Universidad Autónoma de Querétaro

Facultad de Ingeniería

Theoretical Development of a Magnetic Spring-Based Hydroelectric Converter by Vortex-Induced Vibrations

T E S I S

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presenta:

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TESIS

Que como parte de los requisitos para obtener el Grado de

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México

Este trabajo de tesis está dedicado a mi familia.
A mi mamá y mi hermano por ser mi soporte y apoyo incondicional.
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Resumen

Esta tesis presenta el desarrollo teórico de un convertidor de energía hidroeléctrica basado en resortes magnéticos y vibraciones inducidas por vórtices (VIV). Se proponen, modelan y analizan tres sistemas oscilatorios no lineales: un oscilador magnetogravitacional, un sistema oscilatorio de tres imanes y un péndulo rígido con repulsión magnética. Se estudió el comportamiento de los sistemas dinámicos mediante mecánica lagrangiana y hamiltoniana, análisis dimensional, métodos de perturbación asintótica y simulaciones numéricas (Runge-Kutta). Se estudiaron los sistemas en condiciones ideales, amortiguadas y forzadas. La interacción fluido-estructura se incorpora mediante VIV, donde la fuerza de sustentación y el número de Strouhal definen la excitación externa. La estabilidad y el caos se evalúan mediante exponentes de Lyapunov, multiplicadores de Floquet y diagramas de bifurcación. Finalmente, se aplica la ley de inducción de Faraday para estimar la tensión eléctrica generada en las bobinas, demostrando la viabilidad de convertir la oscilación mecánica en energía eléctrica. Los resultados resaltan el papel de los parámetros adimensionales (por ejemplo, el número de Energie, el número de Strouhal, la relación gravedad-inercia) y muestran que la configuración basada en péndulo ofrece el mayor potencial para una recolección de energía eficiente.

Abstract

This thesis presents the theoretical development of a hydroelectric energy converter based on magnetic springs and vortex-induced vibrations (VIV). Three nonlinear oscillatory systems are proposed, modeled, and analyzed: a magneto-gravitational oscillator, a three-magnets oscillator system, and a rigid pendulum with magnetic repulsion. We studied the behavior of the dynamic systems using Lagrangian and Hamiltonian mechanics, dimensional analysis, asymptotic perturbation methods, and numerical simulations (Runge-Kutta). We studied our systems under ideal, damped, and forced conditions. The fluid-structure interaction is incorporated through VIV, where the lift force and Strouhal number define the external excitation. Stability and chaos are assessed via Lyapunov exponents, Floquet multipliers, and bifurcation diagrams. Finally, Faraday's law of induction is applied to estimate the electrical voltage generated in coils, demonstrating the feasibility of converting mechanical oscillation into electrical energy. The results highlight the role of dimensionless parameters (e.g., Energie's number, Strouhal number, gravity-inertia ratio) and show that the pendulum-based configuration offers the highest potential for efficient energy harvesting.

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Introduction

Since Ancient times, the first civilizations of humanity sought energy sources that would facilitate labor and manufacturing processes, such as mills pulled by oxen, horses, or slaves, and typical water mills or even Archimedes' screw; but it wasn't until the Industrial Revolution that, taking advantage of thermodynamics knowledge and flourishing science, humans had the capacity to provide machines with enough energy to reduce both production time and labor.

The method used in the Industrial Revolution was refined over time, building increasingly efficient machines, but using the same type of energy source: fossil fuels, which later would prove to be harmful not only to human health but to the entire biosphere, generating a phenomenon known as climate change, as fuel residues disperse in the environment causing the Earth to absorb more radiation, warming it and changing the climate of different ecosystems around the world.

Due to the above, the United Nations (UN) in 1979 during the 1st World Climate Conference identified climate change as an urgent global problem and urged nations to seek new energy sources, which would not leave such a marked impact on nature. Since then, humanity has been looking for renewable energy sources, some examples of these energies are solar, wind, nuclear, etc.

Renewable energies are at their peak. The best-known renewable energies are wind and solar. But as mentioned by Michaelides, two-thirds of the planet is covered with water, water that is in constant movement and has kinetic energy, which is not usually harnessed, which is why an alternative to other renewable energy sources with great potential is found in water flows and fluid movement ([25]).

This work will study different mechanical systems that implement repulsive magnetic forces to generate an oscillatory system, with the intetion to develop a theoretical analysis related to the conversion of mechanical energy to electrical energy from fluid-structure interaction, since a fluid in motion possesses kinetic energy, which can be transformed into another type of energy, especially electrical energy. To convert kinetic energy into electrical energy, a ***hydroelectric converter*** is required, which is a system that allows us to convert the kinetic energy from the moving fluid into electrical energy.

We are going to work with three different systems or forms to build our hydroelectric converter. The first system consists of two magnets: one fixed and one free to move vertically, where the restoring force comes from the combination of gravity and magnetic repulsion. The second system uses three magnets: two fixed at the ends of a vertical channel and one free to move between them, generating a symmetric oscillation. The third

system is a rigid pendulum with a magnet at its tip, whose oscillation is limited by two fixed magnets placed symmetrically.

The operation of the hydroelectric converter is as follows: when the moving fluid interacts with the structure, it causes it to experience an oscillating pressure force, pushing the structure towards one of the channel walls, where as it approaches, the repulsive forces increase, repelling the structure from the wall in the direction of the other wall, the process is repeated generating a forced oscillation between the structure and the fluid. It is important to highlight that the restitution force present in the system is a magnetic force, which is non-linear.

It is important to know the theoretical behavior of such a system because it allows us to estimate how much energy can be captured and transformed.

We can explain our motivation like:

- Fluid–structure interaction appears naturally in many environments: ocean currents, wind flows, and vortex streets generated around obstacles. These phenomena contain mechanical energy that is often wasted.
- Magnetic–gravitational oscillators offer a simple, low-cost way to capture part of this natural energy without the need for electronics, complex materials, or external power sources.
- By using only magnets, gravity, and fluid motion, this system proposes an alternative pathway toward clean and passive energy harvesting.
- Understanding the dynamics of these oscillators helps us design devices capable of converting environmental flows into useful oscillations, contributing to sustainable technologies and small-scale renewable systems.
- This project aims to analyze, model, and simulate these mechanisms to demonstrate their potential as energy–extraction tools for a cleaner future.

The first, second, and third chapters are theoretical chapters. Here, we present all the theory that we are going to need to work on our systems. Each chapter is focused on a specific topic with the idea of concentrating the same information in the same block.

In the first chapter, we discuss the mathematical techniques we need to apply to the differential equations of our systems to solve and analyze them. We explore the concept and power of differential equations, their classification, and how we can analyze them before to solve it, to know their behavior and stability. Another important topic that we inspect is the Perturbative analysis, and how it is useful to solve differential equations that are commonly unsolvable. We introduce and explain the idea behind dimensional analysis and how transforming systems to a dimensionless system could help in the analysis. Another group of techniques that we address in this chapter is the mechanical analytical techniques (Lagrangian and Hamiltonian analysis). Finally, we talk about numerical solutions for differential equations that are analytically unsolvable, in particular, the Runge-Kutta method.

The second chapter focuses on the oscillation theory. Here, we talk about the different types of oscillations (harmonic, damped, and forced oscillations), the different natures of the restoring forces on an oscillatory system, and how to solve them. In every system, we can obtain the expression of the energy, so we find the expression of the energy for each system. For our systems, which have an external force adding energy, it is important to

understand and apply the idea of resonance, so at the end of this chapter, we study this crucial phenomenon.

As we mentioned previously, we are going to extract energy from the motion of fluids. So, it is important to know how fluids act in nature, and the third chapter of our project is centered on that idea. So we explore the fluid dynamics and the fluid-structure interaction. The fluid-structure interaction generates a kind of vortices that are generated at some period of time, these vortices are the Vortex Induced Vibrations (VIV), and they are so useful because they are the form that we can estimate the energy that a structure or mechanical system can extract from a fluid in motion.

The chapters fourth, fifth, and sixth are focused on solving the different systems that we analyze in our project. Each chapter is focused on a specific system and its solutions.

The fourth chapter is dedicated to the simplest system, the magneto-gravitational system. In this system, we establish our methodology to work in all systems, and we apply the techniques that we explain in the theoretical chapters. This system is characterized by the presence of gravity and the repulsive force between two magnets. And we can find some important dimensionless forms of our system that can help us to describe the movement and the voltage induction in an open circuit.

The fifth chapter extends the analysis to a system with three magnets, where the free magnet oscillates between two fixed magnets. This configuration offers a symmetric potential well and allows us to study the effect of confinement on the oscillation. We analyze the ideal, damped, and forced cases, and we also study the stability and chaos conditions using Lyapunov exponents and bifurcation diagrams. Finally, we model the energy harvesting capability of this system.

The sixth chapter is dedicated to the analysis of a physical pendulum that has a magnet at the bottom of it, and its oscillation is limited by two fixed magnets. Applying the mathematical analysis techniques, we find that this system is the most stable among the three, but also the most complex due to the coupling between gravitational torque and magnetic forces. We derive the dimensionless equations, study the small-angle approximation, and develop a multi-coil model to estimate the total induced voltage, demonstrating that this configuration is the most effective for energy harvesting.

The final chapter, the seventh, is dedicated to the conclusions. In this chapter, we are going to resume and recap the conclusions of the fourth, fifth, and sixth chapters, and we establish how dimensionless numbers and our theoretical tools were effective for this work. We also discuss the main findings, the limitations of our models, and suggest future lines of research to optimize these systems for practical applications in renewable energy generation.

0.1 Objectives

The general objective of our project is to develop a theoretical framework for a hydroelectric converter that transforms the kinetic energy of a moving fluid into electrical energy using vortex-induced vibrations acting on oscillatory systems that use the magnetic force as a restorative force, using analytical mechanics, dimensional analysis, asymptotic approximations, and numerical methods.

0.1.1 Specific objectives

The specific objectives of this project are:

- Derive the equations of motion of our dynamical systems using Lagrangian and Hamiltonian mechanics.
- Non-dimensionalize the systems through dimensional analysis to identify key dimensionless parameters (Energie's number, friction number, Strouhal number, gravity-inertia ratio) and solve the resulting equations using asymptotic approximations and numerical methods (Runge-Kutta).
- Analyze the dynamical stability of the forced systems by computing Lyapunov exponents, Floquet multipliers, and bifurcation diagrams to determine conditions for periodic, stable, or chaotic behavior under vortex-induced vibrations.
- Evaluate the energy harvesting potential by applying Faraday's law to estimate induced voltage in coils, comparing the three proposed configurations to identify the most effective design for converting fluid motion into electrical energy.

CHAPTER 1

Mathematical Analysis Techniques

> Philosophy is written in this grand book, the universe, which stands continually open to our gaze, but it cannot be understood unless one first learns to comprehend the language and read the characters in which it is written. It is written in the language of mathematics, and its characters are triangles, circles, and other geometric figures.
>
> Galileo Galilei, *"The Assayer" ("Il Saggiatore")*, 1623

In this chapter, the various mathematical analyses required to study dynamic systems are presented. Primarily, it delves into the study of differential equations and their methods of resolution, especially the solution by asymptotic series. Additionally, it discusses the study of physical dimensions in engineering problems and how to address them to generate a dimensionless system, meaning applicable to all systems, as well as the use of analytical mechanics (Lagrangian and Hamiltonian mechanics) to understand and characterize the behavior of a system, particularly oscillating systems with forces different from those of Hooke, fluid systems, and those involving fluid-structure interaction.

1.1 Analysis of Differential Equations

Differential equations, as Adkins and Davidson [1] states, are a tool used in multiple problems in science and engineering. They are employed when a description of a measurable quantity is needed in terms of another independent measurable quantity; generally, the independent quantity is time. This can be described using Leibniz's notation, where the differential represents the rate of change of the dependent variable concerning the independent one, which corresponds to a value or function, mathematically expressed as:

$$\frac{df(x)}{dx} = g(x), \tag{1.1.1}$$

where $f(x)$ represents the measurable quantity whose behavior we want to understand (property), x is the independent quantity against which the behavior of $f(x)$ is contrasted (parameter), and $g(x)$ is the behavior of $f(x)$ as x changes. The differential equation is

called *ordinary* if our quantity to measure depends only on one independent quantity, and when it depends on more, it is called *partial* ([27]). An example of a partial differential equation is the following:

$$\frac{\partial f(x,y)}{\partial x} + \frac{\partial f(x,y)}{\partial y} = g(x,y), \tag{1.1.2}$$

where x and y are the parameters, $f(x,y)$ is the property whose description we want to know as x and y vary, and $g(x,y)$ is the known behavior of $f(x,y)$ when the parameters change.

Something important to note is that the properties ($f(x)$) can be expressed in vector form to encompass all the properties of the system in a simple expression. Therefore, the rate of change ($g(x)$) can also be a vector.

$$\frac{d\vec{f}(x)}{dx} = \vec{g}(x), \tag{1.1.3}$$

where $\vec{f}(x)$ is the vector of n parameters that depend on x, and $\vec{g}(x)$ is the vector of the rates of change of those parameters.

For the nature of differential equations, when a differential equation cannot be solved analytically, we can use certain techniques to analyze the system's behavior and, with that, build an approximate solution.

Phase line

When we are working with differential equations, the easiest way to know the behavior of a differential equation is by plotting it, which helps us to find the critical points that can be or not equilibrium points.

The phase line technique is the simplest analysis and, in essence, consists of finding the critical points and in which direction a point in the function will move.

As we know, the critical points of a function are found using the first derivative criteria. In a math form, we have:

$$\frac{df(x)}{dx} = g(x) \quad \text{if} \quad g(x_0) = 0, \tag{1.1.4}$$

where $f(x)$ is the function that we want to understand, x is the independent variable, $g(x)$ is the first derivative of the $f(x)$ function and x_0 is a critical point of the $f(x)$ function. The number of critical points is proportional to the grade of the $g(x)$ function. This can be expressed as.

$$\forall\ g(x) = \sum_{i=0}^{n} a_i x^i : g(x) = 0 \implies \exists\ n \text{ roots or critical points.} \tag{1.1.5}$$

Now that we know the fundamental idea, we can introduce the function of this technique. The phase line consists of putting in a line, as the number line, the critical points of a function, with the intention where the equilibrium points can be. Once we know the equilibrium points of our function, we need to know if they are stable or unstable.

The stability of a point can easily be understood. If a neighbor point of a critical point is going away when the time increases, then the critical point is unstable and is called ***asymptotically unstable***. On the other hand, when the neighbor point is attracted to the critical point, we said that is ***asymptotically stable***.

To understand this technique better, we are going to do an example.

Example

The function that we are going to test is $f(x) = x - 2x^2 - 3x^3 + x^4 + x^{21} + 0.2$, and the grafic of this function is

Phase plane

This analysis is the dimension extension of the phase line analysis because in the phase plane, we can see the inflection points and, more importantly, the behavior of the system. So, we can determine with this analysis whether a system follows a cyclic movement.

For this technique, we require a couple of sets of equations. The equations must have been independent but with some relation between them. In the case of physics, the relation is the energy, the force, and other fundamental physics definitions.

Nullclines

Nullclines are curves in the phase plane that help us to divide the space into different regions that have different behaviors. This technique, as Witelski and Bowen ([44]) highlight, does not provide the solution of the system but lets us know how the trajectories act when they pass through the nullclines curves.

To work with nullclines, we need to start considering that the derivates of our function, at least at a point, are zero, and we are trying to find those points. This can be expressed as

$$\frac{df(x,y)}{dx} = g(x,y) = 0 \ \wedge \ \frac{df(x,y)}{dy} = h(x,y) = 0, \tag{1.1.6}$$

where $f(x,y)$ is our function that we want to analize, x and y are the parameters that affect our function, $g(x,y)$ and $h(x,y)$ are the derivative functios of $f(x,y)$ by each independet variable. That $g(x,y)$ and $h(x,y)$ are equal to zero is so important because in the points where these functions are zero, the trajectories change.

Then, we are going to analyze what nullclines mean. We will begin with the x analysis. $\frac{df(x,y)}{dx} = 0$ is called an x-nullcline and means that the function is completely vertical (because the function does not change in the x direction). For analogy, the analysis of y is the y-nullcline and is when the function has a purely horizontal behavior, or in the math form $\frac{df(x,y)}{dy} = 0$.

The direction of the trajectories can be known with the functions $g(x,y)$ and $h(x,y)$, these functions let us know what the trajectories are outside the nullclines points. The nullcline points are sometimes called "equilibrium points".

Non analitical solution

As Rogan C. and Muñoz G. mentions, in physics, understanding the nature of a force leads us to a motion equation, which in turn derives into a differential equation, which is

generally a partial differential equation. Furthermore, we know from what Thornton and Marion has stated that the motion equation derived from forces or the Lagrangian is a second-order differential equation.

One of the challenges with differential equations, both ordinary and partial, is that often the nature of the problem described mathematically is too complex to be solved analytically ([23]), which is why they are usually solved numerically or through some type of approximation.

Limit cycles

As we mentioned previously, differential equations have the property of modeling the world, and it is important to know what their behavior will be. Some dynamical systems present oscillations that do not depend on the initial conditions, but these oscillations are related to the nature of the phenomena; in other words, some systems begin to oscillate due to an intrinsic property. These behaviors can be shown in the phase plane as a closed trajectory, and they are called ***Limit Cycles.***

Something that is important to highlight is that limit cycles are isolated closed trajectories. Unlike the centers found in linear systems (which come in continuous families), a limit cycle stands alone—nearby trajectories are either attracted to it or repelled from it, but they are not themselves closed orbits. This isolation is a defining characteristic of limit cycles.

Depending on the behavior of nearby trajectories, limit cycles are classified as:

- **Stable (attracting):** All nearby trajectories approach the limit cycle as $t \to +\infty$. These model self-sustained oscillations in real-world systems, such as the beating of a heart or the vibrations of a bridge in strong winds.
- **Unstable (repelling):** Nearby trajectories move away from the limit cycle.
- **Semi-stable:** Attracting from one side and repelling from the other.

The presence of a limit cycle is a hallmark of nonlinearity—linear systems cannot produce isolated periodic orbits. Their study is therefore essential for understanding any phenomenon that exhibits persistent, self-generated oscillations.

Detecting Limit Cycles

Proving the existence and uniqueness of limit cycles is one of the most challenging problems in dynamical systems (it is related to Hilbert's 16th problem). However, several powerful tools exist:

1. **Poincaré–Bendixson Theorem:** This is the most important existence theorem for limit cycles in the plane. It states that if a trajectory is confined to a closed, bounded region of the phase plane that contains no fixed points, then the trajectory must itself approach a closed orbit (a limit cycle). To apply it, one must construct a **trapping region**—an annular region where the vector field points inward on the outer boundary and outward on the inner boundary (or vice versa)—and verify that the region contains no equilibria.
2. **Bendixson's Criterion:** This is a negative criterion. If, in a simply connected domain D, the divergence $\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}$ does not change sign and is not identically zero,

then the system has **no** closed orbits lying entirely in D. This is useful for ruling out limit cycles in specific regions.

3. **Dulac's Criterion:** A generalization of Bendixson's criterion. If there exists a function $B(x, y)$ (a Dulac function) such that $\frac{\partial(Bf)}{\partial x} + \frac{\partial(Bg)}{\partial y}$ has a constant sign in a simply connected region, then no closed orbit can exist entirely in that region.

The analysis of limit cycles is crucial for understanding systems that exhibit sustained oscillations, which can be of any nature, biological, chemical, or otherwise.

Poincaré map

The phase plane, as we have seen, shows us a continuous picture of the dynamics of a system, but it can be complex to analyze particularly if the system presents periodic orbits. The **Poincaré map** is a mathematical technique that reduces the study of a continuous-time system to the study of a discrete-time map.

The construction of a Poincaré map is as follows:

1. **Define a Surface:** In the phase space, we place a surface Σ of codimension one (a line in the 2D plane, a plane in 3D space) that is transverse to the flow of the dynamical system. This surface is called a **Poincaré section**.
2. **Intersect and Record:** For a given periodic orbit, we start at a point x_k on Σ and follow the trajectory until it next intersects Σ. This new point of intersection is called x_{k+1}.
3. **Define the Map:** The Poincaré map $P : \Sigma \to \Sigma$ is defined by the rule $x_{k+1} = P(x_k)$.

This process is illustrated schematically in Figure 1.1 (not shown here). Instead of analyzing the continuous flow, we analyze the discrete sequence of points $\{x_0, x_1, x_2, \dots\}$.

Advantages of the Poincaré Map

- **Dimensionality Reduction:** The study of a continuous n-dimensional flow is reduced to the study of an $(n-1)$-dimensional discrete map. For a limit cycle in the plane (2D), the Poincaré section is a line (1D), making analysis trivial.
- **Stability of Periodic Orbits:** A limit cycle in the continuous system corresponds to a **fixed point** of the Poincaré map. The stability of the limit cycle is directly related to the derivative of the map at that fixed point.

  $$\text{If } |P'(x^*)| < 1 \implies \text{fixed point } x^* \text{ is attracting} \implies \text{limit cycle is stable.}$$
  $$\text{If } |P'(x^*)| > 1 \implies \text{fixed point is repelling} \implies \text{limit cycle is unstable.}$$

- **Detecting Chaos:** For more complex systems (three dimensions or more), the Poincaré map can reveal chaotic behavior. A simple closed curve on the Poincaré section suggests a quasiperiodic orbit (a torus), while a dense set of points with fractal structure is a hallmark of a chaotic attractor.

Lyapunov exponents

The concepts of stability discussed so far (stable/unstable fixed points and limit cycles) are qualitative. **Lyapunov exponents** provide a *quantitative* measure of the sensitivity

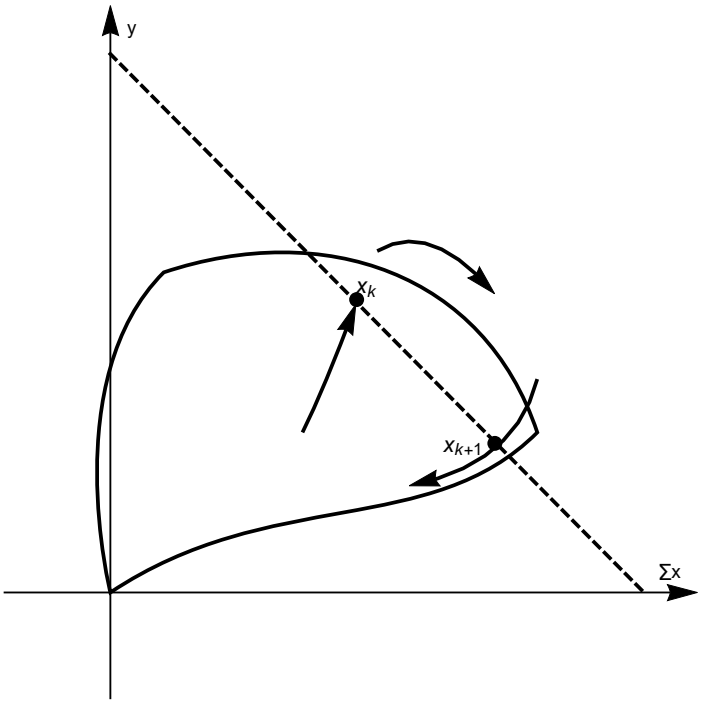


Figure 1.1: Schematic illustration of a Poincaré map. A limit cycle (closed curve) intersects a transverse section Σ at points x_k and x_{k+1}. The flow maps x_k to x_{k+1} after one period.

of a dynamical system to initial conditions. They characterize the rate at which nearby trajectories in phase space diverge or converge.

Imagine two trajectories in phase space starting very close together, separated by an infinitesimal distance $\delta\mathbf{x}(0)$. As the system evolves, this separation will change. For a chaotic system, it will grow exponentially. The (maximal) Lyapunov exponent λ quantifies this growth:

$$\|\delta\mathbf{x}(t)\| \approx e^{\lambda t}\|\delta\mathbf{x}(0)\|. \tag{1.1.7}$$

In an n-dimensional system, there is a spectrum of n Lyapunov exponents, $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$, each describing the average rate of expansion or contraction along a particular direction in phase space.

Interpreting the Lyapunov Spectrum

The signs of the Lyapunov exponents are of paramount importance for classifying the nature of the attractor:

- **Fixed Point (Equilibrium):** All Lyapunov exponents are negative ($\lambda_i < 0$). Perturbations decay in all directions.
- **Limit Cycle (Periodic Orbit):** One Lyapunov exponent is zero ($\lambda_1 = 0$), corresponding to the direction along the flow. All others are negative ($\lambda_{i>1} < 0$).

- **Quasiperiodic Orbit (Torus):** For an m-frequency quasiperiodic orbit, the first m exponents are zero. The rest are negative.
- **Chaotic Attractor:** At least one Lyapunov exponent is positive ($\lambda_1 > 0$). This indicates exponential divergence of nearby trajectories, which is the hallmark of chaos. The sum of all exponents is negative, meaning the dynamics are dissipative and converge to a bounded attractor.

A positive Lyapunov exponent means that even a tiny uncertainty in the initial state will grow rapidly, making long-term prediction impossible—a phenomenon known as the **butterfly effect**.

Numerical Calculation

For most nonlinear systems, Lyapunov exponents cannot be calculated analytically. Then, they are computed numerically by integrating not only the equations of motion for a trajectory but also the **linearized equations of motion** for a set of perturbation vectors.

In conclusion, Lyapunov exponents are the most fundamental diagnostic for chaos. They transform the qualitative idea of "sensitive dependence" into a measurable, calculable quantity that is essential for the rigorous study of nonlinear dynamics.

1.2 Asymptotic or Perturbative Analysis

As presented in the previous section, there is a wide range of differential equations that do not have an analytical solution. For this reason, unsolvable equations are addressed using numerical methods or approximation methods. The most common approximation method for solving differential equations is to generate a solution in the form of a series.

This section introduces asymptotic and perturbative approximations. Furthermore, it presents how these techniques are used for solving equations, particularly differential equations, whether analytically solvable or unsolvable.

1.2.1 The Importance of Asymptotic Approximation

In physics, as in mathematics, there are a limited number of equations (both conventional and differential) that have a solution. To solve problems that lack an exact solution, various techniques are available, such as numerical solutions or approximations. It is in these latter methods that an approximation emerges, namely asymptotic approximation, which involves approximating a set of functions to the real solution.

The main idea of asymptotic approximation is to transition from an expression with the equality sign ($=$) to the asymptotic or similar sign ($\sim$). This is because the equality sign imposes significant restrictions on the possible values obtained, while the asymptotic symbol allows for more flexibility in deriving solutions, permitting the solutions to improve their approximation as the asymptotic function becomes more similar to the original function. This is expressed as follows:

$$f(x) = g(x) \rightarrow f(x) \sim g(x), \tag{1.2.1}$$

where f and g are two functions of x, transitioning from equality between the functions to an approximation of similarity, with the aim of simplifying the treatment between both

functions.

It is important to note that alongside the use of the asymptotic symbol, when using asymptotic approximation, the symbol ϵ is also employed, which represents the perturbative term of the function (that which contains relevant information about the system), with the intention of dividing complex functions into simpler ones for easier manipulation and approximation. Thus, asymptotic approximation is closely related to the perturbative method.

1.2.2 Perturbative Method

The perturbative method consists of decomposing a complex problem into a series of simpler problems; for this purpose, the symbol ϵ is used to represent the perturbative term in the function to be approximated.

This analysis consists of balancing the contribution of any term of a polynomial that is approximated to the solution. The balancing is done for a control parameter ϵ, which tell us how big the contribution of the term in the polynomial to the solution [19].

