta dinámica desde una perspectiva del cálculo diferencial Estocástico. El cálculo diferencial Estocástico surge como una nueva alternativa a los problemas donde se consideran a los eventos con inserción de mateáticas de alto nivel. Dentro de estos tópicos se abordan los procesos de Lévy, un proceso de Lévy consiste en la modelización a partir de movimientos Brownianos o caminatas aleatorias. En este trabajo se presenta la aplicacón de procesos de Lévy a un grupo de acciones de la bolsa mexicana de valores, que por su naturaleza suelen tener menos variabilidad que las bolsas de grandes economías como la Bolsa de NY o la de Asia. La implementación se realiza mediante el software R-Studio con las bases de datos obtenidas de Yahoo-Finance, donde se pueden ver los datos históricos de las acciones. Por último, podrán encontrarse los algoritmos que fueron utilizados para la generación de los resultados.

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CHAPTER 1

Introduction

It may be common knowledge that in 1969, the Central Bank of Sweden created the Nobel prize in economics. A year later Paul Anthony Samuelson was awarded this prize, who coincidentally is the first American to receive this honor.Some years later, 1997 to be exact, Robert Merton and Myron Scholes would also be awarded this prize, sadly the renowned mathematician and Economist Fisher Black who died two years earlier in August 30th 1995.It may be noteworthy to mention that Robert Merton was an assistant to Paul Samuelson at MIT for two years, from 1968 to 1970 during the same time Myron scholes was a teacher's assistant at a MIT as well luck would have it that they would coincide at the same time.

A bit more insight and how this this is it organized the order is as followed. In the next three sections a review of historical, social and economic context is provided in which Louis Bachelier realized his research/investigation. In in the following section a historical review on the phenomenon known as Brownian movement and its conceptual evolution. After we discuss the contributions made by Albert Einstein In the explanation of Brownian movement. Then we will establish a relationship between the work of Samuelson and Bachelier. Following that we discuss the connection between the work of Black, Scholes and Merton and the circumstances that leading up to it. Afterwards a probabilistic model which was developed by Bachelier in order to study the stochastic dynamics of asset price and to determine the value of a call option on said asset. Then covering Brownian movement and we compared those results to those of Bacheliers. The estimate of the options, in which the price of the underlying asset is driven by Geometric Brownian movement, which is what prevents the price of the underlying asset from taking on negative values, which is the case with Bachelier models. Then we present the work of black and scholes in which, under conditions of equilibrium, developed a model to estimate the value a call option and which is driven by Geometric Brownian Motion. In their research, two unknown parameters are considered in the work of Samuelson, which no longer appear in the option price. In this same section, we provide an alternative derivation of Black and Scholes, which employs the CAPM, in order to obtain estimating function. In the following section we present the work of Merton, which extends in various directions the model presented by Black and Scholes, Which include: stochastic interest rates, continuous payment of dividends, American options and a generalization of Samuelson's formula for perpetual options and the valuation of options with barriers. Finally, we presents a set of conclusions.

1.1 Louis Bachelier

Louis bachelier (1870 - 1946), who is of French nationality isn't known as, " the father of modern Financial mathematics," due to his exceptional contributions to financial Theory. Whose doctoral thesis "*Théorie de la Spéculation*",[1] Which he presented in 1900 at the *Sorbonne* in Paris which distinguishes finance as a science, subject to same mathematical rigor as other fields. Louis Bachelier was ahead of his time with the introduction of concepts such as: Brownian movement, Markovian process, conditional expectation, and martingale. what is surprising is that these Concepts would be rediscovered and made popular years later by prominent mathematicians, for example Markovian processes appeared in 1906, the formal notations for conditional expectation was introduced by Kolmogorov om 1933, and the concept of Martingale was developed by Lévy until 1937.

It is important to mention a few of the contributions bacheliers doctoral thesis in the area of mathematical Finance, some of these are: modeling the dynamics of stock prices in the Paris stock exchange through Brownian movement, the first graphic representation of the price of a contract for an option, the formulation of efficient markets, the first estimating formula of an option and the first quantitative definition of "market risk."

Undoubtedly, Louis Bachelier was ahead of his time. When several sub-branches of physics were subjected to mathematical rigor and pure mathematics reached the height of an era, it was impossible to think of a "mathematical theory" that studied the behavior of the prices of financial assets, let alone think of an actual, "mathematical models" [1]that described the movements of said prices. However, Louis Bachelier, who was convinced of the importance of the study of financial markets, through mathematical models, so he continued with his adventure obtaining results that, to this day, continue to be surprising. Unfortunately Bachelier and his work would remain in the shadows for many years. Little is known, even to this date, about this enigmatic and mysterious character. Sadly Louis Bachelier died without any recognition from France's scientific elite of the time. It wasn't until the 1970's when, mainly, from the work of Paul Samuelson, Bachelier's contributions were finally unveiled, albeit long after his death.

Finally, it must be stated that even the great Louis Bachelier took or was inspired by previous works/investigations, i. E. the work of the economist and French financier Jean Joseph Nicolás Regnault, who used the pseudonym of Jules Regnault, "Calcul des chance et philosophie de la Bourse" published in 1863, already mentioned that the average deviation of the prices of the actions were proportional to the square root of the time in which the observations were taken, which is a central property of the Brownian movement.

1.2 Additional information on Louis Bachelier's life

In 1889 when Louis was 19, just out of High school, tragedy struck, both of his parents died. Which was the determining factor on why he abandoned his education, he had to take control and keep his family's business open and support his two younger bothers. Even though he couldn't continue with his education it was running his family's business he was introduced to the world of financial markets, and experience that would mark young Louis for life, sparking his interest in the behavior of different assets which even included product derivatives. Another even that would mark young Louis' life would come in 1891, when he choose to enlist in the military in order to take care of his military service (in some countries military service is required as part of becoming a citizen). It would not be until 1892, at the age of 22 when young Louis would have the opportunity to continue his education at the university *Sorbonne* thanks to a letter of recommendation written by Emile Borel, thanks to this letter young Louis was able to obtain a scholarship. With a four year gap between graduating high school and being accepted to the university it is not surprising that young Louis would struggle. In fact his grades were actually lower then those of his classmates (two noteworthy classmates are: Legenvin and Liénard), but despite these set backs in 1895 he would obtain a B.S in mathematics then in 1897 he would go on to obtain his master's degree in mathematical physics. On the 29th of March, 1900 Bachelier was a doctoral student at *Sorbonne* and he presented his doctoral thesis, *"Thórie de la Spéculation"* in the faculty of science at the Paris Academy. His *sinodales* were none other than Henri Poincaré, Paul Appel, and Joseph Boussinesq. Unfortunately for Bachelier mathematical finance was still not seen as an exact science, so the comments of Hadamard, Borel, Lesbesque, Lévy, and Baire were not expected and in a sence under valued Bachelier's contribution, again in this time in France Physics and pure Mathematics were the main areas of study[1].

1.3 Mathematical Finance and Probability, the lost years

In the XX century, 1970's and early 1980's to be specific, what is known today as mathematical finance was considered an area of interest for the mathematician of that era. And even less considered would be derivative product theory, which was seen as an area lacking mathematical rigor and scrutiny and was far from the areas of interest i.e. algebraic topology, differential geometry, function analysis, complex variable, modern algebra, etc. Which is far from what is seen today, now a days each mathematical department has and area dedicated to mathematical finance and even offer postgraduate degrees.

Probability theory would share a similar, not until Kiyosi Itô (1915 - 2008), who was one of the most noteworthy mathematician of the 20th century who as well a progenitor of the famous stochastic calculus, he stated,

"Since I started my studies in mathematics (beginning of the 1930's) I was really interested in the discovering the statistical laws which resided in random phenomenon. I knew that with probability theory I would be able to discover said phenomenon, hence I dedicated to it. When i was a student there were very few investagor in probablity, a few of them are Andrei Kolmogorov from Rusia and Paul Lévy of France."

Is how his passion was born.

1.4 From Robert Brown to Louis Bachelier, a Brief Historical Review of Brownian Motion

In 1827, a Scottish botanist by the name of Robert Brown (1773 - 1858) while examining a particulates of pollen under a microscope, he observed that when the particle of pollen was suspended in water the particle would continue moving in an erratic manner with no signs of stopping. Brown's original thoughts were that the particle had its own movement, he even considered the possibility of the particle of pollen being "alive". Given the implied possibilities investigation in to this was intensified and soon, and to the dismay of Brown, he obtained evidence where non-organic material i.e. find dust, had the same behavior as did the pollen, which quickly disproved his initial hypothesis. Two years later, 1829, Robert Brown renounces his hypothesis about living particulates and possessing its own movement. He later went on to state,

"... I am unaware what causes a small particle of solid matter being organic or inorganic while suspended in water or oter liquids exhibit erratic or irregular behavior."

After this the investigation was divided in to different facets or areas.

- Attraction or repulsion between suspended particles.
- The instability of equilibrium of the liquid the particles find themselves in.
- The presence of minuscule bubbles in the liquid or on the particles.

As it may no to come to a big surprise most of these were almost immediately discarded and what evidence was collected was inconclusive or not very favorable.