An example of a perturbation series is:

$$x \approx x_0 + x_1\epsilon + x_2\epsilon^2 + \cdots = \sum_{n=0}^{N} x_n\epsilon^n, \tag{1.2.2}$$

where ϵ is the perturbation term and it helps us fit the polynomials to the solution. The approximation improves with more terms in the perturbation's polynomial [20].

We can estimate the derivation of our perturbation series; we only use the derivation of a sum of functions. The first and second derivatives of the approximation series can be shown as...

$$\frac{d}{dt}\left(\sum_{n=0}^{N} x_n\epsilon^n\right) = \sum_{n=0}^{N} \epsilon^n \frac{dx_n}{dt} \quad \text{and} \quad \frac{d^2}{dt^2}\left(\sum_{n=0}^{N} x_n\epsilon^n\right) = \sum_{n=0}^{N} \epsilon^n \frac{d^2x_n}{dt^2} \tag{1.2.3}$$

To use the perturbative method with differential equations, we only need to apply the differentiation to the approximation series and introduce that expression into the differential equation. After that, we need to balance the contribution of every grade of ϵ in terms of the function that we need to approximate, the initial and boundary conditions.

1.2.3 The Radius of Convergence

When dealing with series approximations, it is always necessary to find the radius of convergence of the series, and the perturbative method is no exception. For this reason, it is important to know how to find the radius of convergence.

It is important to note that the radius of convergence of the perturbative method will grow or shrink depending on the term of the function that contains the perturbation.

1.3 Dimensional Analysis

Units are so important in our lives because with them we can interact with the Universe and we are able to generate precise descriptions of natural phenomena.

Units appear in a natural form in our mind when we are doing a measurement or analyzing phenomena, because we need some constant things to be able to compare our world and the Universe with those constants ([30]). Units are created for humans and respond to the necessities of every civilization, which is the reason why there have been a lot of different units a long of human history for the same phenomenon, like the mile and the meter for the distance or the lunar and solar calendars to measure a year.

Over the years, humanity has learned to abstract its ideas and to write them in a mathematical language. So, we can express a measure as the number of times a unit is used, or in mathematical form.

$$\mathbf{A} = n\mathbf{a}, \tag{1.3.1}$$

where $\mathbf{A}$ is the measure that we do, n is the number of times that we use the unit, and $\mathbf{a}$ is the unit. But the problem arises when we remember that every civilization has its particular unit system, so how can we contrast the results of measurement between some civilizations? The answer to this question has three different solutions, which are:

1. Create a world unit system (as the International System of Units), but as we know, not all countries use this unit system.
2. Knowing the conversion factors of every unit in any system of units, as we use to convert the temperature or distance between the International System and the English Units.
3. Taking advantage of math and dimensionless measure, because the numbers are universal and do not change from country to country, describing the nature in terms of some numbers, as the Reynolds' number.

Dimensional analysis is based on the third option, so in this technique, we are going to describe all the dynamics of a physical system in terms of a set of numbers without dimensions. The idea of this analysis is to generate a method to reduce the complexity of physical systems. The main idea of this analysis is the concept of ***similarity***.

Similarity consists in finding the equivalence between two different systems ([34]), but in math, this concept consists of applying a transformation on the variables to reduce the number of independent variables in the system.

To work with this analysis, we need to follow a set of steps that Sonin said are four steps. These steps are:

1. The independent variables: This step is the most important and consists of identifying the complete set of independent variables of the system. In other words, we need to know how many dimensions act in the evolution of our physical system.
2. Dimensional considerations: We need to know the dimensions of every variable in our system. An example is the dimensions of the gravitational acceleration, in this example its dimensions are: $[g] = LT^{-2}$.
3. Dimensionless variables: In this step, we are going to define the dependent variables in terms of the independent variables with the intention of obtaining a set of dimensionless variables.
4. Buckingham's Π-theorem: We apply the Buckingham's Π-theorem. This theorem tells us: That when we dimensionless the physical quantities, the parameters that

really affect the dynamics of the system are reduced and describe all the systems with no importance of the unit system that we want to use.

Example: The energy of the atomic bomb

The most iconic example of the application of the ***Dimensional analysis*** was when a British scientist (G. I. Taylor) based on a set of photos about the Trinity test, he could estimate the energy of the atomic bomb because the set of photos provided the diameter of the explosion and time lapse.

So, recreating Taylor's work using the steps of the dimensional analysis.

Step 1: Independent variables

The elements that act in this problem are: the density of air (ρ), the radius of the explosion, and the time lapse, and we are searching for the energy that the bomb had. So we have four variables, but energy is a dependent variable; on the other hand, the time, the radius, and the density are independent. To express this in terms of math, we have:

- $j = 4$, where j is the number of variables present in the system.
- $n = 3$, where n is the number of independent dimensions (L, T, M). If we had another dimension, as temperature or substance, we would add it to the independent dimensions.
- $k = j - n = 1$, where k is the number of dimensionless groups that this system has.

Step 2: Dimensional considerations

Now, we are going to find the dimension configuration of every variable that we have. These are those configurations:

- Time: $[t] = T$
- Radius: $[r] = L$
- Density: $[\rho] = \frac{M}{L^3} = ML^{-3}$
- Energy: $[E] = \frac{ML^2}{T^2} = ML^2T^{-2}$

Step 3: Dimensionless variables

Once we have the dimensions of every variable, we select some variables, the same number of independent dimensions present in the system, to create our dimension set (we can choose with liberty wherever variables, but in this example we are looking for the energy; then we do not select as base). So, we have:

$$\Pi_i = f(\rho, t, r, E), \tag{1.3.2}$$

where Π is the dimesionless group. The previous expression tells us that the dimensionless group is a function form by the variables. To generate the dimensionless variables, we use the base variables, and we will assume that every variable can be expressed as the product of the powers of our base variables. This can be shown in the followin equation.

$$v = x^{\alpha} y^{\beta} z^{\delta}, \tag{1.3.3}$$

where v represent the variable, x, y and z are the base variables and α, β and δ are the power of each base variable. To find the values of the powers for every variable, we need to generate an equivalence dimension unit between our variable and our base variables. In other words.

$$[v] = M^{a} L^{b} T^{c} = [x]^{\alpha} [y]^{\beta} [z]^{\delta}, \tag{1.3.4}$$

this equation generates a set of equations that help us to know the configuration for every variable. In our example, we have:

$$[v] = [r]^{\alpha} [\rho]^{\beta} [t]^{\delta} \implies [v] = (L)^{\alpha} \left(ML^{-3}\right)^{\beta} (T)^{\delta} \implies [v] = L^{\alpha - 3\beta} M^{\beta} T^{\delta} \tag{1.3.5}$$

Then, every variable in terms of the base variables and its dimensionless form is:

- Time: The powers to generate the time variable are...

$$\alpha = 0, \beta = 0, \delta = 1 \quad \therefore \quad t = t \implies \tilde{t} = 1$$

- Radius: The powers to generate the radius variable are...

$$\alpha = 1, \beta = 0, \delta = 0 \quad \therefore \quad r = r \implies \tilde{r} = 1$$

- Density: The powers to generate the density variable are...

$$\alpha = 0, \beta = 1, \delta = 0 \quad \therefore \quad \rho = \rho \implies \tilde{\rho} = 1$$

- Energy: The powers to generate the energy variable are...

$$\alpha = 5, \beta = 1, \delta = -2 \quad \therefore \quad E = \frac{\rho r^5}{t^2} \implies \tilde{E} = \frac{E t^2}{\rho r^5}$$

Step 4: Buckingham's Π-theorem

Now that we know the dimensionless form of the energy, we can find the energy expression. To do this, we need to remember that the Π group is dimensionless, then it is equal to some constant. So, we can find:

$$\Pi = f(r, t, \rho, E) = \tilde{E} = c \implies \frac{E t^2}{\rho r^5} = c \quad \therefore \quad E = \frac{c \rho r^5}{t^2}. \tag{1.3.6}$$

This is the result that Taylor obtained, and to find the value of the energy, we only need to substitute the values that appear in the photo set. Taylor estimated the energy of the atomic bomb in 10.4 megatons of TNT, and this result was so close to the exact energy that the US army reported.

Conclusion

Dimensional analysis has the property of giving us functions that describe a phenomenon that we do not know more about than the variables that act on it.

The most important thing about dimensional analysis is that we can express all the dynamics of a phenomenon in a general way, without the necessity of indicating the unit system that we use.

1.4 Lagrangian Analysis

The main issue with Newton's laws, as mentioned by Thornton and Marion [37], is that they are written in vector form, causing many problems to acquire a certain degree of complexity and making their manipulation more laborious. For this reason, it is necessary to use an analysis in which the dynamic system retains all its properties but is not expressed in vector form. This is where Joseph-Louis Lagrange made a significant contribution with his work ***Mécanique analytique***; where he defined a function in terms of coordinates (q) and speeds ($\dot{q}$). This can be expressed as.

$$L(q_1, q_2, \ldots, q_n, \dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n), \tag{1.4.1}$$

where q_1, q_2, ..., and q_n represent the n coordinates that characterize the system; $\dot{q}_1$, $\dot{q}_2$, ..., and $\dot{q}_n$ are the speeds of the n coordinates, and L is the function that describes the system and is called the ***Lagrangian*** of the system.

According to Landau and Lifshits [21], the Lagrangian characterizes the dynamics of a physical system if and only if the Lagrangian minimizes the action, or in other words, we apply the ***principle of least action***. Mathematically, it implies that the action (S), which is expressed in the following integral, takes the minimum possible value.

$$S = \int_{t_1}^{t_2} L(q, \dot{q}, t)dt. \tag{1.4.2}$$

The Lagrangian function is defined as the sum of the kinetic energies minus the potential energies of all the particles in the system.

$$L = T - V \implies L = \sum_{i=1}^{n} \frac{1}{2} m_i \dot{q}_i^2 - V(q_1, q_2, \ldots, q_i) \tag{1.4.3}$$

If we use D'Alembert's principle and the calculus of variations, we can find a set of second-order differential equations Soldovieri C., T. ([33]). This differential equation is known as the ***Euler-Lagrange*** equation and is expressed as follows:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = Q_i, \tag{1.4.4}$$

where q_i represents the i-th coordinate, $\dot{q}_i$ the i-th speed, and Q_i the external force or geometric restriction in the i-th direction. This differential equation is the equation of motion for the n-th particle in the i-th direction. It is important to highlight that the complete system has $n \times i$ equations of motion that we need to solve to know all the

system's behavior. For example, for a particle that moves in the three directions (x, y, z), we have to solve three (1×3) equations of motion.

The relevance of the Lagrangian analysis consists in, if we substitute the definition of the Lagrangian in the Euler-Lagrange equation and consider a simple system (without external forces and geometric restrictions), we can see, as it is shown in the following equation, that Euler-Lagrange equations are equivalent to Newton's second law ([21],[33],[37]).

$$\begin{aligned}
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0 &\implies \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \frac{\partial L}{\partial q_i} \\
\frac{d}{dt}\left(\frac{\partial (T-V)}{\partial \dot{q}_i}\right) = \frac{\partial (T-V)}{\partial q_i} &\implies \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) = \frac{\partial V}{\partial q_i} \\
\frac{d}{dt}\left(\frac{\partial}{\partial \dot{q}_i}\left(\frac{1}{2} m_i \dot{q}_i^2\right)\right) = \frac{\partial V}{\partial q_i} &\implies m_i \frac{d}{dt}\left(\dot{q}_i\right) = \frac{\partial V}{\partial q_i} \\
m_i \ddot{q}_i = \frac{\partial V}{\partial q_i} &\implies m_i a_i = -F_i,
\end{aligned} \tag{1.4.5}$$

1.5 Hamiltonian Analysis

Another important analysis in physics is the ***Hamiltonian analysis***. This analysis consists of analyzing the movement in terms of position (coordinates) and momentum, these are called ***canonical variables***. That is the principal difference between the Lagrangian and Hamiltonian analyses. To change the variables in the Lagrangian to the canonical variables, we have the following relation.

$$p_i = \frac{\partial L(q_i, \dot{q}_i, t)}{\partial \dot{q}_i} \implies p_i = \sum_{i=1}^{n} m_i \dot{q}_i - \frac{\partial V(q_1, q_2, \ldots, q_n)}{\partial \dot{q}_i} \quad \therefore \quad p_i = \sum_{i=1}^{n} m_i \dot{q}_i \tag{1.5.1}$$

The Hamiltonian (H) is defined as the Legendre transform of the Lagrangian ([33]), so the Hamiltonian has the following expression.

$$H(q_i, p_i, t) = \sum_{i=1}^{n} \dot{q}_i p_i - L(q_i, \dot{q}_i, t), \tag{1.5.2}$$

this is the general form of the Hamiltonian, without relevance to whether the system is conservative or not. For a conservative system, the Hamiltonian can be expressed as.

$$
\begin{aligned}
H =& \sum_{i=1}^{n} \dot{q}_i p_i - L(q_i, \dot{q}_i, t) \\
=& \sum_{i=1}^{n} \dot{q}_i p_i - \sum_{i=1}^{n} \frac{1}{2} m_i \dot{q}_i^2 + V(q_1, q_2, \ldots, q_i) \\
=& \sum_{i=1}^{n} \dot{q}_i p_i - \frac{1}{2} \sum_{i=1}^{n} \underbrace{m_i \dot{q}_i}_{p_i} \dot{q}_i + V(q_1, q_2, \ldots, q_i) \\
=& \sum_{i=1}^{n} \dot{q}_i p_i \left(1 - \frac{1}{2}\right) + V(q_1, q_2, \ldots, q_i) \\
=& \sum_{i=1}^{n} \frac{1}{2} m_i \dot{q}_i^2 + V(q_1, q_2, \ldots, q_i) \\
\therefore \quad & \boxed{H = T + V};.
\end{aligned}
\tag{1.5.3}
$$

If we apply the principle of least action:

$$
\begin{aligned}
S =& \int_{t_1}^{t_2} L(q, \dot{q}, t) dt = \int_{t_1}^{t_2} \dot{q}p - H(q, p, t) dt \\
\delta S =& 0 = \delta \int_{t_1}^{t_2} \dot{q}p - H(q, p, t) dt \\
0 =& \int_{t_1}^{t_2} \left(\dot{q}\delta p + p\delta\dot{q} - \frac{\partial H}{\partial p}\delta p - \frac{\partial H}{\partial q}\delta q \right) dt \\
0 =& \int_{t_1}^{t_2} \left(\left\{ \dot{q} - \frac{\partial H}{\partial p} \right\} \delta p - \frac{\partial H}{\partial q}\delta q + \underbrace{p\delta\dot{q}}_{-\dot{p}\delta q} \right) dt \\
0 =& \int_{t_1}^{t_2} \left(\left\{ \dot{q} - \frac{\partial H}{\partial p} \right\} \delta p - \left\{ \frac{\partial H}{\partial q} + \dot{p} \right\} \delta q \right) dt,
\end{aligned}
\tag{1.5.4}
$$

to be real, the two parts of the integral need to be zero. Then we obtain two first-order differential equations, that are called ***Hamilton's equations***. These equations are:

$$
\dot{q} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial q}. \tag{1.5.5}
$$

These equations are so relevant, because they are obtained by the principle of least action, like Euler-Lagrange equation, ergo they have the property to describe the dynamics of the system, with the difference that Hamilton's equations are two first-order differential equations. If we expand the number of particles to n and coordinates to i, then we need to solve $2 \times n \times i$ differential equations of first order. This implies that if we use the Hamiltonian analysis, we need to solve the double of equations as in the Lagrangian analysis, but their equations are simpler than in the Lagrangian analysis.

As we have a set of equations, we can put them together in a vector.

$$\begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} \frac{\partial H}{\partial p} \\ -\frac{\partial H}{\partial q} \end{pmatrix}, \tag{1.5.6}$$

this vector represents how a particle changes its position and its momentum over time. Then, the vector has the property to describe all the dynamics presented in the system. For this reason, the Hamiltonian analysis can be plotted and lets us know the behaviour of the system, this is because it uses the phase plane.

1.6 Numerical Solutions of the Equations of Motion

In the previous sections, the system was analyzed using analytical approaches based on the Lagrangian and Hamiltonian formulations. However, due to the inherent nonlinearities present in the potential terms, the resulting equations of motion do not generally admit analytical solutions. To study the temporal evolution of the system, it becomes necessary to employ numerical methods that approximate the solution through discrete time integration.

Numerical integration methods convert the continuous problem into an iterative process, where the state of the system is updated at small time intervals. These algorithms are essential in nonlinear dynamics, as they allow the exploration of regions of phase space that are analytically inaccessible.

Among the most common numerical schemes are:

- **Euler's method**: the simplest explicit scheme, conceptually straightforward but limited in both accuracy and stability.
- **Runge–Kutta methods**: a family of higher-order explicit algorithms that achieve better accuracy while maintaining reasonable computational cost.
- **Implicit methods**: typically used for stiff equations, where explicit methods become unstable.

In what follows, we briefly describe the fourth-order Runge–Kutta method, which is used in this work to integrate the equations of motion corresponding to the nonlinear magnetic pendulum.

1.6.1 The Fourth-Order Runge–Kutta Method

Consider a general first-order ordinary differential equation:

$$\frac{dy}{dt} = f(t, y), \qquad y(t_0) = y_0. \tag{1.6.1}$$

The Runge–Kutta (RK) methods estimate the value of $y(t)$ at a later time by evaluating f at several intermediate points within each time step. The fourth-order Runge–Kutta

method (RK4), one of the most widely used, updates the solution as:

$$\begin{aligned} k_1 &= f(t_n, y_n), \\ k_2 &= f\left(t_n + \frac{h}{2}, y_n + \frac{h}{2}k_1\right), \\ k_3 &= f\left(t_n + \frac{h}{2}, y_n + \frac{h}{2}k_2\right), \\ k_4 &= f\left(t_n + h, y_n + hk_3\right), \\ y_{n+1} &= y_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4), \end{aligned} \tag{1.6.2}$$

where h is the time step.

This scheme provides a local truncation error of order $\mathcal{O}(h^5)$ and a global error of order $\mathcal{O}(h^4)$, which means that halving the step size approximately reduces the overall error by a factor of 16. The method's robustness and precision make it ideal for integrating nonlinear oscillatory systems.

1.6.2 Illustrative Example: Harmonic Oscillator

To illustrate the method, consider the simple harmonic oscillator described by

$$\ddot{x} + \omega^2 x = 0. \tag{1.6.3}$$

This equation can be written as a system of first-order equations:

$$\begin{aligned} \dot{x} &= v, \\ \dot{v} &= -\omega^2 x. \end{aligned} \tag{1.6.4}$$

The RK4 algorithm can be directly applied to this system, yielding a numerical trajectory $x(t)$ that reproduces the analytical oscillatory motion

$$x(t) = A\cos(\omega t + \phi).$$

This example serves as a validation of the numerical implementation before applying the same procedure to the nonlinear magnetic pendulum analyzed in this chapter.

Implementation in Mathematica

```
 = 1;
eqs = {x'[t] == v[t], v'[t] == -^2 x[t]};
ics = {x[0] == 1, v[0] == 0};
sol = NDSolve[{eqs, ics}, {x, v}, {t, 0, 20}];
Plot[Evaluate[x[t] /. sol], {t, 0, 20},
 PlotLabel -> "Harmonic Oscillator (RK4)",
 AxesLabel -> {"t", "x(t)"}]
```

Implementation in Python

```
import numpy as np
import matplotlib.pyplot as plt

def f(t, y, =1):
    x, v = y
    return np.array([v, -**2 * x])

def rk4_step(f, t, y, h):
    k1 = f(t, y)
    k2 = f(t + h/2, y + h*k1/2)
    k3 = f(t + h/2, y + h*k2/2)
    k4 = f(t + h, y + h*k3)
    return y + h*(k1 + 2*k2 + 2*k3 + k4)/6

t0, tf, h = 0, 20, 0.01
t = np.arange(t0, tf, h)
y = np.zeros((len(t), 2))
y[0] = [1, 0]

for i in range(len(t)-1):
    y[i+1] = rk4_step(f, t[i], y[i], h)

plt.plot(t, y[:,0])
plt.xlabel('t'); plt.ylabel('x(t)')
plt.title('Harmonic Oscillator (Runge-Kutta 4)')
plt.show()
```

Both implementations yield results consistent with the analytical solution, demonstrating the accuracy and stability of the Runge–Kutta approach. Once validated, the same algorithm is employed to integrate the nonlinear equations of motion of the magnetic pendulum presented earlier in this chapter.

CHAPTER 2

Oscillation theory

> The same laws that govern the oscillations of a pendulum govern the oscillations of everything, from the motion of a planet to the vibrations of an atom.
> Richard Feynman, *The Feynman Lectures on Physics. Vol I*, 1964

The study of oscillations is important because they describe many phenomena in nature, such as the movement of a pendulum clock, sound, and more complex interactions like the Earth's movement around the Sun or the behavior of light.

An oscillation is defined as the cyclical motion that a particle follows around a point, called the equilibrium position (x_{eq} or x_0), which is generated by an initial perturbation. This can be appreciated by the following figure.

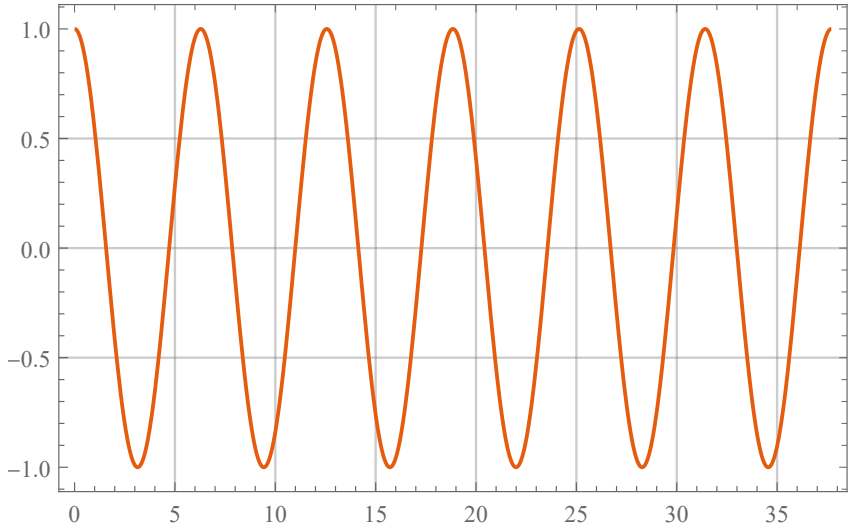


Figure 2.1: An oscillator system.

The oscillations can be categorized into one of three categories:

- ***Harmonic:*** This is the ideal oscillation, where we only consider the restoring force.

- ***Damped:*** This describes a system where we consider both the restoring force and the friction force.
- ***Forced:*** In this category, we consider all the previously mentioned forces, along with the existence of an external force that continuously perturbs the system.

2.1 Harmonic oscillation

A harmonic oscillation is an ideal system, which does not suffer from the effects of external forces or dissipative forces like friction. This means that the system can be described solely by knowing the driving force ($\vec{F}$) and the restoring force present in the system ($\vec{F}_R$).

Using Newton's third law, it can be seen that the sum of both forces must produce mechanical equilibrium, which implies that the restoring force must be opposite to the driving force, leading to the following equation:

$$\vec{F} = -\vec{F}_R \implies \vec{F} + \vec{F}_R = 0, \tag{2.1.1}$$

It is important to note that our equation is a vector equation, which implies that this relationship holds for each of the components of the vector system.

Converting our driving force into a differential equation gives:

$$m\frac{d^2\vec{r}}{dt^2} + \vec{F}_R(\vec{r}) = 0 \implies \sum_{i=1}^{3}\left(m\frac{d^2 r_i}{dt^2} + F_{R\,i}(r_i)\right)\hat{e}_i = 0 \implies \sum_{i=1}^{3}\left(\frac{d^2 r_i}{dt^2} + \frac{F_{R\,i}(r_i)}{m}\right)\hat{e}_i = 0, \tag{2.1.2}$$

where r_i and $F_{R\,i}$ are the i-th component of the position vector and the restoring force vector, respectively, and $\hat{e}_i$ is the direction of the i-th component of the vectors. Therefore, the behavior of the system depends entirely on the form of the components of the restoring force. It is important to highlight that there is no restriction on the restoring force requiring it to consist of a single element (a single force) or to be of the same nature, which implies that our restoring force can be a combination of forces, including mechanical, electrical, magnetic, etc. Thus, the restoring force can be expressed as:

$$F_{R\,i}(r_i) = \sum F_{mec\,i}(r_i) + \sum F_{elec\,i}(r_i) + \sum F_{mag\,i}(r_i) + \cdots \tag{2.1.3}$$

Therefore, it can be concluded that the differential equation, both its nature and whether it is solvable, depends on the expression of the restoring force.

To work with different kinds of restoring forces, we are going to analyze them according to their nature.

2.1.1 Mechanical restoring force

For the mechanical restoring force, we have three different forms of construction. The first involves applying Hooke's law, where the restoring force is applied by a device, like springs, that redirects the restoring force each time the oscillator passes the equilibrium point.

$$F = -k\Delta x, \tag{2.1.4}$$

where k is the restitution's constant and Δx is the displacement of the matter out of the equilibrium point.

The second form associates the force with a set of periodic forces, where the periodic forces have the equilibrium point where the periodic functions are null or where the sum of forces is zero. Then, the restoring force has the following form:

$$F = F_0 \sin(\theta) + F_1 \cos(\theta). \tag{2.1.5}$$

A clear example of this form of restoring force is the gravitational force applied to a pendulum. In that system, the force is expressed as:

$$F = T + F_g = T(\sin(\theta)\hat{e}_x - \cos(\theta)\hat{e}_y) - mg\hat{e}_y \tag{2.1.6}$$

The third form involves constructing a set of forces that change the direction of our oscillator at some distance, such as a set of springs against a wall or the interaction between two tennis players. These forces, which occur intermittently, can be expressed using the Heaviside function ($u(x)$).

$$F = F_1 u(x) + F_2 u(x - x_1). \tag{2.1.7}$$

2.1.2 Electrical restoring force

Before constructing an oscillatory electric system, we are going to recap the electric force interaction. The electric force between two charges is described by Coulomb's law:

$$\vec{F}_e(r) = \frac{k_e q_1 q_2}{r^2}\hat{r}, \tag{2.1.8}$$

where k_e is the proportionality constant, r is the distance between the two charges, q_1 and q_2 are the charges that interact. The electric force can be attractive or repulsive, and this depends on the charges that are interacting. If the charges have different polarities (+- or -+), they generate an attractive force; in the other size, when the charges have the same polarities (++ or –), they generate a repulsive force.

Now that we remember how the electric force acts, we can build an electric system with the property to cause a charge to oscillate in a region of the space. For this, we need three charges: two fixed charges (Q_1 and Q_2) and one mobile (q). To cause the mobile charge has an oscillatory movement, we need it to be between the two fixed charges; and the distance between the fixed charges is d. When q moves to another charge the repulsive force between the charge and q increases, pushing q to r_{eq}, where r_{eq} is the equilibrium point between the repulsive forces $F_{Q_1 q}$ and $F_{Q_2 q}$. This can be visualized by the following scheme.