1.5 From Robert Brown to Albert Einstein

Two centuries would paste with no satisfactory explanation on what caused the Brownian movement. It was not until the beginning of 20th century, when it was proven that the erratic or irregular movement of the pollen particle was caused by random collisions with water molecules.

In 1905, the famous Jewish and German born physicist Albert Einstein published three pivotal works:

- Photoelectric effects.
- Special Relativity.
- Statistical mechanics.

For his first work the Academy of Switzerland awarded him Nobel prize in 1921. For this second contribution he was credited for uniting classical mechanics and electrodynamics and as for his last pivotal work just the satisfaction of being able to answer a problem that has had almost two centuries that has gone with out one i.e. Brownian motion. Einstein proposed an explanation and mathematical foundation of Brownian motion, from which it is derived that average dispersion of movement or its trajectory of the particle suspended in a liquid is proportional to the squared root of time. $\sqrt{t}$. Bachelier predated Einstein with his own mathematical explanation of Brownian movement, Bacheliers explanation was just as elegant but covering or focusing on a completely different to the erratic movement of the pollen particle.

1.6 Paul Samuelson and Louis Bachelier, a reunion between titans

One of the most important limitations of Bacheliers work was that the asset prices could take on negative values and this was flaw wasn't corrected until 1965 by none else than Paul A. Samuelson in his work about estimating the price of warrants does Samuelson mention the work of Bachelier. An interesting side note, when Samuelson visited *Sorbonne* in 1960 he entered the library and chance would have it that he would run into Bachelier's doctoral thesis, an event that would greatly influence later work.

In 1965, Paul Samuelson published his article, "Rational Theory of Warrant Prices" where he introduced the concept of "economic" Brownian motion, what today is known as Geometric Brownian motion. When Samuelson resolved the issue Bachelier had by eliminating the possibility the asset prices can taking on negative values he unwittingly created a few more inconveniences like the appearance of unknown parameters. In the article he published the price of the underlying asset is driven by Geometric Brownian motion and the option price is calculated by taking the present value of the expected value at its maturity date. In this situation the price of the option is determined by two unknown parameters: the first is the average return expected, which is a parameter linked to the risk agents prefer. The second, is the return that will be paid by the option which is used to calculate its present value which is expected when the option reaches maturity.

1.7 Fischer Black, Myron Scholes, and Robert Merton. A dream team.

Fischer Black and Myron Scholes published their article, "The Pricing of Options and Corporate Liabilities." In the Journal of Political Economy in 1973, Admiringly maybe the choice of "journal" was not the best or most adequate for the work. It was held up in review for almost 2 years before being published. Despite the fact that Black and Scholes work had been previously reviewed by Robert Merton, Merton Miller, and Eugene Fama who gave validity to said work. Under the assumption of general equilibrium, Black and Scholes were able to develop a formula that estimated the value of a European option that does not pay dividends, and whose price is driven by a Geometric Brownian Motion. The deficiencies in Paul Samuelson's article were rectified given that there are no unknown parameters in the price of the option and, more importantly, no additional limitations emerged. The perfect formula was born. As if this was not enough in the SAME article Blakc and Scholes proposed an alternative derivation of their formula using the Capital Asset Pricing Model or C.A.P.M for short.

In their article Black and Scholes developed a second order partial differential equation which is also parabolic and linear. Whose solution is the price of a European option when the final condition is the intrinsic value of the option. In Black and Scholes research, this partial differential equation is transformed into the heat diffusion equation, which has explicit solutions. Soon after this Black and Scholes' partial differential equations becomes extremely popular, seeing as it can be a base for estimating a wide and very diverse variety of derivative products, given that for different boundary conditions, their solutions represent the prices of many available derivative products in the stock market.

It goes without saying that Roberts Merton's article, "Theory of Rational Option Pricing," which was published in 1973 in the Bell Journal of Economics and Management Science is very noteworthy given that he (Merton) not only obtained similar results as Black and Scholes but found various extensions. Merton continued his work on valuation of options in a series of truly impressive articles.

For the exceptional contribution to what is known today as, "financial Mathematics in continuous time," Robert Merton and Myron Scholes won the Nobel prize in economics in 1997. Sadly by then Fischer Black had passed away two years earlier. For his (Merton) considerable contribution the Black and Scholes model could easily be called the Black, Scholes, Merton model. In fact this is how this auther refers to this model as such.

Next we will briefly review the works of Bachelier, Samuelson, Black, Scholes, and Merton. The objective is to show the reader the can see the evolution of the ideas and concepts formulated, some assumptions are simplified and we will use conventional notation. Given that the notations used by Bachelier in his work is convoluted and difficult to fallow.

1.8 Bachelier's Thesis.

Bachelier starts out his thesis giving an explanation as description of the products available at the time in the French stock market i.e forwards, options futures, etc. He then continues by developing a probabilistic model that described the movement of a financial asset and established the principal that the conditional expectation of profit for the speculator is zero. Here the term conditional refers to the fact that actual information is taken as given. Bachelier also accepts that the stock market price evolves like a homogeneous Markovian process in time. He follows by demonstrating that the probability distribution function (PDF) associated with this process satisfies what we currently call the Chapman-Kolmogorove equation, he also verifies that Gaussian PDF with an increasing linear variance in time is a solution to this equation. Although he makes no argument for uniqueness, he does offer a few arguments in order to confirm his conclusion. It is also noteworthy to state that Bachelier demonstrated that the family of density functions associated with the process that drives the price also satisfies the heat equation. Lastly the probabilistic model that describes the behavior of the price of a financial asset is used in order to estimate the value of a French option whose earning are paid upon maturity or when the expiry date is reached.

1.8.1 Probabilistic law behind pricing a financial asset.

In this section we will present the probabilistic model developed by Bachelier in order to study the stochastic behavior of the price of a financial asset. Also as previously mention we will use "modern" notation in order to help understand, given that Bacheliers notation is a bit convoluted and hard to fallow.

Let S_{t_1} be the price of a financial asset in time $t_1 > 0$. Now let us assume that S_{t_1} is a random variable (R.V) over a fixed probability space $(\Omega, \mathfrak{F}, \mathbb{P})$. We will also assume that S_{t_1}, has a conditional density function $f_{S_{t_1}|S_0}(s|S_0)$ and we know that the probability that at time t_1, the price of the financial asset, S_{t_1} is located in the interval $[s, s+ds$ given S_0, can be written as:

$$\mathbb{P}[s \leq S_{t_1} \leq s+ds|S_0] = \int_s^{s+ds} f_{S_{t_1}|S_0}(u|S_0)du = f_{S_{t_1}|S_0}(s|S_0)ds + o(ds) \quad (1.1)$$

Where $\frac{o(ds)}{ds} \to 0$ when $ds \to 0$. Next it is is convenient if we represent the value of $f_{S_{t_1}|S_0}(s|S_0)ds$ with $p(s,t|S_0,0)ds$. Thus,

$$\mathbb{P}[s \leq S_{t_1} \leq s+ds|S_0] \approx p(s,t_1|S_0,0)ds \quad (1.2)$$

We also have:

$$\mathbb{P}[u \leq S_{t_2}+s \leq u+du|S_{t_1}=s] = \mathbb{P}[u-s \leq S_{t_2} \leq u-s+du|S_{t_1}=s]$$

$$= f_{S_{t_1+t_2}|S_0}(u|S_{t_1})du + o(du) \quad (1.3)$$

If we denote $f_{S_{t_1+t_2}|S_0}(u|S_{t_1})$ as $p(u, t_1+t_2|s,t_1)du$ we then have that:

$$\mathbb{P}[u \leq S_{t_2}+s \leq u+du|S_{t_1}] \approx \mathbb{P}[u, t_1+t_2|s,t_1]du \quad (1.4)$$

This way, the conditional probability that $S_{t_1+t_2}$ is located in $[u+du]$, given S_0, and can be calculated as:

$$\mathbb{P}[u \leq S_{t_1+t_2} \leq u+du|S_0]$$

$$= \int_{s\in\mathbb{R}} \mathbb{P}[u \leq S_{t_2}+s \leq u+du|S_1=s] * \mathbb{P}[s \leq S_{t_1} \leq s+ds|S_0] \quad (1.5)$$

1.8.2 Chapman-Kolmogorov equation.

Given $\mathbb{P}[s \leq S_{t_1} \leq s+ds|S_0] \approx p(s,t_1|S_0,0)ds$ and $\mathbb{P}[u \leq S_{t_2}+s \leq u+du|S_{t_1}] \approx p(u,t_1+t_2|s,t_1)du$ and working under the assumption that the approximation error is negligible, we can write $\mathbb{P}[u \leq S_{t_1+t_2} \leq u+du|S_0] = \int_{s\in\mathbb{R}} \mathbb{P}[u \leq S_{t_2}+s \leq u+du|S_1=s] * \mathbb{P}[s \leq S_{t_1} \leq s+ds|S_0]$ as

$$p(u,t_1+t_2|S_0,0) = \int_{\infty}^{-\infty} p(u,t_1+t_2|s,t_1) * p(s,t_1|S_0,0)ds \quad (1.6)$$