Then, the electrical restoring force can be expressed as:

$$\vec{F}_R = \vec{F}_{Q_1 q} + \vec{F}_{Q_2 q} = \frac{k_e Q_1 q}{r_1^2}\hat{r} - \frac{k_e Q_2 q}{r_2^2}\hat{r} \implies \vec{F}_R = k_e q\left[\frac{Q_1}{r_1^2} - \frac{Q_2}{r_2^2}\right]\hat{r}. \tag{2.1.9}$$

If $Q_1 = Q_2 = Q$ and the coordinate origin is in one of the fixed charges (in Q_1), we can express the forces in terms of r, where r is the distance of q to the origin. With the previous considerations, our restoring force can be reexpressed as:

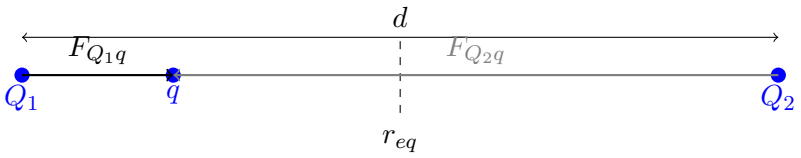


Figure 2.2: Representation of an electric oscillator system, where r_{eq} is the equilibrium point between the repulsive forces F_{Q_1q} and F_{Q_2q}. In our scheme, the intensity of the forces can be visualized in terms of the color intensity, where black is more intense than gray.

$$\begin{aligned} \vec{F}_R &= k_e qQ \left[\frac{1}{r^2} - \frac{1}{(d-r)^2}\right]\hat{r} = k_e qQ \left[\frac{(d-r)^2 - r^2}{r^2(d-r)^2}\right]\hat{r} \\ \implies \vec{F}_R &= k_e qQ \left[\frac{d^2 - 2dr + r^2 - r^2}{r^2(d-r)^2}\right]\hat{r} = k_e qQ \left[\frac{d^2 - 2dr}{r^2(d-r)^2}\right]\hat{r} \\ \therefore \vec{F}_R &= k_e qQd \left[\frac{d-2r}{r^2(d-r)^2}\right]\hat{r}. \end{aligned} \tag{2.1.10}$$

An analog form to attack this phenomenon is introducing the idea of electric fields. The electric field is defined as $\vec{E}(r) = \frac{k_e Q}{r^2}\hat{r}$, so the force in terms of the electric fiel is $\vec{F} = q\vec{E}(r)$. Applying the electric fields in our equation:

$$\vec{F}_R = q\vec{E}_1(r_1) + q\vec{E}_2(r_2) \implies \vec{F}_R = q\left[\vec{E}_1(r) + \vec{E}_2(d-r)\right] \implies \vec{F}_R = q\left[E_1(r) - E_2(d-r)\right]\hat{r}, \tag{2.1.11}$$

where E_1 and E_2 are the electric fields of the fixed charges.

Another way to make an oscillator system with electrical restoring force is when two charges with the same polarity cause a repulsive electric force and a force that acts over the mobile charge.

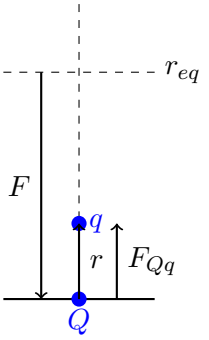


Figure 2.3: Representation of the second electric oscillator system.

Then, the mathematical expression of the system is:

$$\vec{F}_R = \vec{F}_{Qq} + \vec{F} \implies \vec{F}_R = \left(\frac{k_e qQ}{r^2} - F\right)\hat{r}. \tag{2.1.12}$$

We can observe that the repulsive force domains when the distance (r) between the two charges is small, but when the distance is increasing, the attractive force starts to obtain the principal role in the movement. If we use the electric field, our restoring force is.

$$\vec{F}_R = q\vec{E} + \vec{F} \implies \vec{F}_R = (qE(r) - F)\,\hat{r}, \tag{2.1.13}$$

from this last expression, we can see that the direction and intensity of the restoring force ($\vec{F}_R$) depends on r.

2.1.3 Magnetic restoring force

The magnetic case is similar to the electric case, except that we are going to work with magnetic interactions.

In contrast with the electric force, the magnetic force does not have a specific form. The behavior of the magnetic force depends on how the magnetic field is originated. If our magnetic field is generated by an electric current or a charge that is moving, then our magnetic force follows the magnetic part of the Lorentz force.

$$\vec{F} = q\vec{v} \times \vec{B}, \tag{2.1.14}$$

where $\vec{v}$ is the velocity of the charge, $\vec{B}$ is the magnetic field. Something important to highlight is that the magnetic force is a force that is applied in a perpendicular direction to the movement and the magnetic field.

But if we have two permanent magnets, we can build a magnetic analogous to Coulomb's law. For this reason, this equation is sometimes called the magnetic charges force. So, the magnetic force for two permanent magnets is:

$$\vec{F}_M = \frac{\mu_0 m_1 m_2}{4\pi r^2}\hat{r}, \tag{2.1.15}$$

where μ_0 is the magnetic permeability of the free space, $m1$ and $m2$ are the magnetic charges and r is the distance between the two magnets. Something important to see is that if we want to consider the environment where the experiment is done, then we need to change the magnetic permeability of the free space (μ_0) to the magnetic permeability of the environment.

Another way to describe the force between magnets is by using the magnetic energy present in their interaction. Therefore, the equation that models the magnetic force is:

$$\vec{F}_M = \frac{M}{r}\hat{r}, \tag{2.1.16}$$

where M is the magnetic energy in the interaction. This method is useful when we do not know the value of the magnetic charges.

In the same way as the electrical restoring force, the magnetical restoring force has the same two ways to be built. The first is using the interaction between three magnets, two of which are fixed magnets, and the other has the property of moving in the region between the fixed magnets.

The main difference between the magnetic system and the electric system is that magnets have at least two poles, so we need to be cautious with their poles to generate a double

repulsive force in the region. This specific magnets configuration can be shown in the following figure.

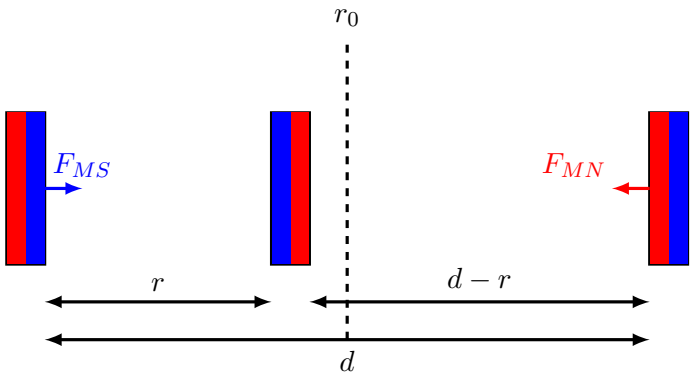


Figure 2.4: Graphic representation of a magnetic system with three permanent magnets that use the repulsive force between magnets to create an oscillatory system.

Expressing the configuration system present in the previous figure (figure 2.4) in an equation.

$$\vec{F}_R = \vec{F}_{MS} + \vec{F}_{MN} \implies \vec{F}_R = \left[\frac{M_S}{r} - \frac{M_N}{d-r}\right]\hat{r}, \tag{2.1.17}$$

where M_S is the magnetic energy present in the interaction by the south poles of a fixed magnet and the mobile magnet and M_N is the energy in the interaction by the north poles. If $M_S = M_N = M$ we can simplify the systema as.

$$\vec{F}_R = M\left[\frac{1}{r} - \frac{1}{d-r}\right]\hat{r} \implies \vec{F}_R = M\left[\frac{d-r-r}{r(d-r)}\right]\hat{r} \implies \vec{F}_R = M\left[\frac{d-2r}{r(d-r)}\right]\hat{r}. \tag{2.1.18}$$

The second magnetic system, like its electric analog system, uses the repulsive force between two magnets and an external force. This can be shown in the figure 2.5.

If we use the magnetic analog of the equation (2.1.12), we can describe our system as.

$$\vec{F}_R = \vec{F}_M + \vec{F} \implies \vec{F}_R = \left[\frac{M}{r} - F\right]\hat{r} \tag{2.1.19}$$

2.1.4 General solution

Now that we know the different types of restoring forces, we can work on the differential equation to find the solution of an oscillatory system. To find the solution, we need to check if the differential equation has a theoretical solution. If it does not have it, then we use an approximation technique. If the equation has a solution, then we use the equation (2.1.2) to find it.

If we define the restitution acceleration as $a_R(r) = \frac{F_R(r)}{m}$. To simplify the calculus, we are going to work with only one dimension, but it is simple to expand it to the other two dimensions. Then, the equation can be rewritten as:

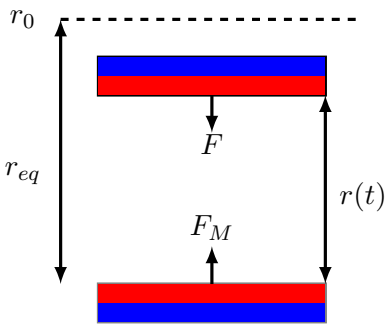


Figure 2.5: Graphic configuration of the magnetic system that uses the magnetic repulsive force to generate a magnetic restoring force.

$$\frac{d^2r(t)}{dt^2} + a_R(r) = 0. \tag{2.1.20}$$

We define the restitution acceleration function as the product of the square of the natural frequency of restitution (ω_0) and the position ($a_R(r) = \omega_0^2 r(t)$). Then, our differential equation is:

$$\frac{d^2r(t)}{dt^2} + \omega_0^2 r(t) = 0. \tag{2.1.21}$$

Now, to solve the equation we need to find the characteristic polynomial of the system. To find it, we use the technique to change a differential equation to an algebraic equation ($s^n = \frac{d^n r}{dt^n}$).

$$\frac{d^2r(t)}{dt^2} + \omega_0^2 r(t) = 0 \implies s^2 + \omega_0^2 = 0 \implies s^2 = -\omega_0^2 \implies \boxed{s_{1,2} = \pm i\omega_0}\,, \tag{2.1.22}$$

Introducing the solution of the characteristic polynomial in the general solution, we obtain.

$$r(t) = Ae^{s_1 t} + Be^{s_2 t} \implies \boxed{r(t) = Ae^{i\omega_0 t} + Be^{-i\omega_0 t}}\,, \tag{2.1.23}$$

we can transform our complex functions to real functions using the complex exponential definition ($e^{ix} = \cos(x) + i\sin(x)$).

$$\begin{aligned} r(t) &= Ae^{i\omega_0 t} + Be^{-i\omega_0 t} \\ &= A\left[\cos(\omega_0 t) + i\sin(\omega_0 t)\right] + B\left[\cos(\omega_0 t) - i\sin(\omega_0 t)\right] \\ &= \underbrace{(A+B)}_{C}\cos(\omega_0 t) + \underbrace{i\,(A-B)}_{D}\sin(\omega_0 t) \end{aligned} \tag{2.1.24}$$

$$\therefore \boxed{r(t) = C\cos(\omega_0 t) + D\sin(\omega_0 t)}\,,$$

where C and D are the amplitudes of the oscillations.

Now, to find the behavior of the system, we need to find the speed function. Then, we need to find the first change of the position function in time.

$$
\begin{aligned}
v &= \frac{dr(t)}{dt} \\
&= \frac{d}{dt}\left(C\cos\left(\omega_0 t\right) + D\sin\left(\omega_0 t\right)\right) \\
&= C\frac{d}{dt}\cos\left(\omega_0 t\right) + D\frac{d}{dt}\sin\left(\omega_0 t\right) \\
\therefore \boxed{v(t) &= -C\omega_0\sin\left(\omega_0 t\right) + D\omega_0\cos\left(\omega_0 t\right)}.
\end{aligned}
\tag{2.1.25}
$$

With the position and speed functions, we can plot a phase plane to know the behavior of a harmonic oscillator system.

2.1.5 Amplitude

The amplitude of the system is obtained using the Pythagorean theorem:

$$
A = \sqrt{C^2 + D^2}. \tag{2.1.26}
$$

It is important to say that the amplitude will be obtained after we used the initial conditions that are going to help us to find the values of C and D.

If the initial conditions are $r(0) = r_0$ and $v(0) = v_0$, then

$$
C = r(0) = r_0, \qquad D = \frac{v(0)}{\omega_0} = \frac{v_0}{\omega_0}.
$$

Therefore, the amplitude in terms of initial data is

$$
\boxed{R = \sqrt{r_0^2 + \left(\frac{v_0}{\omega_0}\right)^2}.}
$$

2.1.6 Energy

One of the most important things in a physical system is energy. For this reason, we need to find the expression of the energy present in our system.

As we know, the total energy of the system is composed of the kinetic and potential energies.

$$
E = T + V \implies E = \frac{1}{2}mv^2 + \frac{1}{2}kx^2 \implies E = \frac{m}{2}\left(v^2 + \frac{k}{m}x^2\right) \implies E = \frac{m}{2}\left(v^2 + \omega_0^2 x^2\right), \tag{2.1.27}
$$

where $\omega_0 \equiv \sqrt{\frac{k}{m}}$. If we replace the speed as the change of the position on time, we can rewrite the energy expression as.

$$
\begin{aligned}
E =& \frac{m}{2}\left[\left(\frac{dx(t)}{dt}\right)^2 + \omega_0^2 x^2(t)\right] \\
=& \frac{m}{2}\left[(-C\omega_0 \sin(\omega_0 t) + D\omega_0 \cos(\omega_0 t))^2 + \omega_0^2 (C\cos(\omega_0 t) + D\sin(\omega_0 t))^2\right] \\
=& \frac{m\omega_0^2}{2}\left[C^2 \sin^2(\omega_0 t) - 2CD\sin(\omega_0 t)\cos(\omega_0 t) + D^2\cos^2(\omega_0 t)\right] \\
&+ \frac{m\omega_0^2}{2}\left[C^2\cos^2(\omega_0 t) + 2CD\sin(\omega_0 t)\cos(\omega_0 t) + D^2\sin^2(\omega_0 t)\right] \\
=& \frac{m\omega_0^2}{2}\left[C^2\left(\sin^2(\omega_0 t) + \cos^2(\omega_0 t)\right) + D^2\left(\sin^2(\omega_0 t) + \cos^2(\omega_0 t)\right)\right] \\
\therefore & \boxed{E = \frac{m\omega_0^2}{2}\left(C^2 + D^2\right) = \frac{m\omega_0^2}{2}A^2}\,.
\end{aligned}
\tag{2.1.28}
$$

This result lets us know that the energy depends on the mass of the oscillator, the frequency, and the amplitude of the movement.

2.2 Damped oscillation

When an oscillator system presents a dissipative force, that system is called a ***damped oscillator***. In this system, the oscillation frequency is perturbed by friction, causing the oscillation to become slightly slower than in the beginning.

The dissipative force is the friction force of the environment and is proportional to the velocity. Then, the dissipative force is expressed as:

$$\vec{F}_d = \beta\vec{v}, \tag{2.2.1}$$

where β is the friction coefficient and $\vec{v}$ is the velocity of the oscillation.

Now, the forces in our system are three: the driving force (F), the friction force of the environment (F_d), and the restitution force (ω_R). These forces interact until the system arrives at equilibrium. Then, the sum of forces is:

$$\vec{F} + \vec{F}_d + \vec{F}_R = 0, \tag{2.2.2}$$

using the definitions of $\vec{F}_d$ and ω_R, let us know the set of differential equations that govern the motion in every direction.

$$m\frac{d^2\vec{r}}{dt^2} + \beta\frac{d\vec{r}}{dt} + \vec{F}_R(\vec{r}) = 0 \implies \sum_{i=1}^{3}\left(m\frac{d^2 r_i}{dt^2} + \beta\frac{dr_i}{dt} + F_{R\,i}(r_i)\right)\hat{e}_i = 0, \tag{2.2.3}$$

finding the equation of motion and dividing it over the mass, the differential equation is...

$$\sum_{i=1}^{3}\left(\frac{d^2 r_i}{dt^2} + \frac{\beta}{m}\frac{dr_i}{dt} + \frac{F_{R\,i}(r_i)}{m}\right)\hat{e}_i = 0 \implies \sum_{i=1}^{3}\left(\frac{d^2 r_i}{dt^2} + 2\gamma\frac{dr_i}{dt} + a_{R\,i}(r_i)\right)\hat{e}_i = 0, \tag{2.2.4}$$

where $\gamma = \frac{\beta}{2m}$ represents the half of the resistance frequency of the environment and $a_R = \frac{F_{R\,i}}{m}$ the restitution acceleration function.

If we define $s^n \equiv \frac{d^{(n)}r}{dt^n}$, we can rewrite our differential equation as a power equation, letting us find the characteristic equation of our homogenous system. For the non differential term ($r = \frac{d^{(0)}r}{dt^0} = s^0$) we can define $\omega_R(s^0) = \frac{a_R(r)}{r}$ as the frequency of the restitution forces.

Then, the characteristic equation in general form is:

$$s^2 + 2\gamma s + \omega_R^2(s^0) = 0 \implies s = -\gamma \pm \sqrt{\gamma^2 - \omega_R^2} = -\gamma \pm \eth, \tag{2.2.5}$$

where $\eth = \sqrt{\gamma^2 - \omega_R^2}$ is the discriminant of the system. Then, the solution of the system is:

$$\boxed{r(t) = e^{-\gamma t}\left[Ae^{\eth t} + Be^{-\eth t}\right]}, \tag{2.2.6}$$

this equation lets us know the amplitude of the oscillation at the time t. It is important to see that the discriminant value ($\eth$) defines the behavior of the system; there are three different values and three possible behaviors.

1. When $\eth < 0$ the oscillation is called ***underdamped***. In this type of oscillation, the amplitude of the oscillation decays gradually over time.
2. When $\eth = 0$ the oscillation is called ***critically damped***. In this kind of oscillation, the system returns quickly to the equilibrium point without oscillating.
3. When $\eth > 0$ the oscillation is called ***overdamped***. In this style of oscillation, the system does not oscillate but returns slowly to the equilibrium point.

2.2.1 Damped frequency

As we saw in the previous part, the only damped oscillation system where the system really has an oscillating behavior is the ***underdamped oscillator***. The underdamped oscillator requires that the discriminant be less than zero ($\eth < 0$), this causes:

$$\eth < 0 \implies \eth = \sqrt{\gamma^2 - \omega_R^2} < 0 \implies \omega_R > \gamma \therefore \eth \in \Im. \tag{2.2.7}$$

The previous solution lets us know that the discriminant is complex, so we can rewrite the discriminant in terms of a complex function.

$$\eth = i\sqrt{\omega_R^2 - \gamma^2} = i\omega_d, \tag{2.2.8}$$

where ω_d is the damped angular frequency of the system and represents how the frequency of restitution changes by the effect of the damping.

The damped frequency can be found using the relation between frequency and angular frequency:

$$f = \frac{\omega}{2\pi} \implies f_d = \frac{\omega_d}{2\pi} = \frac{\sqrt{\omega_R^2 - \gamma^2}}{2\pi}. \tag{2.2.9}$$

Now, we can find the solution of the system in terms of the damped angular frequency, as.

$$r(t) = e^{-\gamma t}\left[Ae^{i\omega_d t} + Be^{-i\omega_d t}\right]. \tag{2.2.10}$$

Using Euler's property by the complex numbers, we can simplify the solution as follows.

$$\begin{aligned} r(t) &= e^{-\gamma t}\left[Ae^{i\omega_d t} + Be^{-i\omega_d t}\right] \\ &= e^{-\gamma t}\left[A\left\{\cos(\omega_d t) + i\sin(\omega_d t)\right\} + B\left\{\cos(\omega_d t) - i\sin(\omega_d t)\right\}\right] \\ \therefore\ r(t) &= e^{-\gamma t}\left[C\cos(\omega_d t) + D\sin(\omega_d t)\right], \end{aligned} \tag{2.2.11}$$

where the constants are $C = A + B$ and $D = i(A - B)$. It is important to highlight that the C constant is a real number meanwhile the D constant is an imaginary number, so the solution of the system is a complex number.

Another important thing to see is that the oscillation decreases when the time increases, causing the term $e^{-\gamma t}$ to represent the dissipation.

2.2.2 Energy

Following the process of the subsection (2.1.6) we can find how the energy is expressed in the system. So finding the velocity and acceleration expressions.

We know that the velocity is :

$$\begin{aligned} v =& \frac{dr(t)}{dt} \\ =& \frac{d}{dt}\left\{e^{-\gamma t}\left[C\cos(\omega_d t) + D\sin(\omega_d t)\right]\right\} \\ =& \frac{d}{dt}\left\{e^{-\gamma t}C\cos(\omega_d t)\right\} + \frac{d}{dt}\left\{e^{-\gamma t}D\sin(\omega_d t)\right\} \\ =& \left[-\gamma Ce^{-\gamma t}\cos(\omega_d t) - \omega_d Ce^{-\gamma t}\sin(\omega_d t)\right] + \left[-\gamma De^{-\gamma t}\sin(\omega_d t) + \omega_d De^{-\gamma t}\cos(\omega_d t)\right] \\ =& e^{-\gamma t}\left[(-\gamma C + \omega_d D)\cos(\omega_d t) - (\omega_d C + \gamma D)\sin(\omega_d t)\right]. \end{aligned} \tag{2.2.12}$$

Now, use the expressions of the kinetic and the potential energy to find the total energy.

$$\begin{aligned} E = T + U \implies E &= \frac{1}{2}mv^2 + \frac{1}{2}kr^2 \\ \implies E = \frac{1}{2}\left[mv^2 + kr^2\right] \implies E &= \frac{m}{2}\left[v^2 + \frac{k}{m}r^2\right], \end{aligned} \tag{2.2.13}$$

using the natural frequency identity ($\omega_0 = \sqrt{\frac{k}{m}}$) we can rewrite the energy expression as:

$$E = \frac{m}{2}\left[v^2 + \omega_0^2 r^2\right], \tag{2.2.14}$$

if we introduce the x and v definitions in the last expression of the energy, we can obtain the following equation.

$$
\begin{aligned}
E &= \frac{m}{2}\left[v^2+\omega_0^2x^2\right] \\
&= \frac{m}{2}\left[e^{-2\gamma t}\left\{(-\gamma C+\omega_d D)\cos(\omega_d t)-(\omega_d C+\gamma D)\sin(\omega_d t)\right\}^2+\omega_0^2e^{-2\gamma t}\left\{C\cos(\omega_d t)+D\sin(\omega_d t)\right\}^2\right] \\
&= \frac{me^{-2\gamma t}}{2}\left[\left\{(-\gamma C+\omega_d D)\cos(\omega_d t)-(\omega_d C+\gamma D)\sin(\omega_d t)\right\}^2+\omega_0^2\left\{C\cos(\omega_d t)+D\sin(\omega_d t)\right\}^2\right],
\end{aligned}
\tag{2.2.15}
$$

from this result, we can see that the energy of the system decays to zero when the time increases, so we can be sure that in some time our system will stop its movement. The speed of decaying depends on the γ's value.

2.2.3 Note

In this project, we will work firstly with underdamped oscillations, but it is important to remember that de frequency ω_R depends on the nature of the force and the number of forces present on the system. So, a general form of the discriminant is:

$$
\eth = \sqrt{\gamma^2-\frac{1}{m^2}\left(\sum\frac{\vec{F}_{mec}(r_i)}{r_i}+\sum\frac{\vec{F}_{elec}(r_i)}{r_i}+\sum\frac{\vec{F}_{mag}(r_i)}{r_i}+\cdots\right)^2}
\tag{2.2.16}
$$

2.3 Forced oscillation

A ***forced oscillation*** is the system where an external force introduces energy into the oscillator with a specific frequency. This external force continuously drives the motion, compensating for the energy lost due to damping, and produces an oscillation that can reach a steady amplitude.

As we mentioned before, the external force acts periodically, so it can be expressed as

$$
\vec{F}_{ex} = \vec{F}\cos(\omega_{ex}t),
\tag{2.3.1}
$$

where $\vec{F}$ is the amplitude of the applied external force and ω_{ex} is the angular frequency of this force. It is important to note that the cosine function may be replaced by any other periodic function depending on the physical characteristics of the system.

Including the external excitation in the damped oscillator, the system is now subject to four forces:

$$
\vec{F}+\vec{F}_d+\vec{F}_R = \vec{F}_{ex}.
\tag{2.3.2}
$$

Using the definitions of each force, we obtain the differential equation that describes the motion of the system:

$$
m\frac{d^2\vec{r}}{dt^2}+\beta\frac{d\vec{r}}{dt}+\vec{F}_R = \vec{F}\cos(\omega_{ex}t) \implies \sum_{i=1}^{3}\left(m\frac{d^2r_i}{dt^2}+\beta\frac{dr_i}{dt}+F_{R\,i}(r_i)\right)\hat{e}_i = \sum_{i=1}^{3}F_i\cos(\omega_{ex}t)\hat{e}_i.
\tag{2.3.3}
$$

Now, dividing the whole expression by the mass and using Einstein's notation, we obtain

$$\frac{d^2r_i}{dt^2}+\frac{\beta}{m}\frac{dr_i}{dt}+\frac{F_{R\,i}(r_i)}{m}=\frac{F_i}{m}\cos(\omega_{ex}t) \implies \frac{d^2r_i}{dt^2}+2\gamma\frac{dr_i}{dt}+a_{R\,i}(r_i)=a_i\cos(\omega_{ex}t), \quad (2.3.4)$$

where $\gamma=\dfrac{\beta}{2m}$ represents half of the resistance frequency of the medium, $a_{R\,i}=\dfrac{F_{R\,i}}{m}$ is the restoring acceleration, and $a_i=\dfrac{F_i}{m}$ is the acceleration associated with the external excitation.

A forced oscillation is a non-homogeneous differential equation, which means that its solution must be divided into two parts: the homogeneous and the particular solutions.

$$\underbrace{\frac{d^2r_i}{dt^2}+2\gamma\frac{dr_i}{dt}+a_{R\,i}(r_i)}_{\text{homogeneous part}}=\underbrace{a_i\cos(\omega_{ex}t)}_{\text{non-homogeneous part}}. \quad (2.3.5)$$

The homogeneous part corresponds to the damped oscillation previously studied, while the particular solution represents the steady-state response caused by the external periodic force. The homogeneous solution can be written as

$$r_h(t)=Ae^{-\gamma t}\cos(\omega_d t+\phi), \quad (2.3.6)$$

where $\omega_d=\sqrt{\omega_0^2-\gamma^2}$ is the damped frequency, A is the amplitude determined by the initial conditions, and ϕ is the initial phase.

Now, we are going to find the particular solution of the system. As the forcing term is a cosine function, we propose a solution of the same form:

$$r_p(t)=C\cos(\omega_{ex}t-\delta), \quad (2.3.7)$$

where C is the amplitude of the steady oscillation and δ represents the phase difference between the external and internal oscillations.

Substituting this expression into the differential equation and simplifying, we obtain

$$C=\frac{a}{\sqrt{(\omega_0^2-\omega_{ex}^2)^2+(2\gamma\omega_{ex})^2}}, \qquad \tan\delta=\frac{2\gamma\omega_{ex}}{\omega_0^2-\omega_{ex}^2}. \quad (2.3.8)$$

Therefore, the complete solution for the forced oscillation is

$$\begin{aligned} r(t)&=r_h(t)+r_p(t)\\ &=Ae^{-\gamma t}\cos(\omega_d t+\phi)+\frac{a}{\sqrt{(\omega_0^2-\omega_{ex}^2)^2+(2\gamma\omega_{ex})^2}}\cos(\omega_{ex}t-\delta). \end{aligned} \quad (2.3.9)$$

The first term represents the *transient response*, which disappears after some time due to the exponential decay, while the second term corresponds to the *steady-state oscillation*, which remains as long as the external force acts on the system. When the external frequency ω_{ex} approaches the natural frequency ω_0, the amplitude reaches its maximum value. This phenomenon is known as **resonance**.

2.3.1 Energy

Now, we are going to obtain the expression of the energy in this system. As before, we need to find the velocity, which is obtained by differentiating $r(t)$:

$$\begin{aligned} v(t) &= \frac{dr}{dt} \\ &= -Ae^{-\gamma t}(\gamma \cos(\omega_d t + \phi) + \omega_d \sin(\omega_d t + \phi)) - C\omega_{ex} \sin(\omega_{ex} t - \delta). \end{aligned} \tag{2.3.10}$$

Using this result, we can write the total mechanical energy as

$$\begin{aligned} E = T + U \implies E &= \frac{1}{2}mv^2 + \frac{1}{2}kr^2 \\ \implies E &= \frac{m}{2}\left[\left(\frac{dr}{dt}\right)^2 + \omega_0^2 r^2\right]. \end{aligned} \tag{2.3.11}$$

In this case, the total energy of the system is not conserved because the external force continuously injects energy, while the damping force dissipates it. During the transient regime, the external source provides energy until the system reaches a dynamic equilibrium, where the energy supplied per cycle equals the energy lost due to damping. This balance produces a steady oscillation with constant amplitude, characterizing the behavior of a forced oscillator.

2.4 Resonance

Resonance represents one of the most remarkable phenomena in the study of oscillatory systems. It occurs when an oscillator interacts with an external periodic excitation, allowing the system to absorb energy efficiently from the external source. In this sense, resonance appears naturally in forced oscillation systems, where an external force continuously acts on the system.

Let us consider the general form of a damped, forced oscillator:

$$m\ddot{x} + b\dot{x} + kx = F_0 \cos(\omega_{ex} t), \tag{2.4.1}$$

where m is the mass of the oscillator, b is the damping coefficient, k is the elastic constant, F_0 is the amplitude of the external force, and ω_{ex} is its frequency.

In the steady-state regime, the particular solution of the system can be expressed as

$$x(t) = A(\omega_{ex}) \cos(\omega_{ex} t - \delta), \tag{2.4.2}$$

where $A(\omega_{ex})$ is the amplitude of the oscillation, which depends on the excitation frequency, and δ is the phase difference between the force and the displacement. The amplitude is given by

$$A(\omega_{ex}) = \frac{F_0/m}{\sqrt{(\omega_0^2 - \omega_{ex}^2)^2 + (2\beta\omega_{ex})^2}}, \tag{2.4.3}$$

where $\omega_0 = \sqrt{k/m}$ is the natural frequency of the system and $\beta = b/(2m)$ is the damping parameter.

From this expression, we can observe that the amplitude $A(\omega_{ex})$ reaches its maximum value when the denominator is minimum, which occurs close to the condition $\omega_{ex} = \omega_0$ (for small damping). This defines the ***resonance frequency*** of the system, given approximately by

$$\omega_{res} \approx \sqrt{\omega_0^2 - 2\beta^2}. \tag{2.4.4}$$

Under resonance, the amplitude of oscillation increases significantly, and the energy transfer between the external source and the oscillator becomes highly efficient. The energy stored in the oscillator is proportional to $A^2(\omega_{ex})$, which explains why even a small external force can produce large oscillations when the system is near resonance.

From an engineering perspective, resonance is a double-edged phenomenon. On one hand, it can lead to structural instabilities or even catastrophic failures if not properly controlled. A classical example is the shattering of a glass when exposed to a sound whose frequency matches its natural vibration mode. Similarly, during earthquakes, buildings whose natural frequencies coincide with those of the seismic waves experience amplified oscillations, sometimes leading to severe damage.

On the other hand, resonance can also be exploited advantageously. In this project, we aim to take advantage of resonance to extract energy from a fluid flow interacting with an oscillatory structure. By tuning the system parameters so that the excitation frequency of the flow matches the natural frequency of the structure, we can enhance the oscillatory response and maximize the energy conversion efficiency.

CHAPTER 3

Fluid-Structure interaction

> Fluid-structure interaction problems require the simultaneous solution of coupled fluid and structural equations with proper interface conditions.
> Farhat, Lesoinne & Le Tallec, 1998

The fluid-structure interaction is a phenomenon that is relevant in engineering, in particular for the engineering of bridges. This is because this interaction reveals how the flow of a fluid influences the dynamics of a structure.

The most famous example of fluid-structure interaction is the Tacoma Narrows Bridge. When a flow of air with a speed between 60 and 70 km caused the collapse of the bridge. This can be explained by the interaction of the air with the bridge. When air impacted the bridge, it generated some force over the bridge, but the bridge changed the direction of the flow, generating some vortices that induced vibrations in the structure. Those vibrations had the same frequency as the natural frequency of the bridge, causing resonance and, finally, the collapse of the bridge.

By the previous example, we can see that the fluid-structure interaction is so relevant for the building of bridges and some buildings. But the most important thing that we can highlight is that we can generate vibrations over structures with the flux of a fluid with a specific speed and density, with the intention of capturing the energy of the fluid and transforming it into another kind of energy.

To better understand this physical phenomenon, we will introduce the fundamental ideas of fluid dynamics and then analyze how the fluid affects an object and its dynamics.

3.1 Fluid dynamics

Matter can be classified into two groups: solids and fluids ([41]). As White said, to determine which group our object of study belongs to, we need to observe how it responds to a shear stress force. A substance is considered a solid if it resists deformation and maintains a fixed shape when subjected to a shear stress. In contrast, a substance is considered a fluid if it continuously deforms (flows) when subjected to a shear stress, no matter how small.

Fluids are classified into two groups: liquids and gases. And to determine which group our fluid belongs to depends on the cohesive force of the molecules. A liquid has a definite volume but takes the shape of its container, and this is because the cohesive forces are strong. In contrast, a gas has neither a definite shape nor volume and expands to fill its container; then the cohesive forces are weak.

When we

Another important thing that we need to consider when we are working with fluids is the dimensions. Previously, we knew that dimensional analysis is so important in physics and engineering, but this technique has a principal role in the physics of fluids. This is because by the principle of ***Dimensional Homogeneity*** lets us transform all the principles and laws of physics that we use in particles, to another kind of phenomenon like fluids. So, Newton's laws can be applied to fluids to find the dynamics that the fluid will follow, certainly with some changes in the expression of the force, which are characterized by its properties.

Now that we know the principal things about fluids, we are going to present the principal properties of fluids.

3.1.1 Properties of fluids

In this work, we will be using fluids, so it is essential to understand the elements that play a crucial role in the physics of the system. Then, we need to understand the properties that characterize fluids. The most important properties that a fluid has are density, pressure, viscosity, compressibility, and Reynolds number, so we are going to expand our knowledge of these properties.

Density (ρ)

Density is a measure of how matter is distributed in a given space. Another way to define density is as mass per unit of space. Then, we can write the mathematical expression as.

$$\rho = \frac{m}{V}, \tag{3.1.1}$$

where m is the mass of the fluid and V is the volume that the fluid occupies. It is important to see that in the definition we said space, then we can use another space that has fewer dimensions, then we have three different densities, and each one responds to the number of dimensions that the space has. These densities are:

1. Linear (λ): This density is when we analyze a unidimensional system, and is commonly used in cables or lines. Its units are $\frac{kg}{m}$ and its are dimensions $\left[\frac{M}{L}\right]$.
2. Surface (σ): This kind of density is applied when our space has two dimensions, length and height. Its units are $\frac{kg}{m^2}$ and its are dimensions $\left[\frac{M}{L^2}\right]$.
3. Volumetric (ρ): We use it when we have a space with three dimensions, length, height, and width; and it is the most common density because we live in a three-dimensional world. Its units are $\frac{kg}{m^3}$ and its dimensions are $\left[\frac{M}{L^3}\right]$.

Density plays an important role in the dynamics of a fluid because it affects buoyancy, low rate, and pressure distribution. And it is used to define another property, the compressibility.

Density appears naturally in the dynamics equation, then using Newton's second law.

$$F = ma \implies F = \rho V a \tag{3.1.2}$$

Pressure (p)

Forces in physics, like density, attend to dimensions and there are three different kind of forces, and they are:

1. Linear force (F_L): It is a force that is applied in one direction.
2. Surface force (F_S): It is the force that is applied to a surface.
3. Body force (F_B): It is the force that acts on all the mass particles of a body at the same time. An example of this kind of force is the gravitational force.

Then, the net force of a system can be expressed in the following form.

$$F = F_L + F_S + F_B \tag{3.1.3}$$

Pressure, as Pascal's law expresses, is a surface force. Then, pressure is the force that a fluid applies to the surface of a body. It is important to highlight that if we put the origin of our coordinate system in the center of the body, the pressure of the fluid is opposite to the resistance force. Sometimes, the pressure of a system can change by the position, or in other words, it is a function of some variable. So, the most complete form for expressing the surface force is:

$$d\vec{F}_S = -pd\vec{S}, \tag{3.1.4}$$

where p is the pressure and $\vec{S}$ is the normal vector to the surface. If we analyze the force in a specific direction (i direction), we obtain.

$$d\vec{F}_S = -pdS\hat{e}_i. \tag{3.1.5}$$

If the force is applied in i direction, the surface is placed in the other directions (j and k). So, if $dS = djdk$, then our force can be expressed as.

$$d\vec{F}_S = [p_i\hat{e}_i]\, djdk, \tag{3.1.6}$$

the force is how the pressure changes at every point of the surface, then we can rewrite the force as follows.

$$d\vec{F}_S = [p(i) - p(i+di)]\, djdk\hat{e}_i, \tag{3.1.7}$$

Using Taylor's series at first order in the i direction.

$$p(i+di) = p(i) + \frac{\partial p}{\partial i}di, \tag{3.1.8}$$

substituting in our force (equation (3.1.7)).

$$dF_{Si} = -\frac{\partial p}{\partial i} didjdk = -\frac{\partial p}{\partial i} dV. \tag{3.1.9}$$

Then, expand the definition of force to every direction.

$$d\vec{F}_S = -\frac{\partial p}{\partial i} dV \hat{e}_i = -\vec{\nabla} p dV, \tag{3.1.10}$$

where $\vec{\nabla}$ is the gradient operator and is defined as $\vec{\nabla} = \partial_i \hat{e}_i$. With the previous definition of surface force and we only have a body force and a surface, we can rewrite the force as:

$$d\vec{F} = dF_B + dF_S \implies d\vec{F} = \rho \vec{a} dV - \vec{\nabla} p dV \therefore \boxed{d\vec{F} = \left(\rho \vec{a} - \vec{\nabla} p\right) dV}. \tag{3.1.11}$$

We can define the force as the way the density force is applied on a volume, which can be expressed in a mathematical form as $d\vec{F} = \vec{f} dV$. Therefore, we can rewrite all in terms of density.

$$\boxed{d\vec{f} = \rho \vec{a} - \vec{\nabla} p}. \tag{3.1.12}$$

By the previous equation, we can appreciate that the density force depends on how the acceleration field interacts with the pressure field, and this defines the motion of the fluid particles.

Viscosity (μ)

Viscosity, as Landau, is the internal friction that a fluid presents. Viscosity affects the flow rate and the ease with which a fluid moves, as White ([41]) expressed, higher viscosity corresponds to a slower flow rate, while lower viscosity corresponds to a more easily flowing fluid.

Viscosity, in a Newtonian fluid, determines the fluid strain rate that is generated by a given shear stress ([41]). It can be interpreted as the rate at which the angle between fluid elements changes over time, or as the rate at which the velocity of the fluid changes with distance.

$$\tau = \mu \frac{d\theta}{dt} = \mu \frac{du}{dy}, \tag{3.1.13}$$

where μ is the viscosity coefficient, θ is the angle, t is the time, u is the velocity of the fluid, y is the direction of the motion and $\frac{du}{dy}$ is the velocity gradient perpendicular to the direction of flow. The viscosity coefficient has dimensions of stress-time ($[\frac{FT}{L^2}]$ or $[\frac{M}{LT}]$).

Something important to highlight is that if the fluid is Newtonian, then viscosity is a thermodynamic property because it depends on temperature and pressure.

Compressibility

Compressibility is the property of particles in a fluid to be more compact under some pressure at some temperature. If the particles are compact, this implies that more particles are in some volume, changing the density of particles. Then, we can define compressibility

as the change of density when we change the pressure at a constant temperature. Therefore, the mathematical expression for this property is.

$$\kappa = -\frac{1}{\rho}\left(\frac{\partial \rho}{\partial p}\right)_T \tag{3.1.14}$$

If we analyze the previous equation, we can see that the differentiation could take two different values, so we can conclude that fluids can be divided into two categories: compressible and incompressible fluids.

- Compressible: When the value of $\kappa \neq 0$, the most common compressible fluid is air.
- Incompressible: It is the fluid that has $\kappa = 0$, the most common incompressible fluid is water.

This property is so important because it determines how a fluid responds to changes in pressure. But if we work in a controllable environment, where the pressure does not change, then we can assume that a fluid is incompressible.

Reynolds number (Re)

This number is dimensionless, and it is used to correlate the behavior of the fluid with its viscosity. Reynolds number is defined as:

$$Re = \frac{\rho u L}{\mu} = \frac{uL}{\nu}, \tag{3.1.15}$$

where u is the velocity scale, L is the length scale of the flow. In the second form ν is the kinematic viscosity defined as.

$$\nu = \frac{\mu}{\rho}. \tag{3.1.16}$$

It is called kinematic because the mass units are canceled, and we only maintain the dimensions of length and time.

The Reynolds number is so important in fluid dynamics that it is very common to estimate it first because knowing the value of the Reynolds number lets us know what kind of flow we are working with.

Now, we are going to classify the flow of the fluid in terms of the Reynolds number.

- Laminar ($Re < 2000$): In this flow, particles move smoothly, in streamlines, and viscous forces dominate.
- Transitional ($2000 < Re < 4000$): In this flow, particles move in an unstable trajectory and can act like turbulent or laminar.
- Turbulent ($Re > 4000$): In this flow, particles have a chaotic movement, with eddies and vortices, and the inertial forces dominate.

Many other properties can be used, but in this work we will not utilize them.

Now, we are going to present the equations that govern the motion of fluids.

3.1.2 Static of fluid

When we are studying physics, we learn that there are two situations in dynamics: static and dynamic. The static in fluids occurs when the particles of the fluid do not have a relative motion between them, which can be the case when the fluid is at rest or when the fluid has a uniform acceleration.

If the force is zero ($f = 0$) the equation (5.3.43)

$$\rho\vec{a} - \vec{\nabla}p = 0 \implies \vec{\nabla}p = \rho\vec{a}. \tag{3.1.17}$$

The previous equation is a vector equation and to solve it, we need to solve the equation in any dimension. To solve it, we are going to use the tensor notation. Then, our equation is now.

$$\begin{aligned} \frac{\partial p}{\partial x_i}\hat{e}_i = \rho a_j \hat{e}_j \implies \frac{\partial p}{\partial x_i}\hat{e}_i(\partial x_i \hat{e}_i) = \rho a_j \hat{e}_j(\partial x_i \hat{e}_i) \\ \implies \partial p = \rho a_j \partial x_i \delta_{ij} \implies \partial p = \rho a_i \partial x_i = \rho\vec{a}\cdot\partial\vec{x}. \end{aligned} \tag{3.1.18}$$

Now, before solving our differential equation, we need to know if our fluid is compressible or incompressible. So, we are going to present the analyses of both kinds of fluids.

Incompressible fluid

As we know from previous sections, an incompressible fluid has a constant density. Then, equation (3.1.18) only depends on the form of the acceleration,

$$p = \rho\int \vec{a}\cdot\partial\vec{x}, \tag{3.1.19}$$

but if we are in a constant acceleration field, we can simplify the equation and obtain.

$$\int \partial p = \rho a_i \int \partial x_i \implies p = \rho a_i x_i + c. \tag{3.1.20}$$

This is important because we can express this in terms of the three dimensions as.

$$p(x, y, z) = \rho(a_x x + a_y y + a_z z) + c = \rho(\vec{a}\cdot\vec{x}) + c. \tag{3.1.21}$$

Compressible fluid

For a compressible fluid, we need to consider that the density is a function of the pressure. Therefore, before calculating the pressure, we need to find the expression for density.

If we suppose that we are working with an ideal gas, we can use the ideal gas law to find the expression of density.

$$\rho = \frac{m}{V} \text{ and } m = nM \implies \rho = \frac{nM}{V}, \tag{3.1.22}$$

where M is the molecular mass, n is the number of moles, and V is the volume of our gas. Substituting in the ideal gas law, we obtain.

$$pV = nRT \implies p = \frac{nRT}{V} \implies p = \frac{\rho RT}{M} \therefore \rho(p) = \frac{pM}{RT}. \tag{3.1.23}$$

Now, using the previous result in the static equation, our equation of behavior is.

$$\vec{\nabla} p = \rho \vec{a} = \frac{M\vec{a}}{RT} p. \tag{3.1.24}$$

It is important to note that our partial differential equation depends on the expression of the components of the acceleration vector. If we consider the acceleration as a constant field in one direction, the solution of our differential equation is as follows.

$$\begin{aligned} \frac{dp}{dx} = \frac{Ma}{RT} p \implies \frac{dp}{p} = \frac{Ma}{RT} dx \\ \implies \int \frac{dp}{p} = \int \frac{Ma}{RT} dx \implies ln(p) = \frac{Max}{RT} + c \\ p(x) = e^{\frac{Max}{RT} + c} \implies \boxed{p(x) = Ce^{\frac{Ma}{RT} x}}. \end{aligned} \tag{3.1.25}$$

As we can see from the expression of pressure in both cases (incompressible and compressible), density plays a crucial role in the pressure of our system and indirectly in the evolution of the system.

3.2 Fluid Dynamics

In order to describe the interaction between a fluid flow and an oscillatory structure, it is essential to recall the fundamental principles that govern the motion of fluids. In this section, we introduce the basic mathematical tools of fluid mechanics, including the material derivative, the conservation laws, and the equations that characterize fluid motion. These tools will be crucial in understanding how vortices are formed and how they later interact with structures.

3.2.1 Kinematics of a Fluid Flow

A fluid velocity field is denoted by $\vec{u}(\vec{x}, t)$, where $\vec{x}$ is the spatial coordinate and t is time. A fluid particle follows a trajectory $\vec{x}(t)$ determined by

$$\frac{d\vec{x}}{dt} = \vec{u}(\vec{x}(t), t). \tag{3.2.1}$$

To compute how a quantity changes as we move with a fluid particle, we introduce the **material derivative**. Given a scalar field $\phi(\vec{x}, t)$, its material derivative is defined as

$$\frac{D\phi}{Dt} = \frac{\partial \phi}{\partial t} + \vec{u} \cdot \vec{\nabla} \phi, \tag{3.2.2}$$

which represents the total temporal rate of change experienced by a particle. When applied to the velocity field we obtain the particle acceleration,

$$\frac{D\vec{u}}{Dt} = \frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \vec{\nabla})\vec{u}. \tag{3.2.3}$$

3.2.2 Conservation of Mass

The conservation of mass can be derived from the Reynolds Transport Theorem applied to a material volume. Expressing that the total mass does not change over time, we obtain

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{u}) = 0. \tag{3.2.4}$$

When the density is constant, the fluid is incompressible and the continuity equation simply becomes

$$\vec{\nabla} \cdot \vec{u} = 0. \tag{3.2.5}$$

3.2.3 Momentum Balance: Euler and Navier–Stokes Equations

Applying Newton's second law to a fluid particle yields

$$\rho \frac{D\vec{u}}{Dt} = \vec{f} + \vec{\nabla} \cdot \boldsymbol{\sigma}, \tag{3.2.6}$$

where $\vec{f}$ are body forces and $\boldsymbol{\sigma}$ is the stress tensor. For an ideal fluid the stress tensor reduces to a pressure term,

$$\boldsymbol{\sigma} = -p\, \vec{I}, \tag{3.2.7}$$

leading to the **Euler equations**,

$$\rho \frac{D\vec{u}}{Dt} = -\vec{\nabla} p + \vec{f}. \tag{3.2.8}$$

If viscosity is included, we assume a Newtonian model where

$$\boldsymbol{\sigma} = -p\, \vec{I} + \mu\big(\vec{\nabla}\vec{u} + \vec{\nabla}\vec{u}^{T}\big), \tag{3.2.9}$$

leading to the **Navier–Stokes equations**,

$$\rho \frac{D\vec{u}}{Dt} = -\vec{\nabla} p + \mu \vec{\nabla}^2 \vec{u} + \vec{f}. \tag{3.2.10}$$

3.2.4 Bernoulli's Principle

Under the assumptions of inviscid, incompressible, steady flow along a streamline, the Euler equation simplifies to

$$\vec{\nabla} \left(p + \frac{1}{2}\rho |\vec{u}|^2 + \rho g z \right) = 0, \tag{3.2.11}$$

which implies that

$$p + \frac{1}{2}\rho |\vec{u}|^2 + \rho g z = \text{constant along a streamline.} \tag{3.2.12}$$

This relation connects velocity, pressure, and elevation, and it plays a central role in understanding how vortex formation is initiated around a bluff body.

3.2.5 Streamlines

Streamlines are curves tangent to the velocity field,

$$\frac{d\vec{x}}{ds} = \vec{u}(\vec{x}, t), \tag{3.2.13}$$

and for steady flows they coincide with particle paths. The structure of streamlines around an obstacle determines where the flow separates and where recirculation regions appear. These regions are the seeds of periodic vortex formation, a phenomenon central to vortex-induced vibrations.

3.3 Fluid–Structure Interaction

When a fluid interacts with a solid body, the motion of the fluid modifies the motion of the structure, and the motion of the structure modifies the motion of the fluid. This mutual influence is what we call **fluid–structure interaction** (FSI). In this project, the solid body is an oscillatory system, so the fluid forces can play the role of an external excitation, damping, or even an added inertia. For this reason, understanding the mathematical form of these interactions is fundamental to predict how the structure behaves when it is surrounded by a moving fluid.

3.3.1 Interface kinematics and dynamics

Since both systems are physically in contact, the first condition that we must impose is *kinematic compatibility*. In simple words, the fluid that touches the structure must move exactly with it. Mathematically, this condition appears as

$$\vec{v}_f|_{\Gamma_{fs}} = \dot{\vec{u}}_s|_{\Gamma_{fs}}, \tag{3.3.1}$$

where Γ_{fs} is the interface between the fluid and the structure. This expression ensures that no fluid particle crosses the surface of the structure and that the surface does not "slip" relative to the fluid.

The second condition is *dynamic compatibility*. The stresses acting on the fluid and on the structure must balance each other at the interface:

$$\boldsymbol{\sigma}_f \vec{n}_f + \boldsymbol{\sigma}_s \vec{n}_s = \vec{0}. \tag{3.3.2}$$

Because the normals satisfy $\vec{n}_f = -\vec{n}_s$, the force exerted by the fluid on the structure is

$$\vec{F}_{\text{fluid}}(t) = \int_{\Gamma_{fs}} \boldsymbol{\sigma}_f \vec{n}\, dS = -\int_{\Gamma_{fs}} p\vec{n}\, dS + \int_{\Gamma_{fs}} \boldsymbol{\tau}_f \vec{n}\, dS, \tag{3.3.3}$$

where the first term is the pressure contribution and the second comes from viscosity. In many engineering applications the pressure term dominates.

3.3.2 Structural dynamics with fluid forcing

If the structure behaves approximately like a single-degree-of-freedom oscillator, then its motion is described by

$$m\ddot{y} + c\dot{y} + ky = F_{\text{fluid}}(t) + F_{\text{ext}}(t), \tag{3.3.4}$$

where $F_{\text{fluid}}(t)$ is the projection of the hydrodynamic force in the direction of motion and $F_{\text{ext}}(t)$ represents any other external excitation. At this point, the equation already shows that the fluid can input energy, extract energy, or modify the resonance properties of the oscillator.

3.3.3 Added mass: an inertial effect of the fluid

One of the most characteristic effects of FSI is the **added mass**. When the structure accelerates, it must push some of the surrounding fluid. This extra effort appears mathematically as if the mass of the structure were larger.

Under the assumptions of inviscid and irrotational flow, the velocity field can be written as $\vec{v} = \nabla\phi$, where ϕ satisfies

$$\nabla^2\phi = 0, \qquad \frac{\partial\phi}{\partial n} = \vec{V}_b \cdot \vec{n} \quad \text{on the surface.} \tag{3.3.5}$$

For small body motions the potential becomes a linear combination of elementary potentials:

$$\phi(\vec{x}, t) = \sum_j \Phi_j(\vec{x})\, V_{b,j}(t).$$

Inserting this expression into the kinetic energy of the fluid gives

$$T_f = \frac{\rho}{2}\int_{\Omega_f} |\nabla\phi|^2\, dV = \frac{1}{2}\sum_{i,j} M_{ij}^{\text{add}}\, V_{b,i} V_{b,j}, \tag{3.3.6}$$

where the coefficients

$$M_{ij}^{\text{add}} = \rho\int_{\Omega_f} \nabla\Phi_i \cdot \nabla\Phi_j\, dV$$

form the so-called **added mass matrix**. This quantity modifies the inertial term of the structure, producing the effective equation

$$(M + M^{\text{add}})\ddot{y} + C\dot{y} + Ky = F_{\text{ext}} + F_{\text{vortex}}. \tag{3.3.7}$$

For a circular cylinder in potential flow, the classical result gives

$$M^{\text{add}} = \rho\frac{\pi D^2}{4},$$

which shows that a non-negligible portion of fluid must always be accelerated together with the body.

3.3.4 Coupled system: fluid and structure

The complete FSI problem is the combination of the Navier–Stokes equations and the structural equation of motion, linked by the interface conditions previously described.

Fluid: Navier–Stokes equations

$$\rho_f \left(\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\vec{v} \right) = -\nabla p + \mu \nabla^2 \vec{v}, \tag{3.3.8}$$

$$\nabla \cdot \vec{v} = 0, \tag{3.3.9}$$

with the condition $\vec{v} = \dot{\vec{u}}_s$ at the fluid–structure interface.

Structure: oscillator with added mass

$$(M + M^{\text{add}})\ddot{y} + C\dot{y} + Ky = \int_{\Gamma_{fs}} (\boldsymbol{\sigma}_f \vec{n}) \cdot \hat{e}_y \, dS + F_{\text{ext}}. \tag{3.3.10}$$

Solving this system requires numerical methods in almost every practical scenario. However, the analytical understanding of added mass, hydrodynamic forcing, and interface constraints provides the conceptual tools needed to analyze more complex fluid–structure phenomena such as vortex-induced vibrations.

3.4 Vortex-Induced Vibrations

Among the most typical and important phenomena in fluid–structure interaction is Vortex-Induced Vibrations (VIV). When a fluid flows around a bluff body (e.g., cylinder, prism, or any object that does not keep the flow attached to it) a periodic and alternating detachment of vortices occurs bottom stream, forming a pattern called the Kármán vortex street, which creates an oscillating lift force on the structure. The lift force acting on the structure is oscillating in time and therefore not constant, which causes the structure to oscillate in response to the alternating vortices.

The main point is that the rate of vortex shedding is not random. For a particular flow rate U and some characteristic length D, the shedding frequency f_s is generally given by the Strouhal number,

$$St = \frac{f_s D}{U}, \tag{3.4.1}$$

where St remains approximately constant for a wide range of Reynolds numbers when dealing with bluff bodies. Therefore, knowing or controlling the flow speed means that we implicitly know the forcing frequency set up by the vortices.

The relevance of VIV develops when the shedding frequency f_s gets close to the natural frequency of the structure f_n. In this flow regime, the force exerted by the fluid motion has a strong coupling with the natural dynamics of the system, generating an effect that is similar to the mechanical resonance. Then, the amplitude of this damped oscillation could increase significantly, even with the external forcing (i.e., the flow) remaining constant.

Mathematically, we often model structural displacement $y(t)$ using the same type of linear oscillator that appears throughout this chapter:

$$m\ddot{y} + c\dot{y} + ky = F_L(t), \tag{3.4.2}$$

where $F_L(t)$ is the unsteady lift force generated by the periodic vortex shedding. A common phenomenological approximation assumes a harmonic lift force,

$$F_L(t) \approx F_0 \sin(2\pi f_s t), \tag{3.4.3}$$

which makes explicit the connection between VIV and forced oscillations.