This expression is known as the Chapman-Kolmogorov equation (for the continuous case). The Only function that satisfies $\mathbb{P}[u \leq S_{t_1+t_2} \leq u+du|S_0] = \int_{s\in\mathbb{R}} \mathbb{P}[u \leq S_{t_2}+s \leq u+du|S_1=s] * \mathbb{P}[s \leq S_{t_1} \leq s+ds|S_0]$ is given by:

$$p(u,t+h|s,h) = q(t)e^{-\pi q(t)^2(u-s)^2} \quad (1.7)$$

Where $q(\cdot)$ is a function of time that will be determined. It should be observed that,

$$\int_{-\infty}^{\infty} p(u,t+h|s,h)du = \int_{-\infty}^{\infty} q(t)e^{-\pi q(t)^2(u-s)^2}du = 1$$

In particular (for simplicity Bachelier assumes $S_0 = 0$).

$$p(s,t_1|S_0,0) = q(t_1) * e^{-\pi q(t_1)^2(s-S_0)^2} \tag{1.8}$$

Also if we observe that if $s = 0$, we then have $p(u,t_1+t_2|s,t) = q(t_1)$

In the same manner,

$$p(0,t_1+t_2|s,t_1) = q(t_2) * e^{-\pi q(t_2)^2(u-s)^2} \tag{1.9}$$

Next we will prove that, $p(u,t+h|s,h) = q(t)e^{-\pi q(t)^2(u-s)^2}$ satisfies $p(u,t_1+t_2|S_0,0) = \int_{\infty}^{-\infty} p(u,t_1+t_2|s,t_1) * p(s,t_1|S_0,0)$ but first observe that from $p(s,t_1|S_0,0) = q(t_1) * e^{-\pi q(t_1)^2(s-S_0)^2}$ and $p(0,t_1+t_2|s,t_1) = q(t_2) * e^{-\pi q(t_2)^2(u-s)^2}$ We have,

$$p(u,t_1+t_2|S_0,0) = \int_{\infty}^{-\infty} q(t_2)e^{-\pi q(t_2)^2(u-s)^2} * q(t_1)e^{-\pi q(t_1)^2(u-s)^2}ds;$$

$$= q(t_1) * q(t_2) * e^{-\pi[q(t_2)^2u^2+q(t_1)^2S_0^2]}\int_{\infty}^{-\infty} e^{-\pi[(q(t_1)^2+q(t_2)^2)s^2+2(q(t_2)^2u+q(t_1)^2S_0)s]}ds \tag{1.10}$$

If we define the change of variable as:

$$w = s * \sqrt{q(t_1)^2+q(t_2)^2} - \frac{q(t_2)^2u+q(t_1)^2S_0}{\sqrt{q(t_1)^2+q(t_2)^2}}$$

Then,

$$p(u,t_1+t_2|S_0,0) = \frac{q(t_1)q(t_2)}{\sqrt{q(t_1)^2+q(t_2)^2}} * e^{-\pi[q(t_2)^2u^2+q(t_1)^2S_0^2]+\dfrac{\pi[q(t_2)^2u+q(t_1)^2S_0]^2}{q(t_1)^2+q(t_2)^2}}$$

Where

$$\int_{\infty}^{-\infty} e^{-\pi w^2}dw = 1$$

consequently,

$$p(u,t_1+t_2|S_0,0) = \frac{q(t_1) * q(t_2)}{\sqrt{q(t_1)^2+q(t_2)^2}} * e^{-\pi\dfrac{q(t_1)^2 * q(t_2)^2}{q(t_1)^2+q(t_2)^2}(u-S_0)^2} \tag{1.11}$$

Now, in virtue of: $p(u,t+h|s,h) = q(t)e^{-\pi q(t)^2(u-s)^2}$, we obtain,

$$p(u,t_1+t_2|S_0,0) = q(t_1+t_2)e^{-\pi q^2(t_1+t_2)^2(u-S_0)^2} \tag{1.12}$$

So then we have

$$q^2(t_1+t_2) = \frac{q^2(t_1) * q^2(t_2)}{q^2(t_1)+q^2(t_2)} \tag{1.13}$$

Now if we set equal the partial derivatives of $p(u,t_1+t_2|S_0,0) = q(t_1+t_2)e^{-\pi q^2(t_1+t_2)^2(u-S_0)^2}$ with respect to t_1 and t_2, or put another way:

$$\frac{\partial q^2(t_1+t_2)}{\partial t_1} = \frac{\partial q^2(t_1+t_2)}{\partial t_2}$$

Which satisfies:

$$\frac{q'(t_1)}{q^3(t_1)} = \frac{q'(t_2)}{q^3(t_2)} \tag{1.14}$$

To put it simply, the quotient $\frac{q'(t)}{q^3(t)}$ is independent of t. Or put another way, it is independent of time. So we observed that, $\frac{q'(t)}{q^3(t)} = a =$ constant. The solution to this differential equation is given by:

$$q(t_1) = \frac{b}{\sqrt{t}} \tag{1.15}$$

Where b is a positive constant. Now if we redefine b as $b = \frac{1}{\sqrt{2\pi}}$ we have,

$$p(u,t+h|s,h) = q(t)e^{-\pi q(t)^2(u-s)^2} = \frac{1}{\sqrt{2\pi t}}e^{\frac{-(u-s)^2}{2t}} \tag{1.16}$$

Now if we take into consideration that $p(s,t+h|S_h,h) = f_{s|S_h}(s|S_h)$ we have:

$$f_{s_t|S_0}(s|S_0) = p(s,t|S_0,0) = \frac{1}{\sqrt{2\pi t}}e^{\frac{-(u-s)^2}{2t}} \tag{1.17}$$

This is a density function Bachelier obtained with $S_0 = 0$ for a financial asset in the French market. Finally we can see that the Chapman-Kolmogrove equation can be written as:

$$f_{S_{t_1+t_2}|S_0}(s|S_0) = \int_{-\infty}^{\infty} f_{S_{t_1+t_2}|S_{t_1}}(s|S_{t_1}) f_{S_{t_1}|S_0}(S_{t_1}|S_0)dS_{t_1} \tag{1.18}$$

1.8.3 Fourier's equation

The first person to describe heat flow was french born Joseph Fourier (1768 - 1830); the heat equation. Which is second order partial differential equation with exact solutions, that describe how heat disperses over time on an infinitely long metal rod after its been heated for an initial time.

Now we must state that in order to save time and not go off a tangent, now if we simplify some of Bachelier's original arguments will be simplified, now let us consider the following equation $p(u, t+h|s, h) = q(t)e^{-\pi q(t)^2(u-s)^2} = \frac{1}{\sqrt{2\pi t}}e^{\frac{-(u-s)^2}{2t}}$ now from that we have

$$p(u, t|s, 0) = \frac{1}{\sqrt{2\pi t}}e^{\frac{-(u-s)^2}{2t}} \tag{1.19}$$

And we obtain

$$\frac{\partial p}{\partial t} = \frac{1}{2}\left(\frac{(u-s)^2}{t^2} - \frac{1}{t}\right) * p(u, t.|s, 0) \tag{1.20}$$

And

$$\frac{\partial^2 p}{\partial^2 t} = \left(\frac{(u-s)^2}{t^2} - \frac{1}{t}\right) * p(u, t.|s, 0) \tag{1.21}$$

There for

$$2 * \frac{\partial p}{\partial t} = \frac{\partial^2 p}{\partial u^2}, \quad -\infty < u < \infty, \quad t \geq 0. \tag{1.22}$$

Now if we define $\mathbb{P}[u, t] = \int_{-\infty}^{\infty} p(u, t|s, 0)ds$, it also holds true, trivially, that:

$$2 * \frac{\partial \mathbb{P}}{\partial t} = \frac{\partial^2 \mathbb{P}}{\partial u^2}, \quad -\infty < u < \infty, \quad t \geq 0. \tag{1.23}$$

Now if we apply a change of variable $t = 2\ddot{A}$ we can evidently see that

$$p(s, 2\tau|u, 0) = \frac{1}{\sqrt{2\pi\tau}}e^{\frac{-1}{4\tau}(s-u)^2}$$

We can write is as

$$V(u, \tau) = \int_{\infty}^{\infty} p(u, 2\tau|s, 0)ds \tag{1.24}$$

It fallows that

$$\frac{\partial V}{\partial \tau} = \frac{\partial^2}{\partial s^2}, \quad -\infty < u < \infty, \quad \tau \geq 0. \tag{1.25}$$

The equations $2 * \frac{\partial p}{\partial t} = \frac{\partial^2 p}{\partial u^2}, \quad -\infty < u < \infty, \quad t \geq 0.$, $2 * \frac{\partial \mathbb{P}}{\partial t} = \frac{\partial^2 \mathbb{P}}{\partial u^2}, \quad -\infty < u < \infty, \quad t \geq 0.$ and $\frac{\partial V}{\partial \tau} = \frac{\partial^2}{\partial s^2}, \quad -\infty < u < \infty, \quad \tau \geq 0.$ have the same format as Fourier's second order partial differential equation $c^2 \frac{\partial \varphi}{\partial t} = \frac{\partial^2 \varphi}{\partial u^2}$ where c is a constant. Especially the equation $\frac{\partial V}{\partial \tau} = \frac{\partial^2}{\partial s^2}, \quad -\infty < u < \infty, \quad \tau \geq 0.$ if we restrict $V(u,0) \equiv 1$.