However, VIV is more subtle than a simple forced harmonic response. When the oscillation amplitude grows, the structure itself modifies the flow, altering the vortex shedding pattern. This makes the problem inherently nonlinear and coupled: the fluid affects the structure, and the structure affects the fluid. One of the clearest manifestations of this coupling is the so-called "lock-in" or "synchronization" phenomenon. In this regime, the shedding frequency f_s no longer follows the Strouhal relation but becomes synchronized with the structural natural frequency f_n. As a result, the oscillation amplitude can remain large over a wide range of flow velocities.

This behavior has profound implications in engineering. On one hand, VIV may compromise structural integrity. Offshore risers, bridge cables, building components, and other slender structures can experience fatigue or catastrophic failure due to long-term VIV. On the other hand, this same phenomenon can be exploited. Because VIV converts kinetic energy from the passing flow into mechanical oscillations, one can design systems that deliberately enter the lock-in regime to maximize energy extraction. In this project, our goal is precisely to leverage VIV to harvest fluid-flow energy through a controlled oscillatory mechanism. Understanding the underlying dynamics is therefore essential to achieve efficient, stable, and predictable energy capture.

3.5 Example

The problem addressed in this document is to find the solution to the equation of motion for a cylinder submerged in a fluid, with the additional condition that the cylinder is attached to springs.

Below is a diagram representing the layout of the system to be solved.

3.5.1 Physical Formulation

We start from the equation of the forced harmonic oscillator, given by:

$$m\ddot{x} + \beta\dot{x} + kx = F\cos(\omega t), \tag{3.5.1}$$

where m is the mass of the cylinder, β is the damping coefficient, k is the elastic restoring constant, F is the external force acting on the system, and ω is the frequency of that external force.

Factoring out the mass of the cylinder leads to:

$$m\left(\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x\right) = F\cos(\omega t), \tag{3.5.2}$$

where $2\gamma = \frac{\beta}{m}$ and $\omega_0 = \sqrt{\frac{k}{m}}$. Since the system is set up for vertical oscillation, we replace x with y and cos with sin:

$$m\left(\ddot{y} + 2\gamma\dot{y} + \omega_0^2 y\right) = F_y\sin(\omega t), \tag{3.5.3}$$

Using the lift coefficient definition $C_L = \frac{F}{\frac{1}{2}\rho u^2 A}$, the equation becomes:

$$m\left(\ddot{y} + 2\gamma\dot{y} + \omega_0^2 y\right) = \frac{1}{2}\rho u^2 A C_L\sin(\omega t), \tag{3.5.4}$$

where A is the area of the cylinder interacting with the fluid, u is the fluid velocity, and ρ is the fluid density. Given that $A = \pi DL$, and introducing a phase shift ϕ in the sinusoid to account for any time instant, the equation of motion is:

$$\boxed{m\left(\ddot{y} + 2\gamma\dot{y} + \omega_0^2 y\right) = \frac{1}{2}\pi\rho u^2 DLC_L\sin(\omega t + \phi)} \tag{3.5.5}$$

3.5.2 Solution of the Differential Equation

To solve differential equation (3.5.5), we first simplify by dividing by the mass:

$$\ddot{y} + 2\gamma\dot{y} + \omega_0^2 y = \frac{1}{2m}\rho u^2 DLC_L\sin(\omega t + \phi) \tag{3.5.6}$$

We seek both the homogeneous and particular solutions.

Homogeneous Solution

The homogeneous part is:

$$\ddot{y} + 2\gamma\dot{y} + \omega_0^2 y = 0 \tag{3.5.7}$$

The associated characteristic polynomial is:

$$m^2 + 2\gamma m + \omega_0^2 = 0 \tag{3.5.8}$$

Its roots are:

$$m_1 = -\gamma + \sqrt{\gamma^2 - \omega_0^2}, \tag{3.5.9}$$

$$m_2 = -\gamma - \sqrt{\gamma^2 - \omega_0^2} \tag{3.5.10}$$

Thus the homogeneous solution is:

$$
\begin{aligned}
y_h(t) &= Ae^{m_1 t} + Be^{m_2 t} \\
&= Ae^{(-\gamma+\sqrt{\gamma^2-\omega_0^2})t} + Be^{(-\gamma-\sqrt{\gamma^2-\omega_0^2})t} \\
\therefore y_h(t) &= e^{-\gamma t}\left(Ae^{\sqrt{\gamma^2-\omega_0^2}\,t} + Be^{-\sqrt{\gamma^2-\omega_0^2}\,t}\right)
\end{aligned} \tag{3.5.11}
$$

Particular Solution

We propose a particular solution of the form:

$$
y_p(t) = C\cos(\omega t) + D\sin(\omega t) \tag{3.5.12}
$$

Its derivatives are:

$$
\dot{y}_p = -\omega C\sin(\omega t) + \omega D\cos(\omega t), \tag{3.5.13}
$$

$$
\ddot{y}_p = -\omega^2 C\cos(\omega t) - \omega^2 D\sin(\omega t) \tag{3.5.14}
$$

Substituting into the differential equation yields:

$$
\left[C\cos(\omega t) + D\sin(\omega t)\right]\left(\omega_0^2 - \omega^2\right) + 2\gamma\omega\left(D\cos(\omega t) - C\sin(\omega t)\right) = \frac{F}{m}\sin(\omega t) \tag{3.5.15}
$$

Separating coefficients leads to:

$$
D\left(\omega_0^2 - \omega^2\right) - 2C\gamma\omega = \frac{F}{m}, \tag{3.5.16}
$$

$$
C\left(\omega_0^2 - \omega^2\right) + 2D\gamma\omega = 0 \tag{3.5.17}
$$

Solving for C and D gives:

$$
C = -\frac{2\gamma\omega F}{m\left((\omega_0^2-\omega^2)^2 + 4\gamma^2\omega^2\right)}, \quad D = \frac{F(\omega_0^2-\omega^2)}{m\left((\omega_0^2-\omega^2)^2 + 4\gamma^2\omega^2\right)} \tag{3.5.18}
$$

Thus:

$$
y_p(t) = -\frac{2\gamma\omega F}{m\left((\omega_0^2-\omega^2)^2 + 4\gamma^2\omega^2\right)}\cos(\omega t) + \frac{F(\omega_0^2-\omega^2)}{m\left((\omega_0^2-\omega^2)^2 + 4\gamma^2\omega^2\right)}\sin(\omega t) \tag{3.5.19}
$$

The complete solution is $y(t) = y_h(t) + y_p(t)$:

$$
y(t) = e^{-\gamma t}\left(Ae^{\sqrt{\gamma^2-\omega_0^2}\,t} + Be^{-\sqrt{\gamma^2-\omega_0^2}\,t}\right) + \frac{F\left[(\omega_0^2-\omega^2)\sin(\omega t) - 2\gamma\omega\cos(\omega t)\right]}{m\left((\omega_0^2-\omega^2)^2 + 4\gamma^2\omega^2\right)} \tag{3.5.20}
$$

Substituting the physical values, we get:

$$y(t) = e^{-\gamma t}\left(Ae^{\sqrt{\gamma^2-\omega_0^2}\,t} + Be^{-\sqrt{\gamma^2-\omega_0^2}\,t}\right) + \frac{\pi\rho u^2 DLC_L\left[(\omega_0^2-\omega^2)\sin(\omega t) - 2\gamma\omega\cos(\omega t)\right]}{2m\left((\omega_0^2-\omega^2)^2 + 4\gamma^2\omega^2\right)}, \tag{3.5.21}$$

where γ is the damping, ω the forcing frequency, ω_0 the natural frequency, m the mass, ρ the fluid density, u the fluid velocity, D the diameter, L the length, and C_L the lift coefficient. Constants A and B depend on initial conditions.

At resonance ($\omega_0 = \omega$), the maximum amplitude is:

$$y(t) = e^{-\gamma t}\left(Ae^{\sqrt{\gamma^2-\omega_0^2}\,t} + Be^{-\sqrt{\gamma^2-\omega_0^2}\,t}\right) - \frac{\pi\rho u^2 DLC_L\cos(\omega_0 t)}{4m\gamma\omega_0}, \tag{3.5.22}$$

Thus, to model the system, we need ρ, u, D, L, m, ω_0, γ, and C_L.

CHAPTER 4

Magneto-gravitational oscillator (Interaction magnet-magnet with gravity presence)

> The long and constant persuasion that all the forces of nature are mutually dependent, having one common origin, or rather being different manifestations of one major power, has made me often think upon the possibility of establishing, by experiment, a connection between gravity and electricity...
>
> Michael Faraday, *"Experimental Researches in Electricity"*, 1850

This physical system is composed of two magnets; one of the magnets is fixed on the ground, and the other is free to move vertically over the fixed magnet. The free magnet is initially displaced from the equilibrium position (x_0) to some amplitude (A). For the gravity presence, the free magnet falls, decreasing the distance between the magnets until the magnetic repulsion force increases enough to push the free magnet, increasing the distance and generating an oscillation.

The magnetic repulsion force F_M is expressed in this problem as a relation between the magnetic energy (M) and the distance between the magnets (x). So the mathematical form of the magnetic force is:

$$F_M = \frac{M}{x}. \tag{4.0.1}$$

As initial conditions for this problem, we have that at the time $t = 0$ the distance between the magnets is a displacement from the equilibrium position and the speed of the free magnet in that initial position is zero. Then, our initial conditions mathematically are:

$$x(0) = x_{eq} + A \quad \wedge \quad \dot{x}(0) = 0. \tag{4.0.2}$$

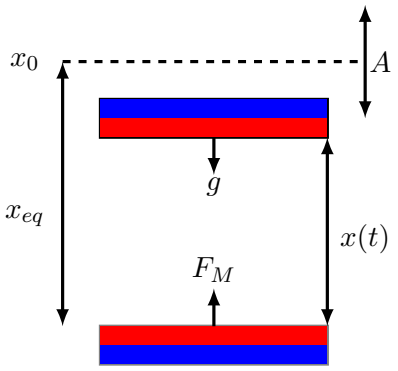


Figure 4.1: Graphic representation of the magnetic gravitational oscillator, where x_0 represents the equilibrium position, x_{eq} the equilibrium distance, A the oscillation amplitude and $x(t)$ the distance between of the two magnets.

The domain of the distance x goes from 0 to ∞, because the free magnet can be too far from the fixed magnet or so close but it can not touch it, so the domain is:

$$\text{Dom}(x) = (0, \infty), \tag{4.0.3}$$

If we use the domain of x and the initial condition of the position, we can find the domain of the amplitude.

$$A = x(0) - x_{eq} \implies A = \begin{cases} -x_{eq} & \text{if } x(0) = 0 \\ \infty & \text{if } x(0) = \infty \end{cases} \implies \text{Dom}(A) = (-x_{eq}, \infty), \tag{4.0.4}$$

this result lets us know that the initial amplitude can go from zero to infinity.

4.1 Ideal system

For the ideal case, the dissipative and external forces do not interact with the physical system, so they are zero. With this, the forces in the system are:

$$F = -F_g + F_M = -mg + \frac{M}{x}, \tag{4.1.1}$$

where F is the force, m is the mass of the free magnet, g the gravitational acceleration, x the distance between the two magnets and M is the constant of the magnetic energy. If the system is in equilibrium, then the net force is zero ($F = 0$) and we can find the equilibrium distance.

$$\frac{M}{x} - mg = 0 \implies \frac{M}{x} = mg \implies \boxed{x_{eq} = \frac{M}{mg}} \quad , \tag{4.1.2}$$

as we can see, the equilibrium distance compares the magnetic energy with the gravitational force; then the increase or decrease in some of these magnitudes affects the equilibrium distance of the system, this lets us modulate the equilibrium distance.

4.1.1 Lagrangian analysis

The relevance of the lagrangian analysis is that we can study the dynamic of the system in terms of its coordinates and degrees of freedom. The lagrangian is described as the sum of the kinetic energies minus the potential energies present in the system.

$$L = T - V \implies L = \frac{1}{2}\sum_i m_i \dot{q_i}^2 - V(q_1, q_2, \ldots, q_i), \tag{4.1.3}$$

where m_i is the ith mass of the system, $\dot{q_i}$ is the ith speed of the system, V is the potential energy function and $q_1, q_2, \ldots, q_i$ are the coordinates of the system.

Our system only have a degree of freedom, so the dynamic of the system is described by one coordinate (x), then our lagrangian has the following expression.

$$L = \frac{1}{2}m\dot{x}^2 - V(x), \tag{4.1.4}$$

where $V(x)$ is the potential energy in our system. To find the potential energy, we are going to use the relation between the work and the potential energy when the system is conservative.

$$\begin{aligned} -\frac{d}{dx}V(x) = F \implies -V(x) = \int F dx \implies -V(x) = -\int mg dx + \int \frac{M}{x} dx \\ \implies -V(x) = -mg\int dx + M\int \frac{dx}{x} \implies -V(x) = -mgx + M\ln(x) \\ \implies -V(x) = -\left[mgx - M\ln(x)\right] \therefore \boxed{V(x) = mgx - M\ln(x)}. \end{aligned} \tag{4.1.5}$$

Substituting this result in our lagrangian function (equation (4.1.4)), we obtain that our lagrangian is:

$$\boxed{L = \frac{1}{2}m\dot{x}^2 - mgx + M\ln(x)}. \tag{4.1.6}$$

With our lagrangian, we can find the energy of the system and the equation of motion of the system.

First, we going to find the energy of the system. The energy of the syste is define as.

$$E = \sum_i \dot{q_i}\frac{\partial L}{\partial \dot{q_i}} - L \implies E = \dot{x}\frac{\partial L}{\partial \dot{x}} - L \quad \therefore \quad \boxed{E = \frac{1}{2}m\dot{x}^2 + mgx - M\ln(x)}. \tag{4.1.7}$$

Now, using the Euler-Lagrange equation we can find the equation of motion of our system. As we know, the external forces are null ($Q = 0$). So the equation of motion for the coordinate x is.

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = Q \implies m\ddot{x} + mg - \frac{M}{x} = 0$$
$$\implies \ddot{x} + g - \frac{M}{mx} = 0 \quad \therefore \quad \boxed{\frac{d^2x}{dt^2} + g - \frac{M}{mx} = 0}. \tag{4.1.8}$$

This equation of motion is relevant because it lets us know the accelerations that generate the movement in our system. It's important to remark that the differential equation does not have an analytical solution.

4.1.2 Hamiltonian analysis

Now that we know the equation of motion, it is important to see if the system really oscillates. To do this we going to use the hamiltonian analysis. To use this analysis we need to construct the hamiltonian of our system.

$$H = E = \frac{1}{2}m\dot{x}^2 + mgx - M\ln(x)$$
$$\boxed{\therefore\ H = \frac{p^2}{2m} + mgx - M\ln(x)}, \tag{4.1.9}$$

where H is the hamiltonian of the system and p is the momentum.

Now, using the Hamilton's equations we can find the dynamic of the system. The Hamilton's equations are:

$$\dot{H} = \begin{pmatrix} \dot{x} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} \dfrac{p}{m} \\ \dfrac{M}{x} - mg \end{pmatrix} \tag{4.1.10}$$

In the Hamilton's equations we can see that the change of momentum depends of position and vice versa. If we plot the solution of the Hamilton's equations we obtain a phase portrait of our system to see the behavior of it.

As we can see in the figure (4.2), the system follows a closed cyclical trajectory, so we can confirm that the system oscillates.

4.1.3 Dimensional analysis

Once we know the behavior of the system, it is important to analyse how. To do that we use de dimensional analysis. To use this analysis, we need to know how many variables have a relevant role in the dynamic of the system. For our systeme we have six variables ($n = 6$), and that variables are: the mass (m), the position (x), the time (t), gravity (g), the amplitude of the oscillation (A) and the magnetic energy (M).

The nature of our system lets us know that there are three physical dimensions ($j = 3$) in which the system acts. The physical dimensionss are: the space, the time and the mass.

Applying the Buckingham's Pi theorem, we can find the dimensionless groups that describe the system. For our problem, we know that we have three dimensionless groups ($k = n - j = 3$).

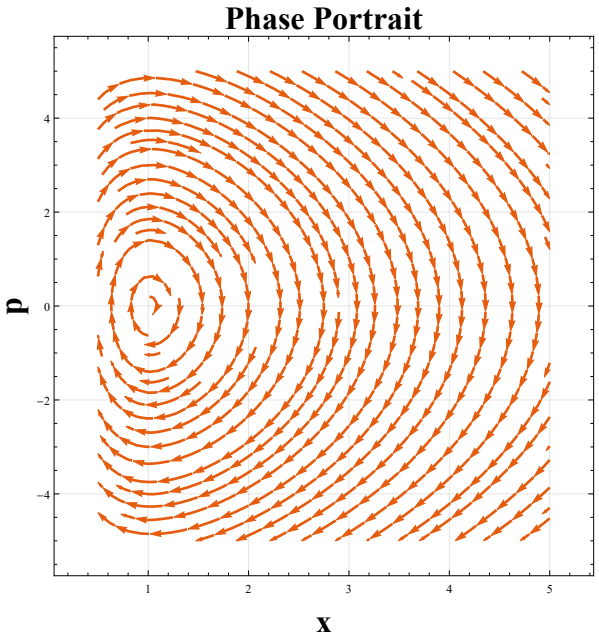


Figure 4.2: Phase portrait with $M = 10$, $g = 9.81$ and $m = 1$

To dimensionless our system, we need to do a dimensional analysis to every variable, so we obtain:

$$[x] = L, [A] = L, [t] = T, [m] = M, [g] = LT^{-2}, [M] = ML^2T^{-2} \tag{4.1.11}$$

If we select the variables m, g y M to dimensionless, we can find that every ith dimensionless set is the product of the powers of our variables that we use to dimensionless. Then, we have:

$$\Pi_i = \delta m^{\alpha} g^{\beta} M^{\lambda}, \tag{4.1.12}$$

where δ is the variable that we want to dimensioless, for our system we need to dimensionless the variables x, t y A; α, β and λ are the powers of the variables (m, g and M, in that order) that dimensionless the ith group.

We can write an equation system that lets us know how can be the powers (α, β and λ) that dimensionless every Pi group.

$$\begin{pmatrix} L \\ M \\ T \end{pmatrix} + \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ 0 & -2 & -2 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \lambda \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \implies \begin{pmatrix} \alpha \\ \beta \\ \lambda \end{pmatrix} = \begin{pmatrix} L + \frac{T}{2} - M \\ L + T \\ -L - \frac{T}{2}. \end{pmatrix} \tag{4.1.13}$$

It is necesary to clarify that every dimensionless group can be reduct to a fraction. The fraction is the relation between the variable of the system and its scale dimensionless factor. For that reason, the dimensionless groups can be write as:

$$\Pi_i = \frac{\delta}{\delta_s} = \tilde{\delta}, \tag{4.1.14}$$

where δ_s is the scale factor of our system's variable and $\tilde{\delta}$ is the dimensionless variable of the system.

Now, we can proceed to analyse the variables (x, t y A) of the system, with the intention to dimensionless the properties and the differential equation of the system.

Dimensionless group of the position:

As we see previously, x has only the space dimension ($L = 1$, $T = 0$, $M = 0$). Then, using the resul of the equation (4.1.13) we can find the powers of our dimensionless variables and the dimensionless group with the scale factor.

$$\begin{pmatrix}\alpha\\ \beta\\ \lambda\end{pmatrix} = \begin{pmatrix}1\\ 1\\ -1\end{pmatrix} \quad \therefore \quad \boxed{\Pi_1 = \frac{x}{x_s} = \frac{xmg}{M} = \frac{x}{x_{eq}} \implies x_s = \frac{M}{mg} = x_{eq}} \quad . \tag{4.1.15}$$

With the previous result we can see that the scale factor of the position is the equilibrium distance. Then, the position can be expressed as the product of the position's scale factor and the dimensionless position, as we can see in the following equation.

$$x = x_s\tilde{x} \tag{4.1.16}$$

If we differentiate the position, the differentiation applies only to the dimensionless position, because the scale factor is a constant. Then, we can express this as:

$$dx = d(x_s\tilde{x}) = x_s d\tilde{x} \implies d^2x = x_s d^2\tilde{x}. \tag{4.1.17}$$

This result is important because we going to use it to dimensionless the differential equation (the equation of motion obtained by the lagrangian analysis).

Dimensionless group of the time:

For the time, we have that the only dimension that affect it is the homologeous ($L = 0$, $T = 1$, $M = 0$). Then, substituting the values of the dimensions in the equation (4.1.13) we obtain the the Pi group and the time's scale factor.

$$\begin{pmatrix}\alpha\\ \beta\\ \lambda\end{pmatrix} = \begin{pmatrix}\frac{1}{2}\\ 1\\ -\frac{1}{2}\end{pmatrix} \quad \therefore \quad \boxed{\Pi_2 = \frac{t}{t_s} = tg\sqrt{\frac{m}{M}} \implies t_s = \frac{1}{g}\sqrt{\frac{M}{m}} = \sqrt{\frac{x_{eq}}{g}}} \quad . \tag{4.1.18}$$

For the previous solution, we can find the expressions for the scale factors of the frequency and the period of oscillation, that are:

$$\omega_s = \frac{1}{t_s} = \sqrt{\frac{g}{x_{eq}}} \quad \wedge \quad T_s = \frac{2\pi}{\omega_s} = 2\pi\sqrt{\frac{x_{eq}}{g}}. \tag{4.1.19}$$

The most important of these expressions is that they are equivalent to the frequency and the period of a pendulum, so we can conclude that this system is analogous to a pendulum.

We obtain this now by using the relation between the time variable with its scale factor and its dimensionless form.

$$t = t_s\tau \implies dt = t_s d\tau \implies dt^2 = t_s^2 d\tau^2 \tag{4.1.20}$$

Dimensionless group of the amplitude:

As the same as the position (x), the amplitude has the same dimensions ($L = 1, T = 0, M = 0$). Then the Pi group of the amplitude and its scale factor are:

$$\begin{pmatrix}\alpha\\ \beta\\ \lambda\end{pmatrix} = \begin{pmatrix}1\\ 1\\ -1\end{pmatrix} \quad \therefore \quad \boxed{\Pi_3 = \frac{A}{A_s} = \frac{Amg}{M} = \frac{A}{x_{eq}} \implies A_s = \frac{M}{mg} = x_{eq}} \quad . \tag{4.1.21}$$

From the above result, we can see that the amplitud, as the position, has a proportional relation with the equilibrium distance. Since the amplitude does not play a principal role in the equation of motion, we are going to use its dimensionless group as the correction factor (ϵ), then:

$$\epsilon = \frac{A}{x_{eq}} = \tilde{A}. \tag{4.1.22}$$

4.1.4 Dimensionless system

With all the dimensionless groups that we obtained previously, we can dimensionless the variables of our equation of motion obtained by the lagrangian (4.1.8). Then, our dimensionless equation of motion is:

$$\begin{aligned}
\frac{d^2x}{dt^2} + g - \frac{M}{mx} = 0 &\implies \frac{d^2(x_s\tilde{x})}{d(t_s\tau)^2} + g - \frac{M}{m(x_s\tilde{x})} = 0,\\
\frac{x_s}{t_s^2}\frac{d^2\tilde{x}}{d\tau^2} + g - \frac{M}{mx_s\tilde{x}} = 0 &\implies \frac{d^2\tilde{x}}{d\tau^2} + \frac{gt_s^2}{x_s} - \frac{Mt_s^2}{mx_s^2\tilde{x}} = 0,\\
\frac{d^2\tilde{x}}{d\tau^2} + \frac{gx_{eq}}{x_{eq}g} - \frac{Mx_{eq}}{mgx_{eq}^2\tilde{x}} = 0 &\implies \frac{d^2\tilde{x}}{d\tau^2} + 1 - \frac{M}{mgx_{eq}\tilde{x}} = 0,\\
\frac{d^2\tilde{x}}{d\tau^2} + 1 - \frac{x_{eq}}{x_{eq}\tilde{x}} = 0 &\implies \boxed{\frac{d^2\tilde{x}}{d\tau^2} + 1 - \frac{1}{\tilde{x}} = 0}.
\end{aligned} \tag{4.1.23}$$

If we multiply every term in our dimensionless differential equation to $\tilde{x}$, we can rewrite it as:

$$\frac{d^2\tilde{x}}{d\tau^2} + 1 - \frac{1}{\tilde{x}} = 0 \implies \boxed{\tilde{x}\frac{d^2\tilde{x}}{d\tau^2} + \tilde{x} - 1 = 0}. \tag{4.1.24}$$

It is relevant to see that our dimensionless differential equation is not homogeneous and non lineal, for this reason we can not solve it directly. Then, we are going to use an approximated solution. The approximated solution that we are going to applied is the asymptotic approximation, using ϵ to perturbate the original solution.

Once that we have our dimensionless differential equation, we are going to dimensionless the initial conditions of the system. Then, applying the dimensionless group of x in the initial condition of the position, we obtain:

$$x(0) = x_{eq} + A \implies x_s\tilde{x}(0) = x_{eq} + A \implies \tilde{x}(0) = \frac{x_{eq} + A}{x_s} \therefore \boxed{\tilde{x}(0) = 1 + \epsilon}. \quad (4.1.25)$$

Doing the same in the domains of x and A, the dimensionless domains are:

$$\begin{aligned} \text{Dom}(x) = (0, \infty) = \text{Dom}(x_s\tilde{x}) = x_{eq}\text{Dom}(\tilde{x}) &\implies \text{Dom}(\tilde{x}) = (0, \infty) \\ \text{Dom}(A) = (-x_{eq}, \infty) = \text{Dom}(A_s\tilde{A}) = x_{eq}\text{Dom}(\epsilon) &\implies \text{Dom}(\epsilon) = (-1, \infty) \end{aligned} \quad (4.1.26)$$

Now, we apply the differential property of x on the equation (4.1.17) to find the dimensionless form of the speed's initial condition.

$$\dot{x}(0) = 0 \implies x_s\dot{\tilde{x}} = 0 \therefore \boxed{\dot{\tilde{x}} = 0}. \quad (4.1.27)$$

4.1.5 Asymptotic approximation

As we have seen previously, the asymptotic approximation expresses the solution of a differential equation as a series. The approximated series or perturbed series is a sum of terms made up of a function (x_n) and a power of the control parameter (ϵ^n), and the objective is to balance the contribution of every term with the target to fit the perturbated series to the analytic solution. Then, the perturbated series is:

$$x \approx x_0 + x_1\epsilon + x_2\epsilon^2 + \cdots = \sum_{n=0}^{N} x_n\epsilon^n, \quad (4.1.28)$$

it is important to highlight that the approximation improves when the number of terms we use in the series is high. Another relevant observation is that we need to expand every function (x) with a series.

Using the perturbated series in our dimensionless equation of motion

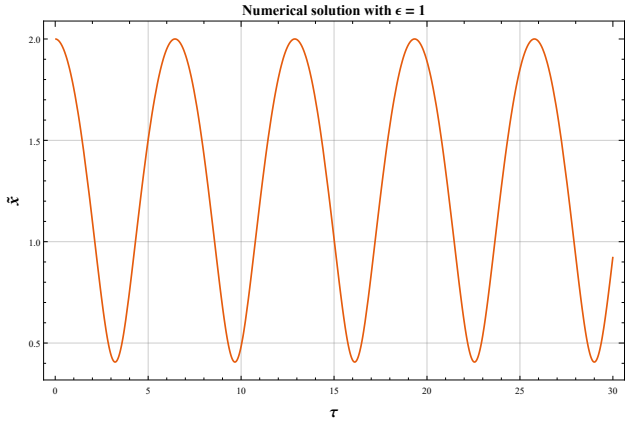
$$\sum_{n=0}^{N}\sum_{m=0}^{N} \epsilon^{n+m}\tilde{x}_n\frac{d^2\tilde{x}_m}{d\tau^2} = 1 - \sum_{j=0}^{N}\tilde{x}_j\epsilon^j. \quad (4.1.29)$$

4.1.6 Numerical solution

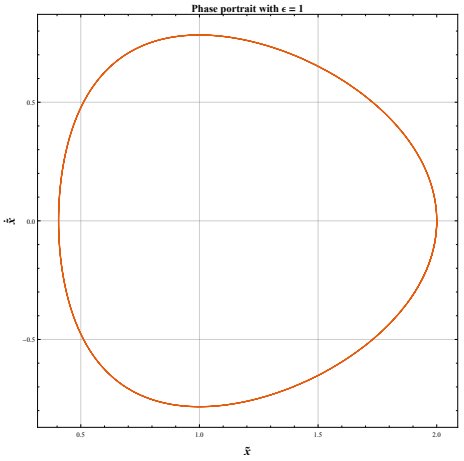
We know that our system does not have an analytical solution. It is for that reason that we need to use a numerical method to find a solution for the system and to contrast it with our perturbation solution.

We are going to use the Runge-Kutta method to find the numerical solution; and to do this, we are going to use the command ***NDSolve*** of the software ***Wolfram Mathematica***.

If we utilize $\epsilon = 1$ in our approximated solution, we get the plot that expresses the evolution of the system by time, and the parametric plot communicates the behavior of the system (this can be seen in figure 4.3).



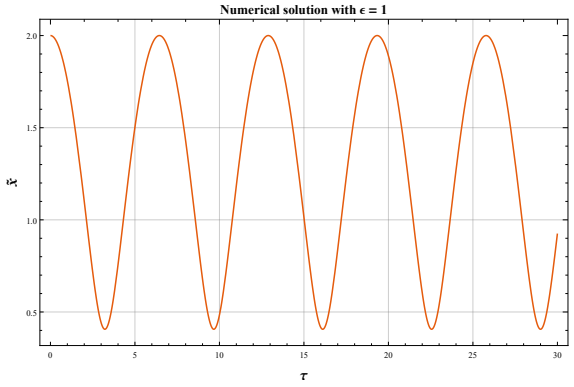
((a)) Time evolution



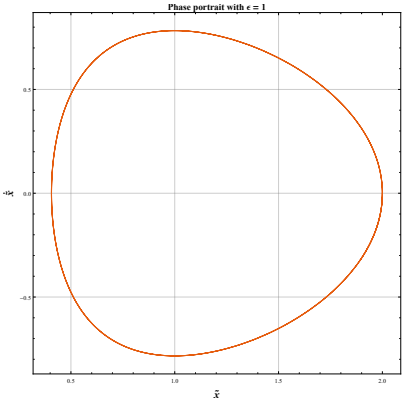
((b)) Phase portrait

Figure 4.3: Plot and parametric plot of the numerical solution with $\epsilon = 1$

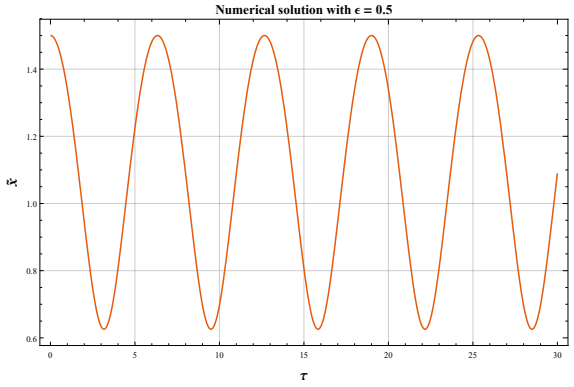
Something pertinent to highlight is how the system changes when we vary the value of ϵ. In figures (4.4 and 4.5), we can see these changes.



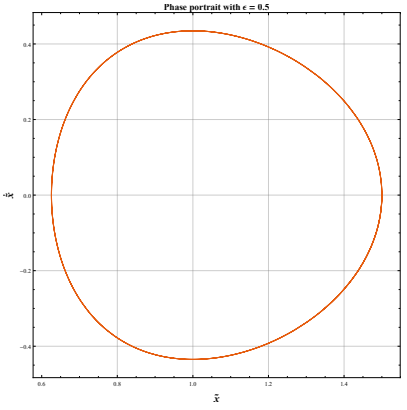
((a)) Time evolution ($\epsilon = 1$)



((b)) Phase portrait ($\epsilon = 1$)



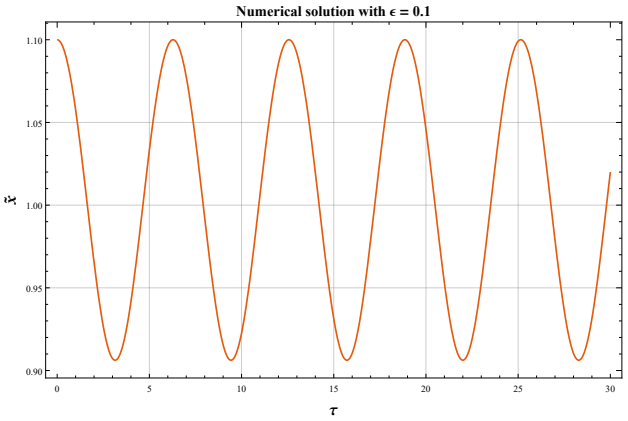
((c)) Time evolution ($\epsilon = 0.5$)



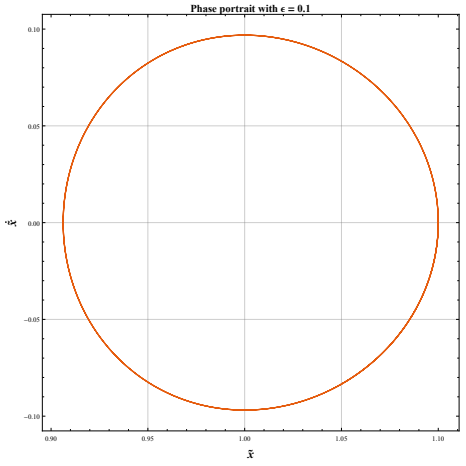
((d)) Phase portrait ($\epsilon = 0.5$)

Figure 4.4: System evolution and phase portraits for $\epsilon = 1$ and $\epsilon = 0.5$.

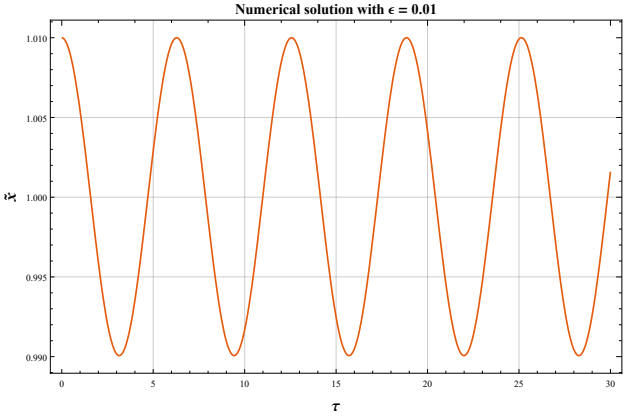
To previously figures named (figure (4.4) and figure (4.5)), we can see that when $\epsilon \approx 0$ is easier to fit a cyclical function, a cosine function.



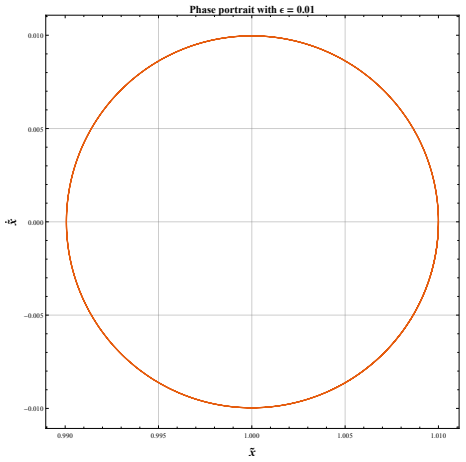
((a)) Time evolution ($\epsilon = 0.1$)



((b)) Phase portrait ($\epsilon = 0.1$)



((c)) Time evolution ($\epsilon = 0.01$)



((d)) Phase portrait ($\epsilon = 0.01$)

Figure 4.5: System evolution and phase portraits for $\epsilon = 0.1$ and $\epsilon = 0.01$.

4.1.7 Approximated solution

Zero order solution

The first step in our asymptotic approximation is to find the constant term of our series, to do this we need to solve the problem at zero order, that implies $\epsilon = 0$. Then the differential equation is:

$$\tilde{x}_0 \frac{d^2}{d\tau^2}\tilde{x}_0 = 1 - \tilde{x}_0 = 0 \implies \boxed{\tilde{x}_0 = 1}\,. \tag{4.1.30}$$

This result lets us know when the system is static, the position of the magnet is in the equilibrium distance.

First order solution

Now that we have the zero-order solution, we are going to find the first-order solution, e.i. , we need all the terms that are multiplied by ϵ. Then, the expression that we need to solve is:

$$\tilde{x}_0 \frac{d^2\tilde{x}_1}{d\tau^2} + \tilde{x}_1 \underbrace{\frac{d^2\tilde{x}_0}{d\tau^2}}_{0} = -\tilde{x}_1 \implies \frac{d^2\tilde{x}_1}{d\tau^2} = -\tilde{x}_1, \tag{4.1.31}$$

something important to highlight is that the natural frequency of the system is one ($\omega_0 = 1$). Solving the last differential equation we find that $\tilde{x}_1 = B_1 \cos(\tau) + B_2 \sin(\tau)$, therefore our series at first order is:

$$\tilde{x}(\tau) \approx 1 + \epsilon(B_1 \cos(\tau) + B_2 \sin(\tau)) \tag{4.1.32}$$

Using the initial conditions we can determinate the values of B_1 and B_2.

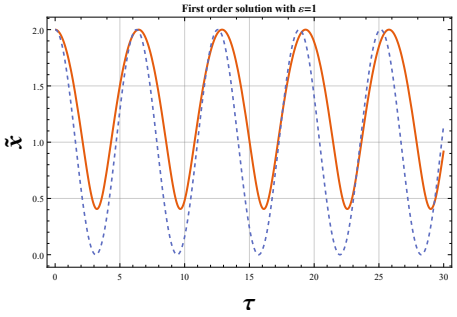
$$\tilde{x}(0) = 1 + \epsilon = 1 + \epsilon(B_1 \cos(0) + B_2 \sin(0)) \implies B_1 = 1, \tag{4.1.33}$$

$$\dot{\tilde{x}}(0) = \epsilon(-B_1 \sin(0) + B_2 \cos(0)) = 0 \implies B_2 = 0. \tag{4.1.34}$$

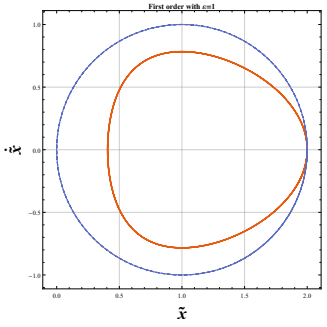
Then, our series at first order has the following form:

$$\tilde{x}(\tau) \approx 1 + \epsilon \cos(\tau). \tag{4.1.35}$$

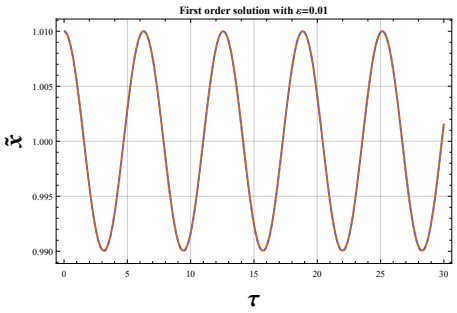
To see how approximate is our solution to the numerical solution, we have plotted both solutions with different values of ϵ to contrast their difference.



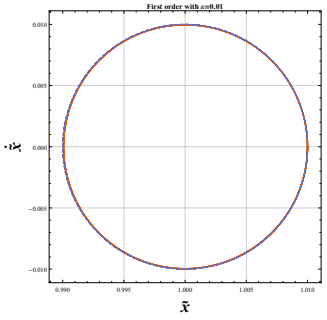
((a)) Time evolution contrast ($\epsilon = 1$)



((b)) Phase portrait contrast ($\epsilon = 1$)



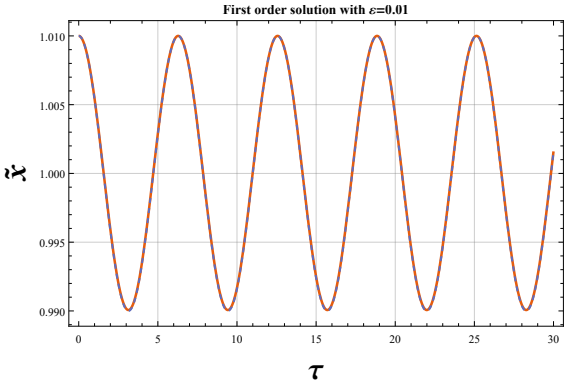
((c)) Time evolution ($\epsilon = 0.01$)



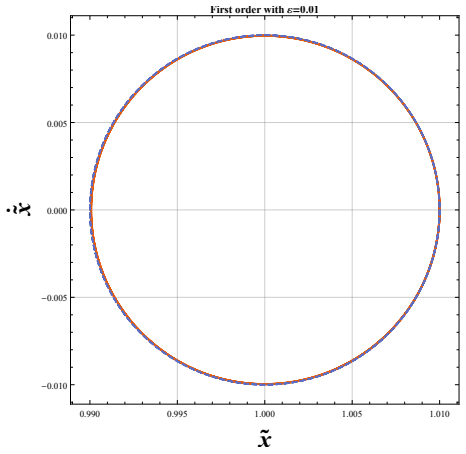
((d)) Phase portrait ($\epsilon = 0.01$)

Figure 4.6: Approximation of the solution to many values of ϵ.

If we analyze the previosly mentioned figure (figure 4.6), we can see when ϵ decrease, our perturbated series fits better with the numerical solution. Then, the first order of the series is a good approximation to a small perturbation ($|\epsilon| \ll 1$).



((a)) Time evolution ($\epsilon = 0.01$)



((b)) Phase portrait ($\epsilon = 0.01$)

Figure 4.7: Contrast of the numerical solution and the first order of the asymptotic solution with $\epsilon = 0.01$.

Second order solution

Once we have the first order solution, we can obtain the term of the second order. To do this we require every term that has a ϵ^2. Then, the dimensionless differential equation that we need to solve is:

$$\tilde{x}_0\frac{d^2\tilde{x}_2}{d\tau^2} + \tilde{x}_1\underbrace{\frac{d^2\tilde{x}_1}{d\tau^2}}_{-\tilde{x}_1} + \tilde{x}_2\underbrace{\frac{d^2\tilde{x}_0}{d\tau^2}}_{0} = -\tilde{x}_2 \implies \frac{d^2\tilde{x}_2}{d\tau^2} = \tilde{x}_1^2 - \tilde{x}_2 = \cos(\tau)^2 - \tilde{x}_2, \tag{4.1.36}$$

solving the differential equation we obtain $\tilde{x}_2(\tau) = \frac{1}{2} - \frac{1}{6}\cos(2\tau) + C\cos(\tau) + D\sin(\tau)$. Therefore, our solution series at second order is:

$$\tilde{x}(\tau) \approx 1 + \epsilon\cos(\tau) + \epsilon^2\left(\frac{1}{2} - \frac{1}{6}\cos(2\tau) + C\cos(\tau) + D\sin(\tau)\right) \tag{4.1.37}$$

Using the initial conditions we can find the values of the constants C and D:

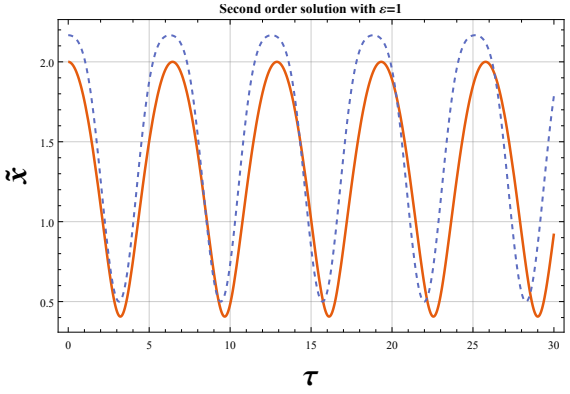
$$\begin{aligned}
\tilde{x}(0) = 1 + \epsilon\cos(0) + \epsilon^2\left(\frac{1}{2} - \frac{1}{6}\cos(0) + C\cos(0) + D\sin(0)\right) &= 1 + \epsilon, \\
1 + \epsilon + \epsilon^2\left(\frac{1}{2} - \frac{1}{6} + C\right) &= 1 + \epsilon, \\
\epsilon^2\left(\frac{1}{2} - \frac{1}{6} + C\right) &= 0, \\
\therefore C &= -\frac{1}{3}.
\end{aligned} \tag{4.1.38}$$

$$\begin{aligned}\dot{\tilde{x}}(0) = -\epsilon \sin(0) + \epsilon^2 \left(\frac{2}{6}\sin(0) - C\sin(0) + D\cos(0)\right) = 0 \\ D\epsilon^2 = 0 \\ \therefore D = 0.\end{aligned} \tag{4.1.39}$$

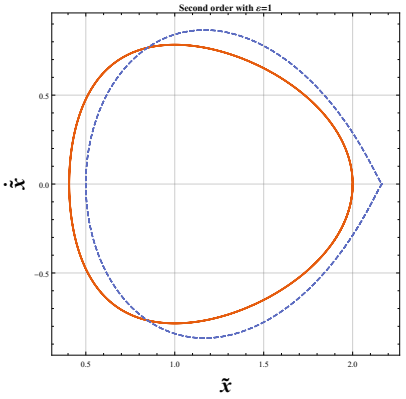
With the previous results the solution at second order is:

$$\tilde{x}(\tau) \approx 1 + \epsilon\cos(\tau) + \frac{\epsilon^2}{6}\left(3 - \cos(2\tau) - 2\cos(\tau)\right). \tag{4.1.40}$$

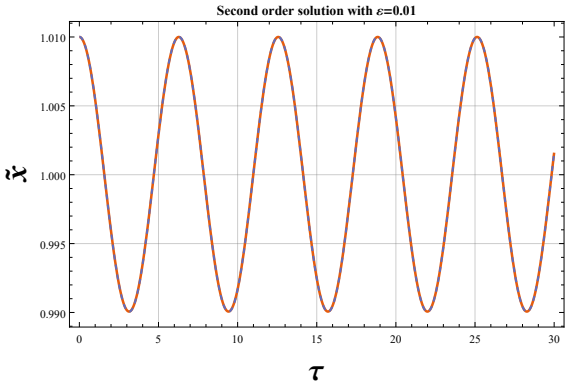
Now, we are going to compare the numerical and asymptotic solutions. To do that, we are going to plot both solutions. Those plots can be seen in figure (4.8). In that figure, we can see that the second order solution is more precise than the first order solution.



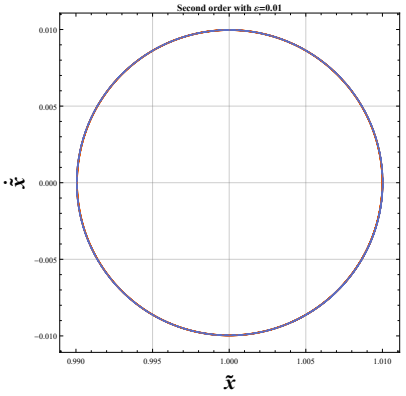
((a)) Time evolution contrast ($\epsilon = 1$)



((b)) Phase portrait contrast ($\epsilon = 1$)



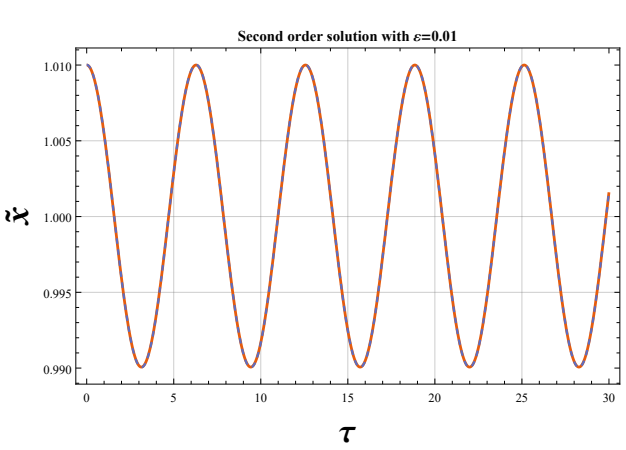
((c)) Time evolution ($\epsilon = 0.01$)



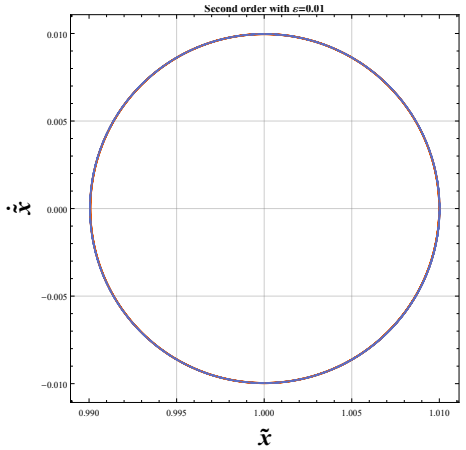
((d)) Phase portrait ($\epsilon = 0.01$)

Figure 4.8: Approximated solution at second order to the numerical solution with different values of ϵ.

We can see when ϵ decrease, the second order series fits better than the first order. The plotting of the second order series with $\epsilon = 0.01$ is:



((a)) Time evolution ($\epsilon = 0.01$)



((b)) Phase portrait ($\epsilon = 0.01$)

Figure 4.9: Comparison of the numerical solution with the second order solution and their parametric plots, with $\epsilon = 0.01$.

If we compare the figures (4.6) and (4.8), we can conclude that the second order solution is more precise than the first order to big values of ϵ. Therefore, to big values of ϵ we need to use more terms in our solution.

4.2 Damped system

For this case, we are going to consider the presence of a dissipative force and the absence of external forces. With these considerations, we have the following system of forces:

$$F = -F_g + F_m - F_f = -mg + \frac{M}{x} - \beta v, \tag{4.2.1}$$

where F is the net force, m is the mass of the free magnet, g is the gravitational acceleration, x the distance between the magnets, β is the mass flow rate coefficient, v the velocity and M is the constant of the magnetic energy.

If we consider the static case, where the net force is null ($F = 0$), we can find the equilibrium distance.

$$\frac{M}{x} - mg - \beta v = 0 \implies \frac{M}{x} = mg + \beta v \implies \boxed{x_{eq} = \frac{M}{mg + \beta v}}, \tag{4.2.2}$$

for the previous result, we can see that the equilibrium distance changes in the function of the velocity. This is because the damping will decrease the velocity of the system, causing the equilibrium point to change until the value of the velocity is zero and we recover the equilibrium position of the ideal system (equation (4.1.2)).

It is important to remark that the domain of the position (x) and the amplitude (A), and the initial conditions of the system are the same of the ideal system.

4.2.1 Lagrangian analysis

As we know, the lagrangian is a functional that acts under the supposition that the system is conservative. For this reason, we can not introduce in it the dissipative term in an

explicit form. To do this, we are going to separate the force's system into two parts, in conservative and in unconservative parts.

$$\underbrace{F + mg - \frac{M}{x}}_{\text{conservative}} + \underbrace{\beta v}_{\text{unconservative}} = 0. \tag{4.2.3}$$

Now that we have separated the force's system, we are going to apply the lagrangian analysis to the conservative part. If we do this the lagrangian of the conservative part is the lagrangian of the ideal system (equation (4.1.6)).

Now, to find the equation of motion of our damping system, we need to union the equation of motion of both parts (conservative and unconservative parts) as the sum of both. We know that the equation of motion of the conservative part is described by the Euler-Lagrange equation and the motion of the unconservative part is described by its differential form. Then, the equation of motion of our damping system is:

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} + \beta v = 0 \implies \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} + \beta \frac{dx}{dt} = 0 \\ \implies m\ddot{x} + mg - \frac{M}{x} + \beta \frac{dx}{dt} = 0 \implies \ddot{x} + g - \frac{M}{mx} + \frac{\beta}{m}\frac{dx}{dt} = 0, \end{aligned} \tag{4.2.4}$$

if we define the β coefficient as $\beta = 2m\phi$, where ϕ is the friction coefficient. Then, the equation of motion has the following form.

$$\ddot{x} + g - \frac{M}{mx} + 2\phi\frac{dx}{dt} = 0 \implies \boxed{\frac{d^2x}{dt^2} + 2\phi\frac{dx}{dt} + g - \frac{M}{mx} = 0}. \tag{4.2.5}$$

4.2.2 Hamiltonian

However, when we introduce damping, the total mechanical energy is no longer conserved, and this breaks the traditional Hamiltonian structure. It is important to remember that a Hamiltonian system must preserve energy unless it depends explicitly on time. For this reason, we cannot construct a time-independent Hamiltonian for a damped oscillator in the usual way.

To solve this problem, we introduce the **Caldirola–Kanai formalism**. This method modifies the Lagrangian with an exponential factor that grows in time. At first sight this may look artificial, but its purpose is simple: the exponential factor "restores" the conservation structure inside the variational formulation, allowing the Euler–Lagrange equations to recover the correct damping term.

Caldirola-Kanai Lagrangian

We start from the equation of motion of the damped oscillator:

$$m\ddot{x} + \beta m\,\dot{x} + V'(x) = 0, \tag{4.2.6}$$

where ϕ represents the damping coefficient. The Caldirola-Kanai proposal consists in defining the Lagrangian

$$L_{CK}(x, \dot{x}, t) = e^{\beta t}\left(\frac{1}{2}m\dot{x}^2 - V(x)\right) = L(x, \dot{x}, t). \tag{4.2.7}$$

Now, the Euler–Lagrange equation naturally generates the correct dissipative term. The exponential factor is the key that allows us to keep the action principle while still describing a system that loses energy.

Canonical momentum

The next step is to compute the canonical momentum associated with the coordinate x:

$$p = \frac{\partial L}{\partial \dot{x}} = m e^{\beta t} \dot{x}. \tag{4.2.8}$$

Here it is important to clarify something: the canonical momentum p is not the same as the physical momentum mv. Instead, both are related by

$$p = mv\, e^{\beta t}. \tag{4.2.9}$$

Solving for the velocity gives

$$\dot{x} = \frac{p}{m} e^{-\beta t}. \tag{4.2.10}$$

Construction of the Hamiltonian

Once we know the canonical momentum, we can construct the Hamiltonian using the definition

$$H(x, p, t) = p\dot{x} - L. \tag{4.2.11}$$

To do this, we first rewrite the Lagrangian in terms of x and p. Using $\dot{x} = \frac{p}{m} e^{-\beta t}$ we obtain

$$L = \frac{p^2}{2m} e^{-\beta t} - V(x) e^{\beta t}. \tag{4.2.12}$$

Substituting into the Legendre transform, we get

$$\begin{aligned} H &= p\left(\frac{p}{m} e^{-\beta t}\right) - \left(\frac{p^2}{2m} e^{-\beta t} - V(x) e^{\beta t}\right) \\ &= \frac{p^2}{2m} e^{-\beta t} + V(x) e^{\beta t}. \end{aligned} \tag{4.2.13}$$

We know that our potential energy $V(x) = mgx - M \ln(x)$. Therefore, the Hamiltonian of the damped oscillator is

$$\boxed{H(x, p, t) = \frac{p^2}{2m} e^{-\beta t} + \left(mgx - M \ln(x)\right) e^{\beta t}.} \tag{4.2.14}$$

Notice that this Hamiltonian depends explicitly on time. This is exactly what we expect in a system that loses mechanical energy.