1.8.4 Markovian Processes

We denote a stochastic processes as a Markovian process if the probability distribution of a future event is solely dependent on actual information and not on previous information. Now if we trivially define the random variable $\Delta^{(h)} S_t = S_{t+h} - S_h$, with $h \geq 0$, the distribution of $\Delta^{(h)} S_t$ only depends on information available at time h, or to put it another way, it only depends on the value of S_h and not on previous values i.e. S_m, where $m, d \leq h$. This implies that $\Delta^{(h)} S_t$ is a Markovian process.

1.8.5 Homogeneous Processes in Time

Now we can observe that from $p(u, t+h|s,h) = q(t) e^{-\pi q(t)^2 (u-s)^2} = \frac{1}{\sqrt{2\pi t}} e^{\frac{-(u-s)^2}{2t}}$, for whichever $h \geq 0$, it satisfies the property:

$$p(u, t+h|s,h) = p(u,t|s,0) = \frac{1}{\sqrt{2\pi t}} e^{\frac{-(u-s)^2}{2*t}} \tag{1.26}$$

If we define the random variable $\Delta^{(h)} S_t = S_{t+h} - S_h$ with an arbitrary $h \geq 0$ we have, from $p(u, t+h|s,h) = p(u,t|s,0) = \frac{1}{\sqrt{2\pi t}} e^{\frac{-(u-s)^2}{2*t}}$, that $\Delta^{(h)}$ and $\Delta^{(0)} S_t$ have the same distribution $N(0,t)$. The distribution $\Delta^{(h)} S_t$ depends on the difference between $t+h$ and h. If we denote $x = u - s$, we then have:

$$p(x,t|0,0) = \frac{1}{\sqrt{2\pi t}} e^{\frac{-x^2}{2t}}$$

If we further generalize, if h and m are non negative arbitrary numbers, then $\Delta^{(h)}$ and $\Delta^{(m)} S_t$ have the same distribution.

1.8.6 Martingales

One of the most important concepts in the study of financial markets is that of efficient markets, which are defined by martigale process. If we observe that condition $E[S_{t+h} - S|S_h] = 0$ we have:

$$E[S_{t+h}|S_h] = S_h \tag{1.27}$$

Which is the best projection of S_{t+h}, given that its actual value is S_h, is precisely S_h. A stochastic process $(S_t)_{t\geq 0}$ which satisfies $E[S_{t+h}|S_h] = S_h$ for all $t, h \geq 0$ which we denote as a martingale.

1.8.7 Brownian Motion

Both theoretical and applied aspects of Brownian motion have been the focus of numerous studies in financial mathematical. It can not be denied that Browning motion is found either implicitly or explicitly in just about every area of continuous-time financial theory. The precursor to what is currently known as Brownian motion is found in Bachelier's work (1900) and before that in Jules Regnault (1863). No if we consider the density function obtained by Bachelier, including the volatility parameter Ã, for the price of a financial asset:

$$f_{S_T|S_0}(s|S_0) = \frac{1}{\sqrt{2\pi t\sigma}} e^{\frac{-s^2}{2\sigma^2 t}}, \quad t \geq 0 \tag{1.28}$$

With $S_0 = 0$. We can observe that the above is equivalent to writing $S_t\ N(0, \sigma t)$. If we define $W_t\ N(0, t)$, then we can write $S_t = \sigma W_t$. In this case, if $t = 0$, then $W_0 \equiv 0$. If we define $\Delta^{(h)} S_t = S_{t+h} - S_h$ and $\Delta^{(h)} = W_{t+h} - W_h, \quad h > 0$, if follows that

$$\Delta^{(h)} S_t = \sigma \Delta^{(h)} W_t \tag{1.29}$$

Finally, notice that if the changes are infinitesimal, equation (1.29) can be replaced by

$$dS_t = \sigma dW_t, \quad dW_t \sim N(0, dt) \tag{1.30}$$

Which is more in line with modern notation.

1.8.8 Estimation of Options (France)

In this section we use the probabilistic model of the price of an asset to value a special type of options, the french options, whose premium is paid upon maturity. A (financial) call option, or call option contract, It is an agreement between two parties that obligates (legally) one of the parties to sell a financial asset, while the counterpart grants the right, but not the obligation, to buy said asset at a preset price at a future date. It is assumed that the sale can only be carried out on the date of maturity. It is customary to say that the buyer takes a long position while the seller a short position.

If we assume the option was established a time t (the present) and that the purchase and sale of the asset at a predetermined price, K, is carried out at a future date, T. The agreed price, K, is called the price of exercise (of the option). It is important to mention that at the time the contract reaches maturity, is when the premium is paid. An agent, who thinks that the price of the asset will increase, can speculate taking a long position on the option. Let S_T be the price of the option at the date of maturity. If $S_T < K$, then the long position does not exercise its right to purchase the call option. While if $S_T > K$, the long position does, obtaining a profit of $S_T - K$. In either case the premium C is paid. The principle that Bachelier establishes how to estimate the value of the option is that the expectation of profit (long position) be zero, better stated:

$$\int_{-\infty}^{K} -Cf_{S_T|S_0}(s|S_0)ds + \int_{K}^{\infty}(s-K-C)f_{S_T|S_0}(s|S_0)ds = 0 \tag{1.31}$$

where

$$f_{S_T|S_0}(s|S_0) = \frac{1}{\sqrt{2\pi t \sigma}} e^{\frac{-(s-S_0)^2}{2\sigma^2 t}}$$

We have introduced the parameter of volatility $\tilde{A}$ in order to give Bachelier's equation a modern touch. Equation $\int_{-\infty}^{K} -Cf_{S_T|S_0}(s|S_0)ds + \int_{K}^{\infty}(s-K-C)f_{S_T|S_0}(s|S_0)ds = 0$ will lead to:

$$\begin{aligned} C &= \int_{K}^{\infty}(s-K)f_{S_T|S_0}(s|S_0)ds \\ &= \int_{K}^{\infty} sf_{S_T|S_0}(s|S_0)ds - KP\{S_T \geq K\} \\ &= \int_{K}^{\infty} sf_{S_T|S_0}(s|S_0)ds - K\Phi(-K/\sigma\sqrt{T}) \end{aligned} \tag{1.32}$$

Where $\Phi(\dot{})$ is the cumulative distribution function of a standard normal random variable. Evidently $\Phi(-K/\sigma\sqrt{T}$ is the probability of exercising the option of buying the option. Of course, if the premium would have to be paid at the moment the contract is struck, the price c, would be given by:

$$c = e^{-rT}\int_{K}^{\infty} sf_{S_T|S_0}(s|S_0)ds - e^{-rT}K\Phi(-K/\sigma\sqrt{T}) \tag{1.33}$$

where r is a constant interest rate and risk free. If we observe that equation $c = e^{-rT}\int_{K}^{\infty} sf_{S_T|S_0}(s|S_0)ds - e^{-rT}K\Phi(-K/\sigma\sqrt{T})$ is independent of the level of risk deemed acceptable by those in the market. On the other hand, the Average expected return on any asset, including the option, is the risk-free interest rate,

r, that is, agents are risk neutral. For this reason, the option price is discounted from the rate r.

1.9 Albert Einstein and the irregular movement of a particle suspended in a liquid and its relation to the heat diffusion equation

In this section we will present the main ideas of Albert Einstein's work on the irregular motion of a particle suspended in a liquid and its relationship with the heat diffusion equation.

Now consider a particle suspended in a stationary liquid. It is assumed that the movement of the particle at different time intervals are independent processes.

$$\Phi(s,0) \equiv p(s,t|0,0) = f_{S_T|S_0}(s|S_0) \tag{1.34}$$

and

$$\Phi(s,0) = \Phi(-s,0)$$

Obviously $\int_{-\infty}^{\infty} \Phi(s,0)ds = 1$.

$$\Phi(u,t+h)du = du\int_{-\infty}^{\infty} \Phi(u-s,t)\Phi(s,0)ds, \tag{1.35}$$

Where $\Phi(u,t+u) = p(u,t+h|0,0)$ and $\Phi(u-s,t) = p(u,t+h|s,t)$ now if h is small enough, we have

$$\Phi(u,t+h) \approx \Phi(u,t) + h\frac{\partial\Phi(u,t)}{\partial t} \tag{1.36}$$

$$\Phi(u-s,t) = \Phi(u,t) - s\frac{\partial\Phi(u,t)}{\partial u} + \frac{1}{2}s^2\frac{\partial^2\Phi(u,t)}{\partial u^2} - \cdots \tag{1.37}$$

$$\Phi(u,t) + h\frac{\partial\Phi(u,t)}{\partial t} = \Phi(u,t)\int_{-\infty}^{\infty} \Phi(s,0)ds + \frac{1}{2}\frac{\partial^2\Phi(u,t)}{\partial u^2}\int_{-\infty}^{\infty} s^2\Phi(s,0)ds \tag{1.38}$$

$$h\frac{\partial\Phi(u,t)}{\partial t} = \frac{1}{2}\frac{\partial^2\Phi(u,t)}{\partial u^2}\int_{-\infty}^{\infty} s^2\Phi(s,0)ds$$

If we define

$$\frac{1}{h}\int_{-\infty}^{\infty} s^2\Phi(s,0)ds = D,$$

we obtain

$$\frac{\partial\Phi(u,t)}{\partial t} = D\frac{\partial^2\Phi(u,t)}{\partial t^2} \tag{1.39}$$

$\Phi(u,0) = 0$ and $\int_{-\infty}^{\infty}\Phi(u,t)du = 1$

$$\Phi(u,t) = \frac{1}{\sqrt{4\pi Dt}}e^{-u^2/4Dt} \tag{1.40}$$

$$\Phi(u,t) = p(u,t|0,0) = \frac{1}{\sqrt{2\pi t}\sigma}e^{-u^2/2\sigma^2 t} \tag{1.41}$$

1.10 Samuelson's Model to estimate the value of European Options

$$dS_t = \alpha S_t dt + \sigma S_t dW_t, \qquad dW_t \sim N(0,dt) \tag{1.42}$$

$$f^{(\alpha)}_{S_T|S_0}(s|S_0) = \frac{1}{\sqrt{2\pi T}\sigma s}exp\{-\frac{1}{2}(\frac{ln(\frac{s}{S_0}) - (\alpha - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}})^2\} \tag{1.43}$$

$(\alpha - \frac{1}{2}\tilde{A}^2)T$

$$f^{(\alpha)}_{S_T|S_0}(s|S_0) = \int_{-\infty}^{\infty} f^{(\alpha)}_{S_{T-U}|S_u}(s|S_u)f^{(\alpha)}_{S_u|S_u}(S_u|S_0)dS_u \tag{1.44}$$

$$E[S_T|S_0] = \int_0^{\infty} sf^{(\alpha)}_{S_T|S_u}(s|S_u)ds = S_0e^{\alpha T} \tag{1.45}$$

$$\begin{aligned} c(S_0,T;\alpha,\beta) &= e^{-\beta T}\int_0^{\infty} max(s-K,0)f^{(\alpha)}_{S_T|S_t}(s|S_t)ds \\ &= e^{-\beta T}\int_K^{\infty}(s-K)f^{(\alpha)}_{S_T|S_t}(s|S_t)ds \end{aligned}$$

$$= e^{-\beta T}\int_K^{\infty} sf^{(\alpha)}_{S_T|S_t}(s|S_t)ds - Ke^{-\beta T}P\{S_T \geq K\} \tag{1.46}$$

$$= e^{-\beta T}\int_K^{\infty} f^{(\alpha)}_{S_T|S_t}(s|S_t)ds - Ke^{-\beta T}\Phi(d)$$

$$d = \frac{ln(\frac{S_0}{K}) + (\alpha - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$$

$$c(S_0,T;\alpha,\beta) = e^{-\beta T}\int_K^\infty (s-K) f^{(\alpha)}_{S_T|S_t}(s|S_t)ds$$
$$= e^{-\beta T}\int_0^\infty (s-K) f^{(\alpha)}_{S_T|S_t}(s|S_t)ds - e^{-\beta T}\int_0^K (s-K) f^{(\alpha)}_{S_T|S_t}(s|S_t)ds$$
$$= e^{-\beta T}(S_0 e^{-\alpha T} - K) + e^{-\beta T}(KP\{S_T \leq K\} - \int_0^K s f^{(\alpha)}_{S_T|S_t}(s|S_t)ds)$$
$$= S_0 e^{(\alpha-\beta)T} - Ke^{-\beta T} + e^{-\beta T}(K(1-\Phi(d)) - \int_0^K s f^{(\alpha)}_{S_T|S_t}(s|S_t)ds)$$

$$= S_0 e^{(\alpha-\beta)T} - Ke^{-\beta T} + e^{-\beta T}(K\Phi(-d) - \int_0^K s f^{(\alpha)}_{S_T|S_t}(s|S_t)ds) \tag{1.47}$$

$$c(S_0,T;\alpha,\beta) = S_0 - Ke^{-\alpha T} + Ke^{-\alpha T}\Phi(-d) - e^{-\alpha T}\int_0^K s f^{(\alpha)}_{S_T|S_t}(s|S_t)ds$$
$$= S_0 - e^{-\alpha T}\int_0^K s f^{(\alpha)}_{S_T|S_t}(s|S_t)ds - Ke^{-\alpha T}\Phi(d)$$

$$= S_0 e^{-\alpha T}\int_K^\infty s f^{(\alpha)}_{S_T|S_t}(s|S_t)ds - Ke^{-\alpha T}\Phi(d) \tag{1.48}$$

$$= S_0\Phi(d+\sigma\sqrt{T}) - Ke^{-\alpha T}\Phi(d)$$

$$c(S_0,\infty;\alpha,\beta) = \frac{(\gamma-1)^{\gamma-1}}{\gamma^\gamma} S_0^\gamma \tag{1.49}$$

$$\gamma = (\frac{1}{2} - \frac{\alpha}{\sigma^2}) + \sqrt{(\frac{1}{2} - \frac{\alpha}{\sigma^2})^2 + 2(\frac{\beta}{\sigma^2} - \frac{\alpha}{\sigma^2})}$$

1.11 Motivation

In the aftermath of each economical crisis both past and present it has become extremely evident that further research is need, given that the current models lack the sophistication to correctly model the current market. The current and most acceptable model, which is the Black-Scholes-Merton has shown quite a few draw back, given that right from the beginning it makes assumptions of the market which are not present in the real world. An example of this is the they assumption that the volatility of a market is constant, and with this most recent financial crisis we were shown the hard way that the volatility is most definitely not constant. Another motivator for this work is that when a financial crisis occurs many hard working people are affected and this is a humble attempt to lessen that burden.

1.12 Problem Formulation

1.13 Objectives

The objective of this work is to demonstrate how Levy processes better model real world stock market behavior and develop a beta version of an open-source desk top application for the operating system Windows that will model a specific real world stock option price in the *Bolsa Mexicana de Valores* (BMV) or in English the Mexican Stock Exchange.

1.13.1 Specific Objectives

The specific objectives of this project are as follows:

- Find where the current models fails.
- Develop and replace the current underlying distribution ($Normal(\mu, \sigma^2)$) with a Lévy process.
- Run the necessary simulations.
- Do a statistical study to see, if any, levels of significance.
- Study the results.
- Conclusions and discussions.

1.14 Thesis Structure

The thesis is organized as follows:

- Chapter 2 is a review on theories, lemmas, concepts and ideas covered throughout this work.
- Chapter 3 is dedicated on showing how we went about proving that the underlying Normal distribution fails. Mathematical proofs are given to demonstrate our findings.
- Chapter 4 we propose our model and and run preliminary simulations.

CHAPTER 2

Preliminary information

This chapter is dedicated to give further context of this thesis. Brief definitions of mathematical theories, lemmas as well as terms, concepts, and ideas needed to fully understand this work. As well as a few references if additional review is required.

2.1 Background

2.1.1 Financial Assets

***Bolsa Mexicana de Valores* (Mexican Stock Exchange)**

Bolsa Mexicana de Valores, S.A.B. de C.V. (BMV)- is a financial entity that operates by concession of the Mexican equivalent of the Internal Revenue Service (I.R.S) (*Sectretaría de Hacienda y credito público*) and must abide by *Ley del Mercado del Valores*, or Stock Market Law which governs said market.

The main objective of the BMV is to seek market development, facilitate market transactions, promote its expansion and competitiveness, through the following functions:

- Establish facilities, logistics and means that facilitate the interactions between supply and demand of stock options, hedge funds among other financial instruments that are registered in *Registro Nacional de Valores* or National Securities Registry (RNV).
- Compile and maintain as well as openly publish information about any and all financial entities and instruments registered in the BMV.
- Establish norms and regulations that ensure a the integrity of the BMV. As well as overseeing any and all operations to ensure this standard is upheld. They also have the power to imply disciplinary sanction if there is a breach of conduct.

Stocks

In this modern economic world it isn't unheard of a company being owned by its shareholder, pro rata investments and in turn these shares provide partial ownership of the company. Company's issue these stocks in order to raise funds. They also reflect the companies earning power as well as the value of the company's actual assets. Theses shares are more colloquially known as "stocks", these shares are traded and quoted in what we know as stock exchanges. The concept of stock exchanges is nothing new the earliest one was seen almost 500 years ago in Atwerp.

Indices

Also refered to as index fallows the value of a basket of bonds, stocks (i.e. BMV, S&P 500, EUROSTOXX 600), etc. Also derivative instruments on these indices can be used for hedge funds or hedging, which is essentially covering against risk. Additionally institutional funds, pension funds for example, who often manage large and diverse portfolios try to imitate specific stock indices and who use derivatives on stock instruments as a tool to aid in their portfolio management.

The reader should keep in mind that this thesis will be applied in the BMV index, from January 4, 2008 until December 4, 2018 which was obtained from yahoo finance.

Dividends

Are individual share earnings which are distributed among the share holders and is paid in proportion of their individual holdings. And are often pain either in currency or additional shares in the company. For our purposes we will look at assets which do not pay dividends, if time and recourse allow we will look at how to incorporate dividend paying assets in the model.