Hamilton's equations and the phase portrait

Now we apply Hamilton's equations. The first one is

$$\dot{x} = \frac{\partial H}{\partial p} = \frac{p}{m} e^{-\beta t}. \tag{4.2.15}$$

The second one is

$$\dot{p} = -\frac{\partial H}{\partial x} = -V'(x)\, e^{\beta t}. \tag{4.2.16}$$

Therefore, the phase portrait of the Caldirola–Kanai oscillator is determined by the system

$$\boxed{\begin{aligned} \dot{x} &= \frac{p}{m} e^{-\beta t}, \\ \dot{p} &= -\left(mg - \frac{M}{x} \right) e^{\beta t}. \end{aligned}} \tag{4.2.17}$$

Something important to highlight is that this Hamiltonian describes the conservative system and does not reflect the physical behavior of the system, but lets us know how the energy depends on time.

If we want to see the behavior of the system by a phase portrait, we can use our Lagrangian. Then, our phase portrait is.

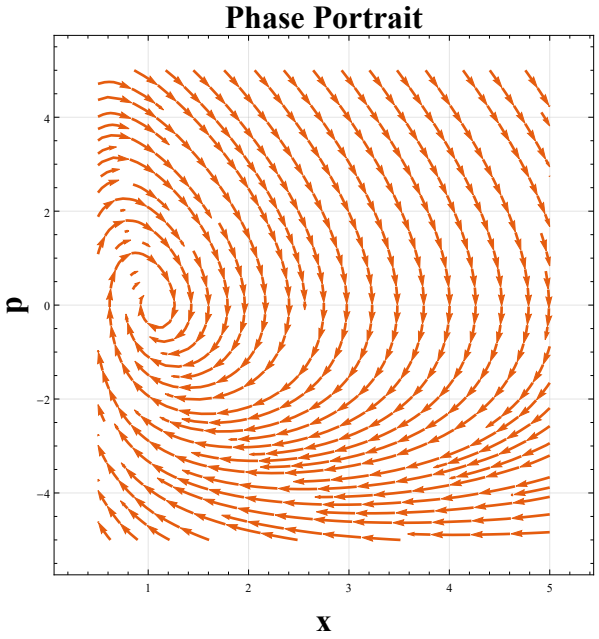


Figure 4.10: Phase portrait with $M = 10$, $g = 9.81$, $\phi = 0.75$ and $m = 1$

4.2.3 Dimensional analysis

We are going to use the same dimensionless groups that we find in the ideal system (section 4.1.3), but we only need to find the dimensionless group of the friction number.

As in the ideal case, we use the variables m, g, and M to dimensionless our system. Doing the dimensional analysis of these variables and the friction number.

$$[\phi] = T^{-1}, [m] = M, [g] = LT^{-2}, [M] = ML^2T^{-2} \tag{4.2.18}$$

With this, we can find the dimensionless groups of the friction number as the product of the powers of our variables that we use to dimensionless. Then, we have:

$$\Pi_i = \delta m^{\alpha} g^{\beta} M^{\lambda}, \tag{4.2.19}$$

where δ is the variable that we want to make dimensionless. Now, using the equation (4.1.13), we can find the powers of our variables to build our dimensionless groups. This equation is:

$$\begin{pmatrix} \alpha \\ \beta \\ \lambda \end{pmatrix} = \begin{pmatrix} L + \frac{T}{2} - M \\ L + T \\ -L - \frac{T}{2}. \end{pmatrix} \tag{4.2.20}$$

Now, we can proceed to analyze our new variable (ϕ) of the system.

Friction group

Now, we are going to work with the friction number (ϕ) that only works with one dimension ($L = 0$, $T = -1$, $M = 0$). Then, using our equation to find the powers (equation (4.3.8)), we obtain that our dimensionless group and its scale factor are.

$$\begin{pmatrix} \alpha \\ \beta \\ \lambda \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ -1 \\ \frac{1}{2} \end{pmatrix} \quad \therefore \quad \boxed{\Pi_4 = \frac{\phi}{\phi_s} = \frac{\phi}{g}\sqrt{\frac{M}{m}} \implies \phi_s = \frac{1}{t_s} = g\sqrt{\frac{m}{M}} = \sqrt{\frac{g}{x_{eq}}}} \quad . \tag{4.2.21}$$

This result is particular because it is the frequency of the pendulum. Then, the frequency is a factor of damping. This could be explained like the friction acts more when the frequency is higher.

Now, we define the dimensionless friction number. To do this, we use the dimensionless group of the frequency ($\Pi_4 = \Phi$). Now, we can express the frequency in terms of the dimensionless frequency, as we can see in the following expression.

$$\Pi_4 = \frac{\phi}{\phi_s} \implies \phi = \phi_s \Phi = \frac{1}{t_s}\Phi \tag{4.2.22}$$

4.2.4 Dimensionless system

Now, using the dimensional groups, we are going to dimensionless the equation of motion (4.2.5). Then, we obtain:

$$\begin{aligned}
\frac{d^2x}{dt^2} + 2\phi\frac{dx}{dt} + g - \frac{M}{mx} &= 0 \\
\implies \frac{d^2(x_s\tilde{x})}{d(t_s\tau)^2} + 2\phi\frac{d(x_s\tilde{x})}{d(t_s\tau)} + g - \frac{M}{mx_s\tilde{x}} &= 0 \\
\implies \frac{x_s}{t_s^2}\frac{d^2\tilde{x}}{d\tau^2} + \frac{2(\phi_s\Phi)x_s}{t_s}\frac{d\tilde{x}}{d\tau} + g - \frac{M}{mx_s\tilde{x}} &= 0 \\
\implies \frac{d^2\tilde{x}}{d\tau^2} + 2\Phi\frac{d\tilde{x}}{d\tau} + \frac{gt_s^2}{x_s} - \frac{Mt_s^2}{mx_s^2\tilde{x}} &= 0 \\
\implies \frac{d^2\tilde{x}}{d\tau^2} + 2\Phi\frac{d\tilde{x}}{d\tau} + 1 - \frac{M}{mgx_{eq}\tilde{x}} &= 0 \\
\implies \frac{d^2\tilde{x}}{d\tau^2} + 2\Phi\frac{d\tilde{x}}{d\tau} + 1 - \frac{1}{\tilde{x}} &= 0.
\end{aligned} \tag{4.2.23}$$

Multiplying all the terms by $\tilde{x}$.

$$\frac{d^2\tilde{x}}{d\tau^2} + 2\Phi\frac{d\tilde{x}}{d\tau} + 1 - \frac{1}{\tilde{x}} = 0 \implies \tilde{x}\frac{d^2\tilde{x}}{d\tau^2} + 2\Phi\tilde{x}\frac{d\tilde{x}}{d\tau} + \tilde{x} - 1 = 0. \tag{4.2.24}$$

It is important to highlight that the domains of $\tilde{x}$ and ϵ and the initial conditions are the same of the ideal system.

4.2.5 Asymptotic solution

Using the asymptotic series (4.1.28) in our differential equation, we can find the perturbed series that allows us to approximate the solution.

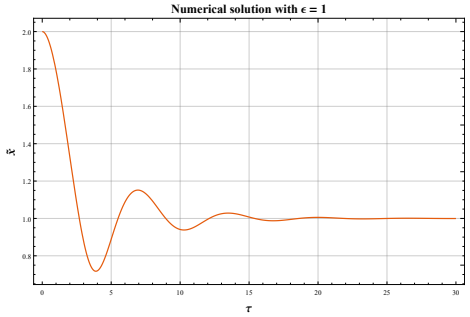
$$\begin{aligned}
&\sum_{i=0}^{N}\left(\epsilon^i\tilde{x}_i\right)\left(\sum_{j=0}^{N}\epsilon^j\frac{d^2\tilde{x}_j}{d\tau^2}\right) + 2\Phi\left(\sum_{k=0}^{N}\epsilon^k\tilde{x}_k\right)\left(\sum_{\ell=0}^{N}\epsilon^\ell\frac{d\tilde{x}_\ell}{d\tau}\right) + \sum_{m=0}^{N}\epsilon^m\tilde{x}_m - 1 = 0 \\
&\implies \sum_{i=0}^{N}\sum_{j=0}^{N}\epsilon^{i+j}\tilde{x}_i\frac{d^2\tilde{x}_j}{d\tau^2} + 2\Phi\sum_{k=0}^{N}\sum_{\ell=0}^{N}\epsilon^{k+\ell}\tilde{x}_k\frac{d\tilde{x}_\ell}{d\tau} + \sum_{m=0}^{N}\epsilon^m\tilde{x}_m - 1 = 0.
\end{aligned} \tag{4.2.25}$$

4.2.6 Numerical solution

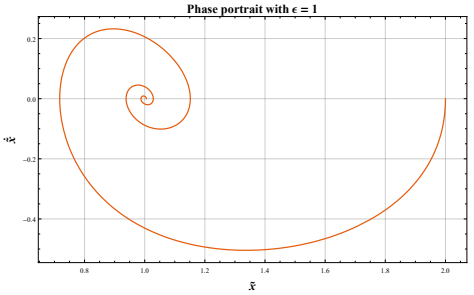
As we have seen in the previous section, we used the Runge-Kutta method to find the numerical solution; the only difference is that in our differential equation, we add the damping term.

As we can see, the damping term caused a decrease in the amplitude of the oscillation. For this numerical simulation, we are going to use $\Phi = \frac{1}{4}$.

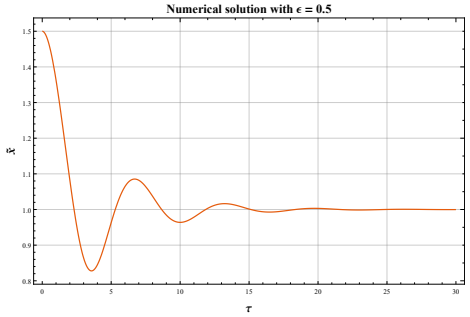
The same as in the ideal case, we will show the evolution of the system when ϵ changes its value. We are going to use the following values of ϵ: 1, 0.5, 0.1, and 0.01. This can be shown in figures 4.11 and 4.12



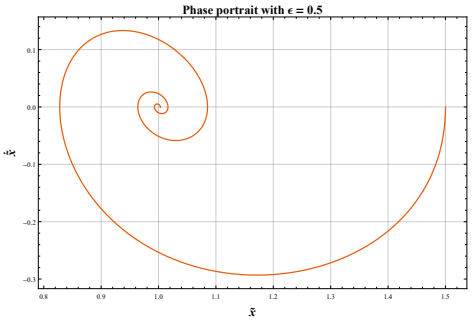
((a)) Time evolution ($\epsilon = 1$)



((b)) Phase portrait ($\epsilon = 1$)

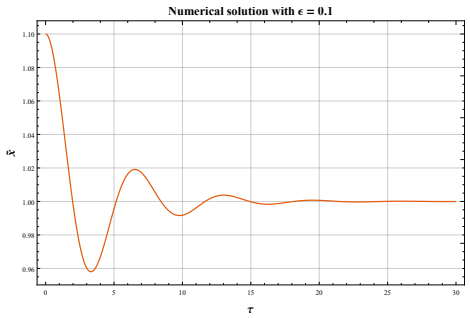


((c)) Time evolution ($\epsilon = 0.5$)

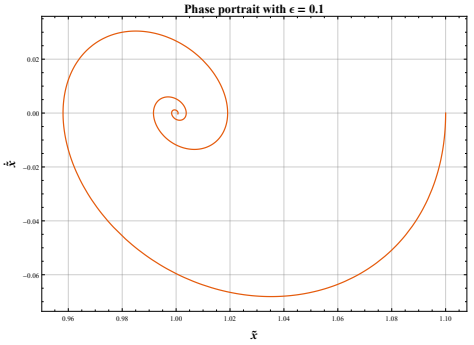


((d)) Phase portrait ($\epsilon = 0.5$)

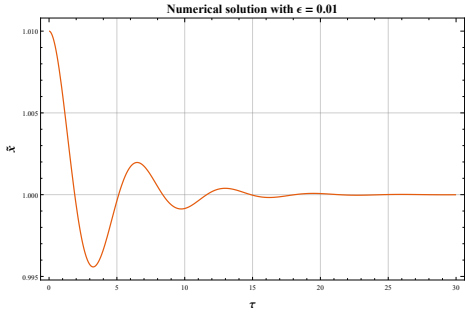
Figure 4.11: System evolution and phase portraits for $\epsilon = 1$ and $\epsilon = 0.5$.



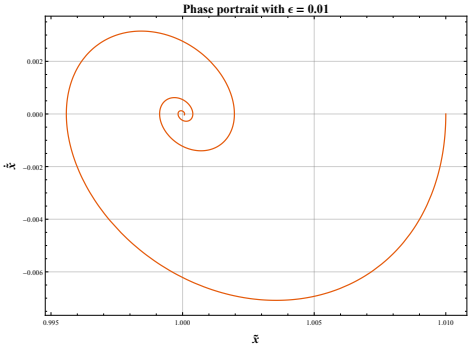
((a)) Time evolution ($\epsilon = 0.1$)



((b)) Phase portrait ($\epsilon = 0.1$)



((c)) Time evolution ($\epsilon = 0.01$)



((d)) Phase portrait ($\epsilon = 0.01$)

Figure 4.12: System evolution and phase portraits for $\epsilon = 0.1$ and $\epsilon = 0.01$.

4.2.7 Approximated solution

Zero order solution

To generate our approximation, we need to find the different order terms of our approximated series. So firstly, we are going to find the constant term, or in other words, the zero-order term. Then, we require using all the terms that have ϵ^0.

$$\begin{aligned} \epsilon^{0+0}\tilde{x}_0\frac{d^2\tilde{x}_0}{d\tau^2} + 2\Phi\epsilon^{0+0}\tilde{x}_0\frac{d\tilde{x}_0}{d\tau} + \epsilon^0\tilde{x}_0 - 1 = 0 \\ \tilde{x}_0\underbrace{\frac{d^2\tilde{x}_0}{d\tau^2}}_{0} + 2\Phi\tilde{x}_0\underbrace{\frac{d\tilde{x}_0}{d\tau}}_{0} + \tilde{x}_0 - 1 = 0 \\ \therefore \tilde{x}_0 = 1. \end{aligned} \tag{4.2.26}$$

Then, our series approximation at zero-order is:

$$\tilde{x}(t) \approx 1 \tag{4.2.27}$$

First order solution

Now, to build our first-order term of the series approximation, we only need the terms that have a multiple of ϵ^1 and solve the respective differential equation.

$$\begin{aligned} \sum_{i=0}^{1}\sum_{j=0}^{1}\epsilon^{i+j}\tilde{x}_i\frac{d^2\tilde{x}_j}{d\tau^2} + 2\Phi\sum_{k=0}^{1}\sum_{\ell=0}^{1}\epsilon^{k+\ell}\tilde{x}_k\frac{d\tilde{x}_\ell}{d\tau} + \sum_{m=0}^{1}\epsilon^m\tilde{x}_m - 1 = 0 \\ \epsilon^{1+0}\tilde{x}_1\frac{d^2\tilde{x}_0}{d\tau^2} + \epsilon^{0+1}\tilde{x}_0\frac{d^2\tilde{x}_1}{d\tau^2} + 2\Phi\epsilon^{1+0}\tilde{x}_1\frac{d\tilde{x}_0}{d\tau} + 2\Phi\epsilon^{0+1}\tilde{x}_0\frac{d\tilde{x}_1}{d\tau} + \epsilon^1\tilde{x}_1 = 0 \\ \tilde{x}_1\underbrace{\frac{d^2\tilde{x}_0}{d\tau^2}}_{0} + \underbrace{\tilde{x}_0}_{1}\frac{d^2\tilde{x}_1}{d\tau^2} + 2\Phi\left(\tilde{x}_1\underbrace{\frac{d\tilde{x}_0}{d\tau}}_{0} + \underbrace{\tilde{x}_0}_{1}\frac{d\tilde{x}_1}{d\tau}\right) + \tilde{x}_1 = 0 \\ \frac{d^2\tilde{x}_1}{d\tau^2} + 2\Phi\frac{d\tilde{x}_1}{d\tau} + \tilde{x}_1 = 0. \end{aligned} \tag{4.2.28}$$

The differential equation that we have obtained is the differential equation of a damped oscillator. To solve this differential equation, we need the characteristic equation.

$$s_{1,2} = \frac{-2\Phi \pm \sqrt{4\Phi^2 - 4}}{2} \implies s_{1,2} = -\Phi \pm \underbrace{\sqrt{\Phi^2 - 1}}_{\Delta}, \tag{4.2.29}$$

from the characteristic solutions, we can see that our determinant Δ defines the type of oscillator that we have. If we change the signs in our determinant, we can build a new determinant, so we obtain.

$$\Delta = \sqrt{\Phi^2 - 1} = \sqrt{-1(1 - \Phi^2)} = i\underbrace{\sqrt{1 - \Phi^2}}_{\eth} \implies \Delta = i\eth, \tag{4.2.30}$$

the new determinant $\eth$, depends of Φ. The oscillator is underdamped when $\Phi < 1$, is critically damped when $\Phi = 1$, and is overdamped when $\Phi > 1$. Then our characteristic solution can be expressed with a complex term as.

$$s_{1,2} = -\Phi \pm \Delta = -\Phi \pm i\eth \tag{4.2.31}$$

If we solve the differential equation, and we apply Euler's complex relation, we find that our solution is.

$$\begin{aligned} \tilde{x}_1(\tau) =& Ae^{s_1\tau} + Be^{s_2\tau} \\ =& e^{-\Phi\tau}\left[Ae^{i\eth\tau} + Be^{-i\eth\tau}\right] \\ \therefore \tilde{x}_1(\tau) =& e^{-\Phi\tau}\left[A\cos(\eth\tau) + B\sin(\eth\tau)\right]. \end{aligned} \tag{4.2.32}$$

Now, applying the initial conditions ($\tilde{x}_1(\tau) = 1$ and $\dot{\tilde{x}}_1(0) = 0$), we can find the values of our constants. Then, with the initial position ($\tilde{x}_1(\tau) = 1$), we have the folowing result.

$$\begin{aligned} \tilde{x}_1(0) =& 1 \\ e^0\left[A\cos(0) + B\sin(0)\right] =& 1 \\ 1\left[A + 0\right] =& 1 \\ A =& 1 \end{aligned} \tag{4.2.33}$$

Finding the speed expression.

$$\begin{aligned} \dot{\tilde{x}}_1(\tau) =& \frac{d}{d\tau}(\tilde{x}_1(\tau)) = \frac{d}{d\tau}\left\{e^{-\Phi\tau}\left[A\cos(\eth\tau) + B\sin(\eth\tau)\right]\right\} \\ =& -\Phi e^{-\Phi\tau}\left[A\cos(\eth\tau) + B\sin(\eth\tau)\right] + \eth e^{-\Phi\tau}\left[-A\sin(\eth\tau) + B\cos(\eth\tau)\right] \\ \therefore \dot{\tilde{x}}_1(\tau) =& e^{-\Phi\tau}\left[(\eth B - \Phi A)\cos(\eth\tau) - (\eth A + \Phi B)\sin(\eth\tau)\right] \end{aligned} \tag{4.2.34}$$

Using the initial speed condition ($\dot{\tilde{x}}_1(0) = 0$) we obtain.

$$\begin{aligned} \dot{\tilde{x}}_1(0) =& 0 \\ 0 =& e^0\left[(\eth B - \Phi A)\cos(0) - (\eth A + \Phi B)\sin(0)\right] \\ 0 =& \eth B - \Phi A \\ \therefore B =& \frac{\Phi}{\eth} = \frac{\Phi}{\sqrt{1-\Phi^2}}, \end{aligned} \tag{4.2.35}$$

something important to see of the B constant is that $\Phi \neq 1$, because if $\Phi = 1$ the solution is undefined. So we can not work with a critically damped oscillator.

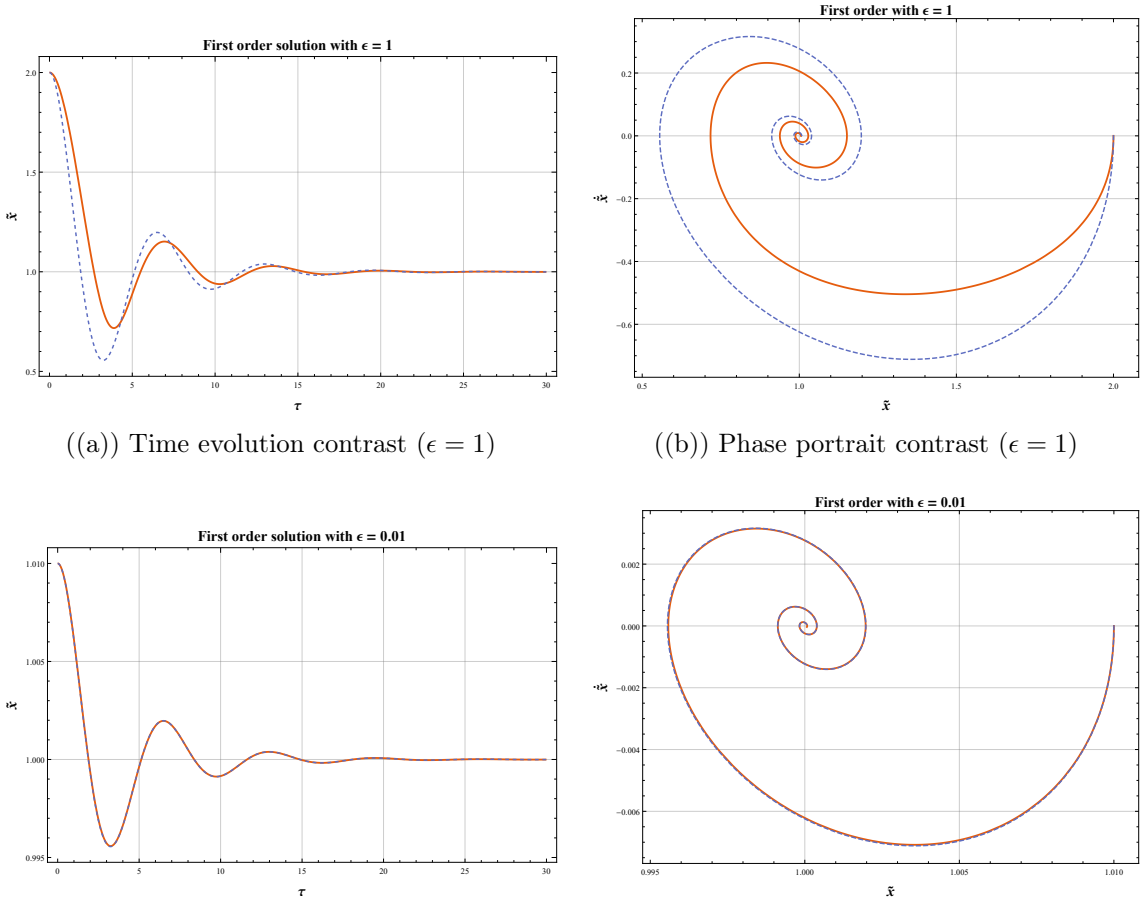
If we substitute the value of the B constant in the approximation of the first order term, we have it as.

$$\tilde{x}_1(\tau) = e^{-\Phi\tau}\left[\cos\left(\eth\tau\right) + \frac{\Phi}{\eth}\sin\left(\eth\tau\right)\right]. \tag{4.2.36}$$

And our approximated series solution at first order is:

$$\tilde{x} \approx \tilde{x}_0 + \epsilon\tilde{x}_1 = 1 + \epsilon e^{-\Phi\tau}\left[\cos\left(\eth\tau\right) + \frac{\Phi}{\eth}\sin\left(\eth\tau\right)\right]. \tag{4.2.37}$$

Now, we plot the numerical and our approximation to contrast the effectiveness of our approximation, we can see this in the following figure.



((a)) Time evolution contrast ($\epsilon = 1$)

((b)) Phase portrait contrast ($\epsilon = 1$)

((c)) Time evolution ($\epsilon = 0.01$)

((d)) Phase portrait ($\epsilon = 0.01$)

Figure 4.13: Approximation of the solution of $\epsilon = 1$ and $\epsilon = 0.01$.

Second-order solution

To find the second-order term of our approximated series, we only need all the terms that contain ϵ^2. So, checking the equation (4.2.25) we find the following terms.

$$\begin{aligned} 0 =& \epsilon^{2+0}\tilde{x}_2\frac{d^2\tilde{x}_0}{d\tau^2} + \epsilon^{1+1}\tilde{x}_1\frac{d^2\tilde{x}_1}{d\tau^2} + \epsilon^{0+2}\tilde{x}_0\frac{d^2\tilde{x}_2}{d\tau^2} \\ &+ 2\Phi\left[\epsilon^{2+0}\tilde{x}_2\frac{d\tilde{x}_0}{d\tau} + \epsilon^{1+1}\tilde{x}_1\frac{d\tilde{x}_1}{d\tau} + \epsilon^{0+2}\tilde{x}_0\frac{d\tilde{x}_2}{d\tau}\right] + \epsilon^2\tilde{x}_2. \end{aligned} \tag{4.2.38}$$

Using the definition of $\tilde{x}_0 = 1$, we can simplify our equation to.

$$\tilde{x}_1 \frac{d^2\tilde{x}_1}{d\tau^2} + \frac{d^2\tilde{x}_2}{d\tau^2} + 2\Phi \left[\tilde{x}_1 \frac{d\tilde{x}_1}{d\tau} + \frac{d\tilde{x}_2}{d\tau}\right] + \tilde{x}_2 = 0 \tag{4.2.39}$$

Now, using the definition of $\tilde{x}_1$ and its derivatives, we can rewrite our differential equation as.

$$\begin{aligned} \tilde{x}_1 \left[-\tilde{x}_1 - 2\Phi \frac{d\tilde{x}_1}{d\tau}\right] + \frac{d^2\tilde{x}_2}{d\tau^2} + 2\Phi \left[\tilde{x}_1 \frac{d\tilde{x}_1}{d\tau} + \frac{d\tilde{x}_2}{d\tau}\right] + \tilde{x}_2 = 0 \\ \implies -\tilde{x}_1^2 + \frac{d^2\tilde{x}_2}{d\tau^2} + 2\Phi \frac{d\tilde{x}_2}{d\tau} + \tilde{x}_2 = 0 \\ \therefore \frac{d^2\tilde{x}_2}{d\tau^2} + 2\Phi \frac{d\tilde{x}_2}{d\tau} + \tilde{x}_2 = \tilde{x}_1^2 \end{aligned} \tag{4.2.40}$$

Finding the characteristic equation of the homogeneous part.

$$\begin{aligned} 0 =& \frac{d^2\tilde{x}_2}{d\tau^2} + 2\Phi \frac{d\tilde{x}_2}{d\tau} + \tilde{x}_2 \implies s_{1,2} = -\Phi \pm \sqrt{\Phi^2 - 1} = -\Phi \pm \Delta \\ \therefore s_{1,2} =& -\Phi \pm \sqrt{(-1)(1-\Phi^2)} = -\Phi \pm i\eth \end{aligned} \tag{4.2.41}$$

Then, the solution of the homogeneous part is.

$$\begin{aligned} \tilde{x}_{2h} =& Ae^{s_1\tau} + Be^{s_2\tau} = Ae^{-\Phi\tau + i\eth\tau} + Be^{-\Phi\tau - i\eth\tau} = e^{-\Phi\tau}\left[Ae^{i\eth\tau} + Be^{-i\eth\tau}\right] \\ \therefore \tilde{x}_{2h} =& e^{-\Phi\tau}\left[A\cos(\eth\tau) + B\sin(\eth\tau)\right] \end{aligned} \tag{4.2.42}$$

Now, solving the particular solution of our system. Applying the method of undetermined coefficients, we solve for the steady-state particular response $\tilde{x}_{2p}(\tau)$. Utilizing the algebraic identity of the subamortized regime to simplify the denominators, we obtain a remarkably compact and physically meaningful particular solution:

$$\tilde{x}_{2p}(\tau) = e^{-2\Phi\tau}\left[\mathcal{C}_{\text{dc}} + \mathcal{C}_{\sin}\sin(2\eth\tau) + \mathcal{C}_{\cos}\cos(2\eth\tau)\right] \tag{4.2.43}$$

where the constant coefficients $\mathcal{C}_{\text{dc}}$, $\mathcal{C}_{\sin}$, and $\mathcal{C}_{\cos}$ are explicitly determined by the dimensionless friction number Φ and the frequency parameter $\eth$ as:

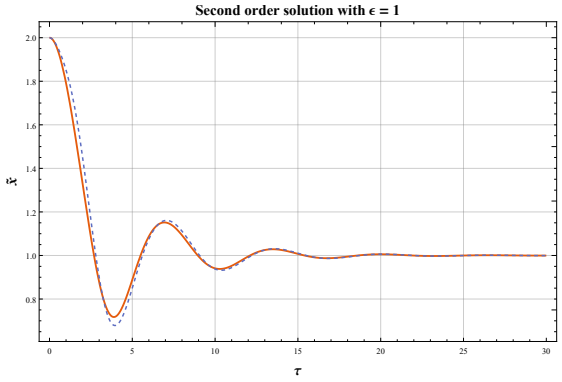
$$\mathcal{C}_{\text{dc}} = \frac{1}{2\eth^2} = \frac{1}{2(1-\Phi^2)} \tag{4.2.44}$$

$$\mathcal{C}_{\sin} = \frac{\Phi(8\Phi^2 - 5)}{\eth(9 - 8\Phi^2)} \tag{4.2.45}$$

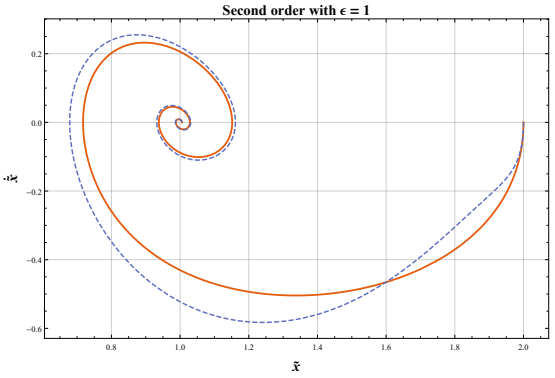
$$\mathcal{C}_{\cos} = \frac{-16\Phi^4 + 18\Phi^2 - 3}{2\eth^2(9 - 8\Phi^2)} \tag{4.2.46}$$

Here, $\mathcal{C}_{\text{dc}}$ dictates the non-oscillatory exponential decay of the transient mean position, while the harmonic terms $\mathcal{C}_{\sin}$ and $\mathcal{C}_{\cos}$ capture the energy transfer into the second harmonic ($2\eth$) generated by the quadratic non-linearity of the magnetic restoring force.