The Stock Price Processes

Throughout this theses we will be using a continuous-time processes to model the price processes of our asset be it a stock or index. We will denote this asset price processes also known as the stock price processes by:

$$S = S_t, t \geq 0 \quad S_t \text{ gives us the price at time } t \geq 0$$

We will need to compare investment in different securities and in order to do this we look at the returns or relative price changes over the time $s > 0$:

$$\frac{S_{t+s} - S_t}{S_t}$$

Even though this last method is valid we will be focusing on the logarithmic return or log-return:

$$log(S_{t+s}) - log(S_t) \quad \text{or} \quad log(\frac{S_{t+s}}{S_t})$$

The main reason that we will be utilizing this method to calculate the returns is because if we wish to calculate the log return of a period of length k x s it would be the same thing if we calculate the log return of K periods of length s:

$$(log(S_{t+s}) - log(S_t)) + (log(S_{t+2s}) - log(S_{t+s})) + \cdots + (log(S_{t+ks}) - log(S_{t+(k-1)s})) = log(S$$

Another reason why other authors also use this method is because in most models the stock price S_t will be modeled by an exponential of some basic stochastic process. And to be quite frank for continuous-time processes (like the ones we will be using in this thesis), returns with continuous compounding log returns are the natural choice.

2.1.2 Derivative Securities

Options

Is a financial instrument that gives one the *right but not the obligation* to makea specific transaction at/by a specified date and at a specified price. And you could view an option as a sale of privileges by one group or party to another. The person or group who the sells the option is called the option seller/writer. The person who buys is called the option buyer, no surprises there.

Types of Options

It it is noteworthy to state that there are many different types of options that exist, here we will give a description of the basic types.

- *Call* options gives one the right to buy.
- *Put* options gives one the right to sell.
- *European* options one the right to buy/sell on the specified date, the date of maturity or expiration.

Markets

Mercado de dinero o monetario ("money market") - issues and advertises what is known as "instruments" of credit in short-term contracts with very low risk, and whose return is known in advance. Examples of these "instruments" are:

- *Certificados de Tesorería de la Federacíon (CETES)* - In the need to finance additional projects and to help control the amount of currency in circulation and interest rates the federal government emits by way of BVM investment contracts which do not have the highest return but is one of the only items that has a guaranteed return.

- *Pagaré bancario* (Bank promissory note) - In a sense it is an I.O.U that banks or financial institutions emit in B.M.V that offers an immediate payout once the the contract expires, again not with the highest returns but the fact that come rain or shine the payout in full will be done when the contract is up.

- *Bonos de Desarrolo del Gobieron Federal* (Government bonds) - These are similar to the ones above but with the difference is that they are auctioned off by Mexico's central bank (Banco de México) and are usually purchased by other banks or financial institutions.

- *Petrobonos* (Petro-bonds) - As the name may imply they are bonds but that are honored and emitted by the national petroleum which was state owned but has been privatized.

Mercado de Capitales ("capital market") - In this market the contracts are high risk with higher payouts but are usually long-term. and the interest rate can either be fixed or vary with the market. An example of what is emitted, bought, sold or negotiated in this market:

- *Acciones* (Stock)- Are just as the name implies is stock in a company, and just like in the U.S. stock holder do not have direct control of the company they do share in the equity and benefit from the profits obtained.

2.1.3 Stochastic Processes

To serve our purpose we will assume that, T sometimes called the parameter set is an interval, i.e. $T = [a,b], [a,b), [a,\infty)$ for $a > b$. We will also assume that X is *continuous-time* processes.

Definition:

A *Stochastic process* X is a collection of random variables

$$(X_t, \quad t \in T) = \quad (X_t(\omega), \quad t \in T, \quad \omega \in \Omega)$$

define on some space Ω.

It may be obvious, but the index t of the random variable X_t is referred to as *time.* We will adopt this as well for our purpose.

Definition:

A Stochastic processes X is a two variable function.

For a fixed instant of time t, is a random variable:

$$X_t = X_t(\omega), \qquad \omega \in \Omega.$$

For a fixed random outcome $\omega \in \Omega$, it is a function of time:

$$X_t = X_t(\omega), \quad t \in T.$$

This function is called *realization, a trajectory* or a *sample path* of the process X.

2.1.4 Martingales

The underlying idea of a martingales is a fair game whose net winnings are evaluated through conditional expectation. This is worth mentioning given that an understanding of martingales notation is indispensable in order to understand Itô stochastic integral. Now assume that $(\mathcal{F}_t, t \geq 0)$ is a collection of $\sigma-$fields in the same space as Ω further more that all of the $\mathcal{F}_t$'s are a subset of a larger $\sigma-$field $\mathcal{F}$ on Ω.

Definition:

The collection $(\mathcal{F}_t, t \geq 0)$ of $\sigma-$field on Ω is called a *filtration*, if

$$\mathcal{F}_s \subset \mathcal{F}_t, \quad \forall\, 0 \leq s \leq t.$$

Ergo a filtration is an increasing stream of data. If $(\mathcal{F}_n, n = 0, 1, 2, \ldots)$ is a sequence of $\sigma-$field on Ω and $\mathcal{F}_t \subset \mathcal{F}_{n+1}\ \forall\, n$, we call $(\mathcal{F}_n)$ a *filtration* as well.

It must be mentioned that for our application, a filtration is frequently associated with stochastic processes.

Definition:

The stochastic process $Y = (Y_t, t \geq 0)$ is said to be *adapted to the filtration* $(\mathcal{F}_t, t \geq 0)$ if:

$$\sigma(Y_t) \subset \mathcal{F}_t \quad \forall t \geq 0$$

The stochastic process Y is always adapted to the *natural filtration generated* by Y :

$$\mathcal{F}_t = \sigma(Y_s, s \leq t).$$

Hence this stochastic process adaptation Y means that Y_t's do not carry more information than $\mathcal{F}_t$,

Definition:

If $Y = (Y_n, n = 0, 1, 2, \ldots)$ is a discrete-time process we define adaptedness is an analogous way: for a filtration $(\mathcal{F}_n, n = 0, 1, 2, \ldots)$ we require that $\sigma(Y_n) \subset \mathcal{F}_n$

Definition:

The stochastic process $X = (X_t, t \geq 0)$ is called a *continuous-time martingale with respect to the filtration* $(\mathcal{F}_t, t \geq 0)$, we write $(X, (\mathcal{F}_t))$, if:

- $E|X_t| < \infty \quad \forall\, t \geq 0$
- X is adapted to $(\mathcal{F}_t)$
- $$E[X_t|\mathcal{F}_s] = X_s \quad \forall\, 0 \leq s < t,$$
  i.e. X_s is the best prediction of X_t given $\mathcal{F}_s$.

It is also possible to define a discrete-time martingale $X = (X_n, n = 0, 1, 2, \ldots)$. In this case, we adapt the defining property as fallows:

$$E[X_{n+k}|\mathcal{F}_n] = X_n, \quad k \geq$$

We show that it suffices to require for $k = 1$.

$$E[X_{n+1}|\mathcal{F}_n] = E[E(X_{n+2}|\mathcal{F}_{n+1})|\mathcal{F}_n] = E[X_{n+2}|\mathcal{F}_n]$$

$$= E[E(X_{n+3}|\mathcal{F}_{n+2})|\mathcal{F}_n] = E[X_{n+3}|\mathcal{F}_n]$$

$$\vdots$$

$$= E[E(X_{n+k}|\mathcal{F}_{n+(k-1)})|\mathcal{F}_n] = E[X_{n+k}|\mathcal{F}_n]$$

Now we defined a martingale in the discrete-time case.

Definition:

The stochastic process $X = (X_n, n = 0, 1, 2, \ldots)$ is called a *discrete-time martingale with respect to the filtration* $(\mathcal{F}_n, n = 0, 1, 2, \ldots)$, we write $(X, (\mathcal{F}_n))$, if:

- $E|X_t| < \infty \quad \forall\, n = 0, 1, 2, \ldots$
- X is adapted to $(\mathcal{F}_n)$

- $$E[X_{n+1}|\mathcal{F}_n] = X_n \quad \forall\, n = 0, 1, 2, \ldots$$

  i.e. X_n is the best prediction of X_{n+1} given $\mathcal{F}_n$.

It is not difficult to see that the defining property can be rewritten in the form:

$$E[Y_{n+1}|\mathcal{F}_n] = 0; \quad \text{where} \quad Y_{n+1} = X_{n+1} - X_n, \quad n = 0, 1, 2, \ldots$$

The sequence (Y_n) is then called a *martingale difference sequence with respect to the filtration* $(\mathcal{F}_n)$.
In what fallows, we often say that "$(X_t, t \geq 0)$, repectivley$(X_n, n = 0, 1, 2, \ldots)$, is a martingale" without pointing out which filtration we use. This will clear from the context:

"A martingale has a remarkable property that its expectation function is constant."

Indeed, using the definition property $E[X_t|\mathcal{F}_s] = X_s$ for $s < t$ and Rule 2, we obtain:

$$EX_s = E[E(X_t|\mathcal{F}_s)] = EX_t, \quad \forall\, s \text{ and } t.$$

This provides an easy way to provide that a stochastic process is not a martingale. For example, if B is a Brownian motion, $EB_t^2 = t \quad \forall\, t$. Hence (B_t^2) cannot be a martingale. However, we cannot use this means to prove that a stochastic process is a martingale.

2.1.5 The Stochastic Integral

The Itô Stochastic Integral for Simple Processes

We start the investigation of the Itô stochastic integral for a class of process $X = (X_t, t \geq 0)$ is adapted to Brownian motion if X is adapted to $(\mathcal{F}_t, t \geq 0)$. This means that, for every t, X_t is a function of the past and present of Brownian motion.

Definition:

Lets now consider the following deterministic differential equation:

$$dx(t) = a(t, x(t))dt, \qquad x(0) = x_0.$$

The desire is to provoke randomness in this equation and the best way to do this is by randomize the initial condition. This will cause the solution $x(t)$ to become a stochastic process $(X_t, t \in [0, T])$

$$dX_t = a(t, X_t)dt, \qquad X_0(\omega) = X(\omega).$$

For our purposes, the randomness in the differential equation is introduced by an additional random noise term

$$dX_t = a(t, X_t)dt + b(t, X_t)dB_t, \qquad X_0(\omega) = X(\omega).$$

If the above equation is interpret as the stochastic integral equation

$$X_t = X_0 + \int_0^t a(s, X_s)ds + \int_0^t b(s, X_s)dB_s, \qquad 0 \leq t \leq T. \quad (5.1)$$

Which is known as the *Itô stochastic differential equation*. Where the first integral is the Riemann integral and the second is the Itô stochastic integral [23].

Definition 0.2:

A *strong solution to the Itô stochastic differential equation* () is a stochastic process $X = (X_t, t \in [0, T])$ which satisfy the following conditions:

- X is adapted to Brownian motion, i.e. at time t it is a function of $B_s, s \leq t$.
- The integrals occurring in () are well defined as Riemann or Itô stochastic integrals, respectively.
- X is a function of the underlying Brownian sample path of the coefficient functions $a(t, x)$ and $b(t, x)$.

Definition 0.3:

A strong or weak solution X of the Itô stochastic differential equation (5.1) is called a *diffusion*. In particular, Taking $a(t, x) = 0$ and $b(t, x) = 1$ in (5.1), it is obvious that Brownian motion is a diffusion process. (We will only consider strong solutions of the Itô stochastic differential equations.)

Definition 0.4:

Brownian motion B is called the *driving process* of the itô stochastic differential equation.

2.1.6 Black-Scholes option pricing formula.

Definition 0.5:

The geometric Brownian motion in the form

$$X_t = f(t, B_t) = X_0 e^{(c-0.5\sigma^2)t+\sigma B_0} \quad ()$$

Gives us the price X_t of a stock (*risk asset*) at time t

$$X_t = x_0 + c\int_0^t X_s ds + \sigma \int_0^t X_s dB_s, \quad (5.3)$$

where formally it can be written as:

$$dX_t = cX_t dt + \sigma X_t dB_t \qquad ()$$

Translating this equation in a unsophisticated way, and have it on $[t, t + dt]$

$$X_{t+dt} - X_t = cX_t dt + \sigma X_t dB_t$$

And in the same manner,

$$\frac{X_{t+dt} - X_t}{X_t} = cdt + \sigma dB_t$$

The left-hand side of this equation informs us that *cdt* is a linear trend which is disrupted by σdB_t which is a stochastic noise term. The *mean rate of return* is denoted by the constant $c > 0$ and the *volatility* is $\sigma > 0$

Known in financial theory as a *bond* which a non-risky asset. Considering the initial investment of β_0 in bonds will return the amount of

$$\beta_0 = \beta_0 e^{rt}$$

at time t.

As mentioned previously the initial investment has been continuously compounded with a constant interest rate $r > 0$ which is not realistic given that interest rates change over time. Something that is note worthy is that β satisfies the deterministic integral

$$\beta_t = \beta_0 + r \int_0^t \beta_s ds \qquad ()$$

The *portfolio* is made up of certain amount of shares of stock denoted by a_t and bonds denoted by b_t. Which is considered as stochastic processes adapted to the Brownian motion and is given the name of a *trading strategy* to the pair

$$(a_t, b_t), \qquad t \in [0, T].$$

Wealth also known as the *value of your portfolio* denoted by

$$V_t = a_t X_t + b_t \beta_t$$

Let us allow a_t and b_t to be any real number. If $a_t < 0$ then this means *short sale* of stock, i.e at time t is when the stock is sold. If $b_t < 0$ means that you *barrow* money at the bond's interest rate r. Another thing to take into consideration is that in a real world setting you would have to pay *transaction costs* which apply to operations on stock and sale, in this example theses costs will be omitted in the name of simplicity.

Additional assumptions are made: the first of which is that a_t and b_t are not bounded which would in theory imply that you could have an infinite amount of capital but on the negative side the same amount of debt. Again this is an over simplification, which is done just to make things a bit "easier". Second, we assume that no money is spent, i.e. the portfolio does not decrease due to *consumption*. The last assumption made, is that the trading strategy (a_t, b_t) is *self-financing*. Which means that any increase of wealth, which is denoted as V_t in only due to the change in price of X_t and β_t of the assets owned.

The *self-financing condition* is formulated in terms of differentials

$$dV_t = d(a_t X_t + b_t \beta_t) = a_t dX_t + b_t d\beta_t$$

Which we understand in the Itô sense of the relation

$$V_t - V_0 = \int_0^t d(a_s X_s + b_s \beta_s) = \int_0^t a_s dX_s + \int_0^t b_s d\beta_s$$

If we replace dX_s with $cX_s ds + \sigma X_s dB_s$ (5.3), and if we replace $d\beta_s$ with $r\beta_{s,}$(5.4). Thus the value of the portfolio V_t at time t is exactly equal to the sum of the initial investment V_0 and any capital gains from stock and bonds up to time t.

CHAPTER 3

Methodology

3.1 Black-Scholes Model

In this section we will discus and develop the most familiar continuous-time, continuous-variable stochastic process for stock prices [6]. A good understanding of this type of process is the fist part in understanding on pricing

3.2 Specification

A specification should tell the reader what the software system is required to do.

3.3 Design

The design then gives the top-level details of how the software system meets the requirement.

3.4 Implementation

3.4.1 Modeling Stochastic Rates

The initial dilemma

Consider the amount of money M(t) at time t invested in a bank account that pays interest at a constant rate r. The differential equation which models this problem is:

$$dM(t) = rM(t)dt$$

Given the initial investment $M(0) = M_0$, the account balance at time t is given by the solution of the equation, $M(t) = M_0 e^{(}rt)$.

The model in the real world

In the real world the interest rate r is not constant. It may be assumed constant only for a very small amount of time, such as one day or one week. The interest rate changes unpredictably in time, which means that it is a stochastic process. This can be modeled in several different ways. For instance, we may assume that the interest rate at time t is given by the continuous stochastic process $r_t = r + \sigma W_t$, where $\sigma > 0$ is a constant that controls the volatility of the rate, and W_t is a Brownian motion process. The process rt represents a diffusion that starts at $r_0 = r$, with constant mean $E[r_t] = r$ and variance proportional with the time elapsed, $Var[r_t] = \sigma 2t$. With this change in the model, the account balance at time t becomes a stochastic process Mt that satisfies the stochastic equation:

$$dM_t = (r + \sigma W_t)M_t dt \quad , t \geq 0$$

Solving the equation

In order to solve this equation, we write it as $dM_t - r_t M_t dt = 0$ and multiply by the integrating factor $e^{\int_0^t r_s ds}$ We can check that

$$d(e^{-\int_0^t r_s ds}) = -e^{-\int_0^t r_s ds} r_t dt$$

$$dM_t d(e^{-\int_0^t r_s ds}) = 0$$

since $dt^2 = dtdW_t = 0$. Using the product rule, the equation becomes exact:

$$d(M_t e^{-\int_0^t r_s ds}) = 0$$

Integrating yields the solution

$$M_t = M_0 e^{-\int_0^t r_s ds} = M_0 e^{-\int_0^t (r+\sigma W_s) ds}$$

$$= M_0 e^{rt+\sigma Z_t}$$

where $Z_t = \int_0^t Wsds$ is the integrated Brownian motion process. Since the moment generating function of Z_t is $m(\sigma) = E[e^{\sigma Z_t}] = e^{\frac{\sigma^2 t^3}{6}}$, we obtain

Conclusion

We shall make a few interesting remarks. If $M(t)$ and M_t represent the balance at time t in the ideal and real worlds, respectively, then

$$E[Mt] = M_0 e^{rt} e^{\frac{\sigma^2 t^3}{6}} > M_0 e^{rt} = M(t)$$

.

This means that we expect to have more money in the account of an ideal world rather than in the real world account. Similarly, a bank can expect to make more money when lending at a stochastic interest rate than at a constant interest rate. This inequality is due to the convexity of the exponential function. If $X_t = rt + \sigma Z_t$, then Jensen's inequality yields

$$E[e^{X_t}] \geq e^{E[X_t]} = e^{rt}.$$

3.4.2 Langevin's Equation

We shall consider another stochastic extension of the equation. We shall allow for continuously random deposits and withdrawals which can be modeled by an unpredictable term, given by αdW_t, with α constant. The obtained equation:

$$dM_t = rM_t dt + \alpha dW_t, \quad t \geq 0$$

is called *Langevin's equation*.

We shall solve it as a linear stochastic equation. Multiplying by the integrating factor e^{-rt} yields

$$d(e^{-rt}M_t) = \alpha e^{-rt} dW_t$$

Integrating we obtain

$$e^{-rt}M_t = M_0 + \alpha \int_0^t e^{-rs} dW_s$$

Hence the solution is

$$Mt = M0ert + \alpha \int_0^t e^{r(t-s)} dW_s$$

This is called the *Ornstein-Uhlenbeck process*. Since the last term is a Wiener integral, by Proposition we have that Mt is Gaussian with the mean

$$E[M_t] = M_0 e^{rt} + E[\alpha \int_0^t e^{r(t-s)} dW_s] = M_0 e^{rt}$$

and variance

$$Var[M_t] = Var[\alpha \int_0^t e^{r(t-s)} dW_s] = \frac{\alpha^2}{2r}(e^{2rt} - 1)$$

.

It is worth noting that the expected balance is equal to the ideal world balance $M_0 e^{rt}$. The variance for t small is approximately equal to $\alpha^2 t$, which is the variance of αW_t.

If the constant α is replaced by an unpredictable function $\alpha(t, Wt)$, the equation becomes

$$dM_t = rM_t dt + \alpha(t, W_t)dW_t, \quad t \geq 0$$

.

Using a similar argument we arrive at the following solution:

$$M_t = M_0 e^{rt} + \int_0^t e^{r(t-s)}\alpha(t, W_t)dW_s$$

This process is not Gaussian. Its mean and variance are given by:

$$E[M_t] = M_0 e^{rt}$$

$$Var[M_t] = \int_0^t e^{2r(t-s)} E[\alpha^2(t, W_t)]ds$$

In the particular case when $\alpha(t, W_t) = e^{\sqrt{2r}W_t}$, with $\lambda = \sqrt{2r}$, we can work out an explicit form of the solution

$$M_t = M_0 e^{rt} + \int_0^t e^{r(t-s)} e^{\sqrt{2r}W_t} dW_s$$

$$= M_0 e^{rt} + e^{rt}\int_0^t e^{-rs} e^{\sqrt{2r}W_t} dW_s$$

Solving the equation

3.4.3 Stock Prices with Rare Events

In order to model the stock price when rare events are taken into account, we shall combine the effect of two stochastic processes:

- the Brownian motion process W_t, which models regular events given by infinitesimal changes in the price, and which is a continuous process;
- the Poisson process N_t, which is discontinuous and models sporadic jumps in the stock price that corresponds to shocks in the market.

Since $E[dN_t] = \lambda dt$, the Poisson process N_t has a positive drift and we need to "compensate" by subtracting λt from N_t. The resulting process $M_t = N_t - \lambda t$ is a martingale, called the compensated Poisson process, that models unpredictable jumps of size 1 at a constant rate λ. It is worth noting that the processes W_t and M_t involved in modeling the stock price are assumed to be independent.

Let $S_{t-} = \lim S_u$ denote the value of the stock before a possible jump occurs at time t. To set up the model, we assume the instantaneous return on the stock, $\frac{dS_t}{S_{t-}}$, to be the sum of the following three components:

- the predictable part μdt;

- the noisy part due to unexpected news σdW_t;
- the rare events part due to unexpected jumps ρdM_t,

where μ, σ and ρ are constants, corresponding to the drift rate of the stock, volatility and instantaneous return jump size (In this model the jump size is constant; there are models where the jump size is a random variable, see [**?**]) adding yields:

$$\frac{dS_t}{S_t-} = \mu dt + \sigma dW_t + \rho dM_t$$

Hence, the dynamics of a stock price, subject to rare events, are modeled by the following stochastic differential equation:

$$dS_t = \mu S_{t-}dt + \sigma S_{t-}dW_t + \rho S_{t-}dM_t$$

It is worth noting that in the case of zero jumps, $\rho = 0$, the previous equation becomes the classical stochastic equation

Using that W_t and M_t are martingales, we have:

$$E[\rho S_t dM_t|\mathcal{F}] = \rho S_t E[dM_t|\mathcal{F}] = 0$$

$$E[\sigma S_t dM_t|\mathcal{F}] = \sigma S_t E[dM_t|\mathcal{F}] = 0$$

This shows the unpredictability of the last two terms, i.e. given the information set $\mathcal{F}_\sqcup$ at time t, it is not possible to predict any future increments in the next interval of time dt. The term $\sigma S_t dW_t$ captures regular events of insignificant size, while $\rho S_t dM_t$ captures rare events of large size. The "rare events" term, $\rho S_t dM_t$, incorporates jumps proportional to the stock price and is given in terms of the Poisson process N_t as:

$$\rho S_{t-}dM_t = \rho S_{t-}d(N_t - \lambda t) = \rho S_{t-}dN_t - \lambda\rho S_{t-}dt$$

Substituting into equation yields:

$$dS_t = (\mu - \lambda\rho)S_{t-}dt + \sigma S_{t-}dW_t + \rho S_{t-}dN_t$$

The constant λ represents the rate at which the jumps of the Poisson process N_t occur. This is the same as the rate of rare events in the market, and can be determined from historical data.

The following result provides an explicit solution for the stock price when rare events are taken into account

Proposition

The solution of the stochastic equation is given by:

$$S_t = S_0 e^{(\mu+\lambda\rho-\frac{\sigma^2}{2})t+\sigma W_t}(1+\rho)^{N_t}$$

where
μ is the stock price drift rate.
σ is the volatility of the stock.
λ is the rate at which rare events occur.
ρ is the size of jump in the expected return when a rare events occurs.

Proof: We shall construct first the solution and then show that it verifies the equation. If $t_k$denotes the kth jump time, then $N_{t_k} = k$. Since there are no jumps before t_1, the stock price just before this time is satisfying the stochastic differential equation:

$$dS_t = (\mu - \lambda\rho)S_t dt + \sigma S_t dW_t$$

with the solution given by the usual formula:

$$S_{t_{1-}} = S_0 e^{(\mu+\lambda\rho-\frac{\sigma^2}{2})t_1+\sigma W_{t_1}}$$

Since $\frac{dS_{t_1}}{S_{t1-}} = \frac{S_{t_1} - S_{t_{1-}}}{S_{t_1}}$, then $S_{t_1} = (1+\rho)S_{t_{1-}}$. Substituting in the aforementioned formula yield:

$$S_{t_1} = S_{t_{1-}}(1-\rho) = S_0 e^{(\mu+\lambda\rho-\frac{\sigma^2}{2})t_1+\sigma W_{t_1}}(1+\rho)$$

Since there is no jump between t_1 and t_2, a similar procedure leads to

CHAPTER 4

Results and Discussion

4.1 Results

At this point of the this thesis it is still early to make any definitive claims, the reason why is it took rigorous mathematical analysis of the theory in order to see exactly were the underlying distribution fails. During our investigation we have concluded UP TO THIS point that the underlying Normal distribution is not an adequate distribution.

Table 4.1: Major index and corresponding information.

Index	Mean	SD	Skewness	Kurtosis
BMV	.000279	.019181	.260399	2.585409
S&P 500	.000235	.012653	-0.368830	11.0583
STOXX Europe 600	5.1497E-8	.012669	-.254147	6.161053

In Table 4.1 we have obtained the financial information via yahoo finance form January 4, 2008 up until December 4, 2018 of three major indices *Bolsa Mexicana de Valores*, S&P 500, and the EUROSTOXXX 600, we have calculated thier respective log returns once we have obtained this, we move to calculating the mean, Standard Deviation, Skewness, and Kurtosis for each of the indices to show empirically that in fact the Normal distribution is a poor choice, given that the Normal distribution has a skewness of 0 and a kurtisis of EXACTLY 3, as we can see this in not the case in any of these indices.

4.2 Discussion

Interpret and explain your results and justify your approach by answering your research problem.

4.3 Significance/Impact

Provide an explanation of the work's significance, its potential benefits and its overall impact. The impact of the project should describe the following:

4.3.1 Social Impact

1. With this model we hope to lessen the burden felt by the working class when a financial crisis occurs.
2. If with this model we could better predict or foresee financial crisis with enough foresight that people could trade accordingly and in turn lessen financial loss.

4.3.2 Environmental Impact

1. Impact 1
2. etc. ..

4.3.3 Economic Impact

1. Impact 1
2. etc. ..

4.4 Publications

Provide your list of publications obtained during the project period, such as conference proceedings, journal publications and patents.

4.5 Future Work

- Estimating the parameters for simulation is not a simple process, so it is important using some software and according to literature do not always work well for simulations. That's why working on a tool to generate these parameters can be very useful.
- Compare Levy's process with neural networks, genetic algorithms, and other methodologies related.
- The literature mentions that not every database is always well received with a Levy process, inquire into the subject and generate some criteria and introduce it to the simulation can help improve and thus know which database is most suitable for simulation through this process.

CHAPTER 5

Conclusion

The objectives of the work were fulfilled which were to develop a simulation for the prices of the shares and understand the topic better. We obtained results that prove it is possible to simulate share's prices with Levy's process. Stochastic differential equations are undoubtedly a great tool that can be used in many areas. One of the main objectives was to seek simulation with an action through a Levy process, however, the execution time of the algorithm was set aside. This leaves us with the possibility of looking for a more efficient algorithm with which the simulation can be performed.

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