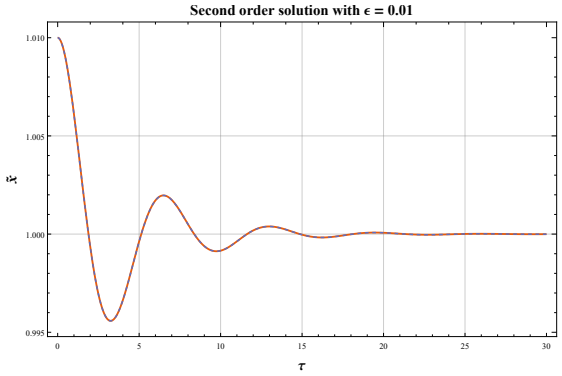
Plotting our solution and contrasting with the numerical solution we obtain the following figures.



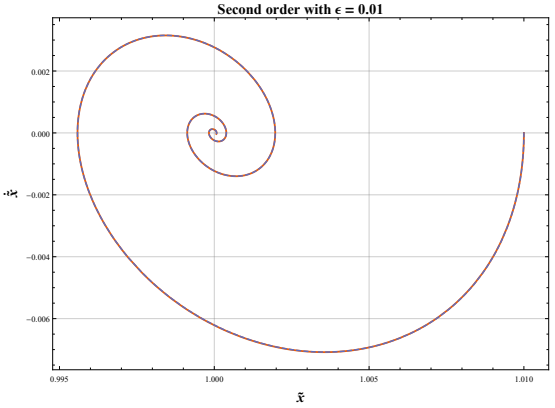
((a)) Time evolution contrast ($\epsilon = 1$)



((b)) Phase portrait contrast ($\epsilon = 1$)



((c)) Time evolution ($\epsilon = 0.01$)



((d)) Phase portrait ($\epsilon = 0.01$)

Figure 4.14: Approximation of the solution of $\epsilon = 1$ and $\epsilon = 0.01$.

4.3 Forcing system

In this case, we consider the presence of all kinds of forces (net force, dissipative force, and an external force, which is periodic). Then, our system of forces has the following aspect:

$$F = -F_g + F_m - F_f + F_{ex} = -mg + \frac{M}{x} - \beta v + F_0 \cos(\omega t), \tag{4.3.1}$$

where F is the net force, m is the mass of the free magnet, g is the gravitational acceleration, x the distance between the magnets, β is the mass flow rate coefficient, v the velocity, M is the constant of the magnetic energy, F_0 is the magnitude of the external force, and ω is the frequency of the external force.

If we consider the static case, where the net force is null ($F = 0$), we can find the equilibrium distance.

$$\frac{M}{x} - mg - \beta v + F_0 \cos(\omega t) = 0 \implies \frac{M}{x} = mg + \beta v - F_0 \cos(\omega t) \implies \boxed{x_{eq} = \frac{M}{mg + \beta v - F_0 \cos(\omega t)}}, \tag{4.3.2}$$

for this result, we can see that the equilibrium distance changes in the function of the velocity and time. In other terms, we have a dynamic equilibrium point. But at some time the oscillation will be stable, causing our system to obtain an equilibrium point like the ideal oscillator (equation (4.1.2)). This equilibrium point only appears when the system passes the region of instability.

This system has the same domain as the ideal and damping cases.

4.3.1 Lagrangian analysis

To find the Lagrangian of this case, we are going to use the definition of a general Lagrangian, which is expressed in equation (1.4.4). To work with it, we need to find the conservative forces (forces that come to a potential) and the external forces.

$$\underbrace{F + mg - \frac{M}{x}}_{\text{conservative}} + \underbrace{\beta v - F_0 \cos(\omega t)}_{\text{unconservative}} = 0 \implies Q = F_0 \cos(\omega t) - \beta v. \tag{4.3.3}$$

With this, using the potential that we found in the ideal case (equation (4.1.5)) lets us use the Lagrangian of the system (equation (4.1.6)).

Now, to find the equation of motion of our forced system, we are going to use the Euler-Lagrange equation (equation (1.4.4)):

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = Q \implies \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} + \beta \frac{dx}{dt} = F_0 \cos(\omega t) \\ \implies m\ddot{x} + mg - \frac{M}{x} + \beta \frac{dx}{dt} = F_0 \cos(\omega t) \implies \ddot{x} + g - \frac{M}{mx} + \frac{\beta}{m}\frac{dx}{dt} = \frac{F_0}{m}\cos(\omega t), \end{aligned} \tag{4.3.4}$$

if we define the β coefficient as $\beta = 2m\phi$, where ϕ is the friction coefficient. Then, the equation of motion has the following form.

$$\ddot{x} + g - \frac{M}{mx} + 2\phi\frac{dx}{dt} = \frac{F_0}{m}\cos(\omega t) \implies \boxed{\frac{d^2x}{dt^2} + 2\phi\frac{dx}{dt} + g - \frac{M}{mx} = \frac{F_0}{m}\cos(\omega t)} \tag{4.3.5}$$

4.3.2 Dimensional analysis

We are going to use the same dimensionless groups that we used in the ideal and damping cases, but we only need to find the dimensionless groups of the external force. The external force has two new elements, the magnitude of the force and the frequency.

As in the previous cases, we use the variables m, g, and M to dimensionless our system. So firstly, we need to do the dimensional analysis for these variables and the variables of the external are:

$$[F] = MLT^{-2}, [\omega] = T^{-1}, [m] = M, [g] = LT^{-2}, [M] = ML^2T^{-2} \tag{4.3.6}$$

With this, we can find the dimensionless groups of the external force variables as the product of the powers of our variables that we use to dimensionless. Then, we have:

$$\Pi_i = \delta m^{\alpha} g^{\beta} M^{\lambda}, \tag{4.3.7}$$

where δ is the variable that we want to make dimensionless. Now, using the equation (4.1.13) we can find the powers of our variables to build our dimensionless groups. This equation is:

$$\begin{pmatrix} \alpha \\ \beta \\ \lambda \end{pmatrix} = \begin{pmatrix} L + \frac{T}{2} - M \\ L + T \\ -L - \frac{T}{2}. \end{pmatrix} \tag{4.3.8}$$

Now, we can proceed to analyze the new variables (F and ω) of the system. We are going to start with the force.

Force group

As we see previously, F use the three dimension ($L = 1$, $T = -2$, $M = 1$). Then, using the result of the equation (4.3.8), we obtain that our dimensionless group and its scale factor are.

$$\begin{pmatrix} \alpha \\ \beta \\ \lambda \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix} \quad \therefore \quad \boxed{\Pi_5 = \frac{F_0}{F_s} = \frac{F_0}{mg} \implies F_s = mg} \quad . \tag{4.3.9}$$

This result is logical, because the principal force that initiates the movement is the gravitational force.

To use the dimensionless group of the Force in our equation, we need to define a symbol that lets us work easily, and this definition is $\Pi_5 = \tilde{F}$

$$\Pi_5 = \frac{F_0}{F_s} \implies F_0 = F_s \tilde{F} = mg\tilde{F} \tag{4.3.10}$$

Frequency group

Now, we are going to work with the frequency (ω) that only works with one dimension ($L = 0$, $T = -1$, $M = 0$). Then, using our equation to find the powers (equation (4.3.8)), we obtain that our dimensionless group and its scale factor are.

$$\begin{pmatrix} \alpha \\ \beta \\ \lambda \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ -1 \\ \frac{1}{2} \end{pmatrix} \quad \therefore \quad \boxed{\Pi_6 = \frac{\omega}{\omega_s} = \frac{\omega}{g}\sqrt{\frac{M}{m}} \implies \omega_s = \frac{1}{t_s} = g\sqrt{\frac{m}{M}} = \sqrt{\frac{g}{x_{eq}}}} \quad . \tag{4.3.11}$$

This result is so important because it is the frequency of the pendulum. So, in other words, our oscillator is analogous to a pendulum.

Now, we define the dimensionless frequency. To do this, we use the dimensionless group of the frequency ($\Pi_6 = \Omega$). Now, we can express the frequency in terms of the dimensionless frequency, as we can see in the following expression.

$$\Pi_6 = \frac{\omega}{\omega_s} \implies \omega = \omega_s \Omega = \frac{1}{t_s}\Omega \tag{4.3.12}$$

4.3.3 Dimensionless system

Using all the dimensional groups that we found previously, we are going to dimensionless our equation of motion (4.3.5). Then, we obtain:

$$\begin{aligned}
&\frac{d^2x}{dt^2} + 2\phi\frac{dx}{dt} + g - \frac{M}{mx} = \frac{F_0}{m}\cos(\omega t) \\
\implies &\frac{d^2(x_s\tilde{x})}{d(t_s\tau)^2} + 2(\phi_s\Phi)\frac{d(x_s\tilde{x})}{d(t_s\tau)} + g - \frac{M}{mx_s\tilde{x}} = \frac{(F_s\tilde{F})}{m}\cos((\omega_s\Omega)(t_s\tau)) \\
\implies &\frac{x_s}{t_s^2}\frac{d^2\tilde{x}}{d\tau^2} + \frac{2\Phi x_s}{t_s^2}\frac{d\tilde{x}}{d\tau} + g - \frac{M}{mx_s\tilde{x}} = \frac{F_s\tilde{F}}{m}\cos(\Omega\tau) \\
\implies &\frac{d^2\tilde{x}}{d\tau^2} + 2\Phi\frac{d\tilde{x}}{d\tau} + \frac{gt_s^2}{x_s} - \frac{Mt_s^2}{mx_s^2\tilde{x}} = \frac{t^2F_s\tilde{F}}{x_sm}\cos(\Omega\tau) \\
\implies &\frac{d^2\tilde{x}}{d\tau^2} + 2\Phi\frac{d\tilde{x}}{d\tau} + 1 - \frac{M}{mgx_{eq}\tilde{x}} = \tilde{F}\cos(\Omega\tau) \\
\implies &\frac{d^2\tilde{x}}{d\tau^2} + 2\Phi\frac{d\tilde{x}}{d\tau} + 1 - \frac{1}{\tilde{x}} = \tilde{F}\cos(\Omega\tau),
\end{aligned} \tag{4.3.13}$$

Multiply all terms of our differential equation by $\tilde{x}$.

$$\frac{d^2\tilde{x}}{d\tau^2} + 2\Phi\frac{d\tilde{x}}{d\tau} + 1 - \frac{1}{\tilde{x}} = \tilde{F}\cos(\Omega\tau) \implies \tilde{x}\frac{d^2\tilde{x}}{d\tau^2} + 2\Phi\tilde{x}\frac{d\tilde{x}}{d\tau} + \tilde{x} - 1 = \tilde{x}\tilde{F}\cos(\Omega\tau). \tag{4.3.14}$$

It is important to highlight that our differential equation does not have an analytical solution, so we only explore the behavior of the system through a numerical analysis.

4.3.4 Asymptotic solution

Before using the asymptotic procedure, we are going to simplify our differential equation. This could be expressed as it can show in the following expression.

$$\tilde{x}\frac{d^2\tilde{x}}{d\tau^2} + 2\Phi\tilde{x}\frac{d\tilde{x}}{d\tau} - \tilde{x}\tilde{F}\cos(\Omega\tau) + \tilde{x} - 1 = 0 \implies \tilde{x}\left(\frac{d^2\tilde{x}}{d\tau^2} + 2\Phi\frac{d\tilde{x}}{d\tau} - \tilde{F}\cos(\Omega\tau) + 1\right) - 1 = 0 \tag{4.3.15}$$

Now, using the asymptotic series (4.1.28), we can find the perturbed series that allows us to approximate the solution.

$$\left(\sum_{i=0}^{N}\epsilon^i\tilde{x}_i\right)\left(\sum_{j=0}^{N}\epsilon^j\frac{d^2\tilde{x}_j}{d\tau^2} + 2\Phi\sum_{k=0}^{N}\epsilon^k\frac{d\tilde{x}_k}{d\tau} - \sum_{\ell=0}^{N}\epsilon^\ell\tilde{x}_\ell\tilde{F}\cos(\Omega\tau) + 1\right) - 1 = 0. \tag{4.3.16}$$

4.3.5 Numerical solution

In the impossibility of finding an analytical solution, we use a numerical method to understand the behavior that our system will follow.

Dynamical system stability

To analyze the behavior of our system, we can use our techniques for differential equations. We can not use the phase portrait technique because our dynamical system depends directly on time. Then we are going to use the Poincaré map to find the stability of our system. It is important to highlight that the Poincaré map is very useful in the stroboscopy space, where $\tau = T = \frac{2\pi}{\Omega}$

Firstly, we need to reduce our second-order differential equation to a set of first-order differential equations. The forcing phase relation is $\tilde{\theta} = \Omega\tau$. These equations are:

$$\tilde{v} = \frac{d\tilde{x}}{d\tau} \tag{4.3.17}$$

$$\dot{\tilde{v}} = \tilde{F}\cos(\tilde{\theta}) - 2\Phi\tilde{v} - 1 + \frac{1}{\tilde{x}} \tag{4.3.18}$$

$$\Omega = \frac{d\tilde{\theta}}{d\tau} \tag{4.3.19}$$

These equations can be expressed as a vectorial system as:

$$\tilde{r} = \frac{d}{d\tau}\begin{pmatrix}\tilde{x}\\ \tilde{v}\\ \tilde{\theta}\end{pmatrix} = \begin{pmatrix}\tilde{v}\\ \tilde{F}\cos(\tilde{\theta}) - 2\Phi\tilde{v} - 1 + \frac{1}{\tilde{x}}\\ \Omega\end{pmatrix}, \tag{4.3.20}$$

where $\tilde{r}$ is the vector that models our system. If we want to know the stability of our system, we need the non-autonomous system, which implies that we do not need to attend to the time. Then our vector $\tilde{r}$ is really formed only by the first and second terms. To find the stability we need the Jacobian matrix of our non-autonomous system. Which is:

$$\mathbf{J}(\tilde{x},\tilde{v}) = \begin{pmatrix}\frac{\partial\tilde{r}_1}{\partial\tilde{x}} & \frac{\partial\tilde{r}_1}{\partial\tilde{v}}\\ \frac{\partial\tilde{r}_2}{\partial\tilde{x}} & \frac{\partial\tilde{r}_2}{\partial\tilde{v}}\end{pmatrix} \implies \mathbf{J}(\tilde{x},\tilde{v}) = \begin{pmatrix}0 & 1\\ -\frac{1}{\tilde{x}^2(\tau)} & -2\Phi\end{pmatrix} = \mathbf{J}(\tau). \tag{4.3.21}$$

With the Jacobian of the system we can find the Monodromy matrix (that is, the transformation of our Jacobian matrix to the Poincaré discrete time space), and this matrix follows the next:

$$\dot{\mathbf{M}}(\tau) = \mathbf{J}(\tau)\mathbf{M}(\tau), \quad \mathbf{M}(0) = \mathbf{I}. \tag{4.3.22}$$

This system is so complicated to solve analytically. Then we are going to use Liouville's formula:

$$\det(\mathbf{M}(\tau)) = \det\mathbf{M}(t_0)e^{\int_{t_0}^{\tau}\operatorname{tr}(\mathbf{J}(\tau))d\tau}. \tag{4.3.23}$$

Now, integrating from 0 to T, to find the. Then we obtain:

$$\begin{aligned} \det(\mathbf{M}(T)) &= \det(\mathbf{M}(0)) e^{\int_0^T \mathrm{tr}(\mathbf{J}(\tau)) d\tau} \\ \det(\mathbf{M}(T)) &= \det(\mathbf{I}) e^{\int_0^T -2\Phi d\tau} \\ \therefore \boxed{\det(\mathbf{M}(T)) = e^{-2\Phi T}} &. \end{aligned} \tag{4.3.24}$$

With this result, we can find the Floquet multipliers, which are the eigenvalues of the Monodromy matrix ($\mathbf{M}(T)$) and they are fundamental to determine the stability of the system. Using the definition of Floquet multipliers, we obtain.

$$\rho 1 \cdot \rho 2 = \det(\mathbf{M}(T)) = e^{-2\Phi T}, \tag{4.3.25}$$

where ρ_1 and ρ_2 are the Floquet multipliers. The Floque multipliers let us know if the trajectory that the system follows is stable or unstable. To know this, we have the following table:

Condition	**How many?**	**System behavior**
$\rho < 1$	All	Stable limit cycle (perturbations decay)
$\rho = 1$	At least one (others < 1)	Marginal stability (bifurcation point)
$\rho > 1$	At least one	Unstable limit cycle (perturbations grow)

Table 4.1: Different behaviors of the system in terms of the Floquet multipliers

If $\Phi > 0$, that is the case of a damped system, the product of the Floquet multipliers is always equal or less than one. That implies that the system has a limit cycle. To show that a limit cycle exists, we now make a Poincaré map.

To make our Poincaré map, we are going to use the Runge-Kutta method in our dynamical system, and we are going to save the values of $(\tilde{x}, \tilde{v})$ when we pass for a multiple of T. Doing this, we obtain the following Poincaré map

Lyapunov exponents

While Floquet multipliers characterize the stability of a periodic orbit, many systems, in particular the forced nonlinear oscillators, can exhibit chaotic behavior where no periodic orbit exists. In such cases, a more general tool is needed: the ***Lyapunov exponents***. The behavior of the system in terms of the Lyapunov exponents follows the next table:

Condition	**How many?**	**System behavior**
$\lambda < 0$	All	Stable dynamics
$\lambda = 0$	At least one (others < 0)	Marginal stability
$\lambda > 0$	At least one	Chaotic dynamics

Table 4.2: Different behaviors of the system in terms of the Lyapunov exponents

When a system is stable, there exists a relation between the Floquet multipliers and the Lyapunov exponents. That relation is:

$$\lambda_i = \frac{1}{T} \ln(\rho_i) \tag{4.3.26}$$

Now, using the Lyapunov exponents equation:

$$\lambda_1 + \lambda_2 = \frac{1}{T}\ln(\rho_1) + \frac{1}{T}\ln(\rho_2) = \frac{1}{T}\ln(\rho_1 \cdot \rho_2) \tag{4.3.27}$$

Using the value of our Floquet product (5.3.32)

$$\lambda_1 + \lambda_2 = \frac{1}{T}\ln\left(e^{-2\Phi T}\right) \implies \lambda_1 + \lambda_2 = -2\Phi \tag{4.3.28}$$

Using the condition of stability:

$$\lambda_1 + \lambda_2 = -2\Phi < 0 \implies \Phi > 0, \tag{4.3.29}$$

this result implies that if the friction number (Φ) increases, the stability increases too. But the amplitude of the oscillation decreases. Another important thing that we can find is the values of Lyapunov exponents that give us stable system

$$\lambda_1 + \lambda_2 = -2\Phi \implies \lambda_1 = -(2\Phi + \lambda_2), \tag{4.3.30}$$

we know that a stable system needs $\lambda_i < 0$, then:

$$\lambda_1 < 0 \implies -(2\Phi + \lambda_2) < 0 \implies 2\Phi + \lambda_2 > 0 \implies \lambda_2 > -2\Phi, \tag{4.3.31}$$

but both Lyapunov exponents need to be less than one; therefore:

$$-2\Phi < \lambda_2 < 0. \tag{4.3.32}$$

Now, we can find the limit for the first Lyapunov exponent domain for a stable system:

$$\begin{aligned} &\lambda_1 = -(2\Phi + 0) = -2\Phi \\ &\lambda_1 = -(2\Phi - 2\Phi) = 0 \\ &\therefore -2\Phi < \lambda_1 < 0. \end{aligned} \tag{4.3.33}$$

We can see that both Lyapunov exponents have the same domain if the system is stable, but it is important to highlight that they can have different values; the only thing that they need to follow is that the sum of both is -2Φ as is presented in the stability condition equation 5.3.36

The previous result indicates that chaos in the system is completely related to the force ($\tilde{F}$) and frequency ($\tilde{\Omega}$) parameters.

Now that we have our domain for the Lyapunov exponents, we can start to plot them and see under what kind of parameters the system is stable. It is important to remember that we use a numerical method to find the Lyapunov exponents, and they change as time increases.

For a particular value of $\tilde{F}$, Φ and Ω, we can find the value of time where the system begins to be stable or not. An example of this is the figure 4.15

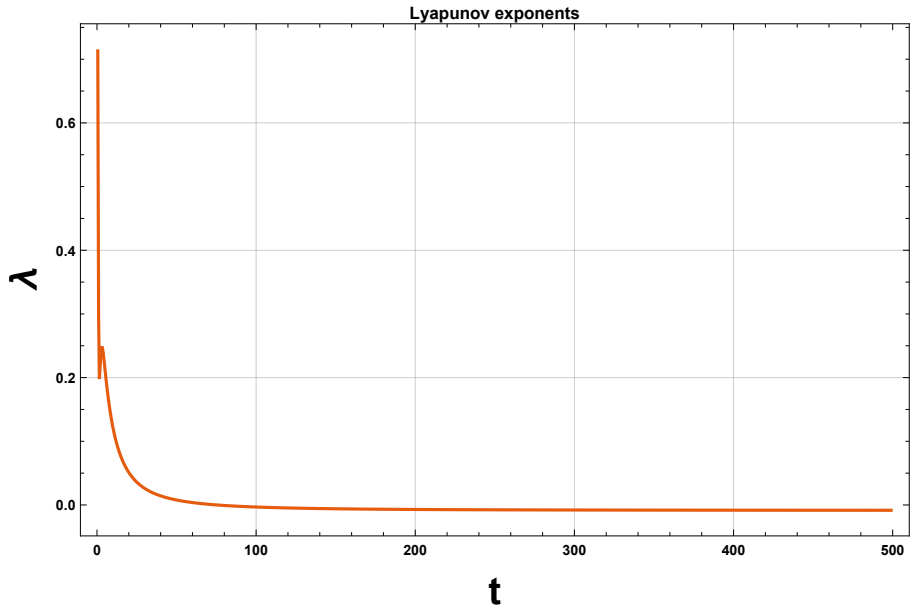


Figure 4.15: Change of the Lyapunov exponents in terms of time for: $\tilde{F} = 1$, $\Phi = 0.1$, and $\Omega = 1$

In the previous figure we can see how the system starts acting like a chaotic system, but with the pass of the time, the system begins to be stable. This tells us that our system with those specific parameters is close to the parameters that generate chaos in the system. So we need to see what kind of values of the force number can generate chaos.

So we decide to run our numerical method to see how Lyapunov exponents change when the value of the force increases. This can be shown in a plot as follows:

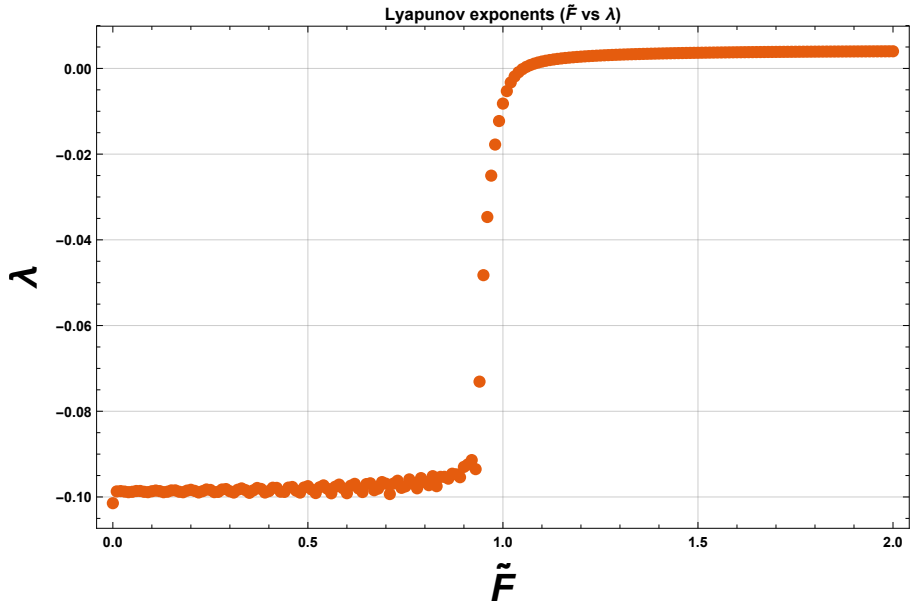


Figure 4.16: Value of Lyapunov exponents when $\tilde{F}$ change it value: $\Phi = 0.1$, and $\Omega = 1$

As we can see in the previous figure, for our system with $\Phi = 0.1$ and $\Omega = 1$, when the

force crosses the horizontal axis, the system begins to be chaotic. To see this in a better way, we do a bifurcation plot of our dynamical system. The bifurcation plot with these parameters is:

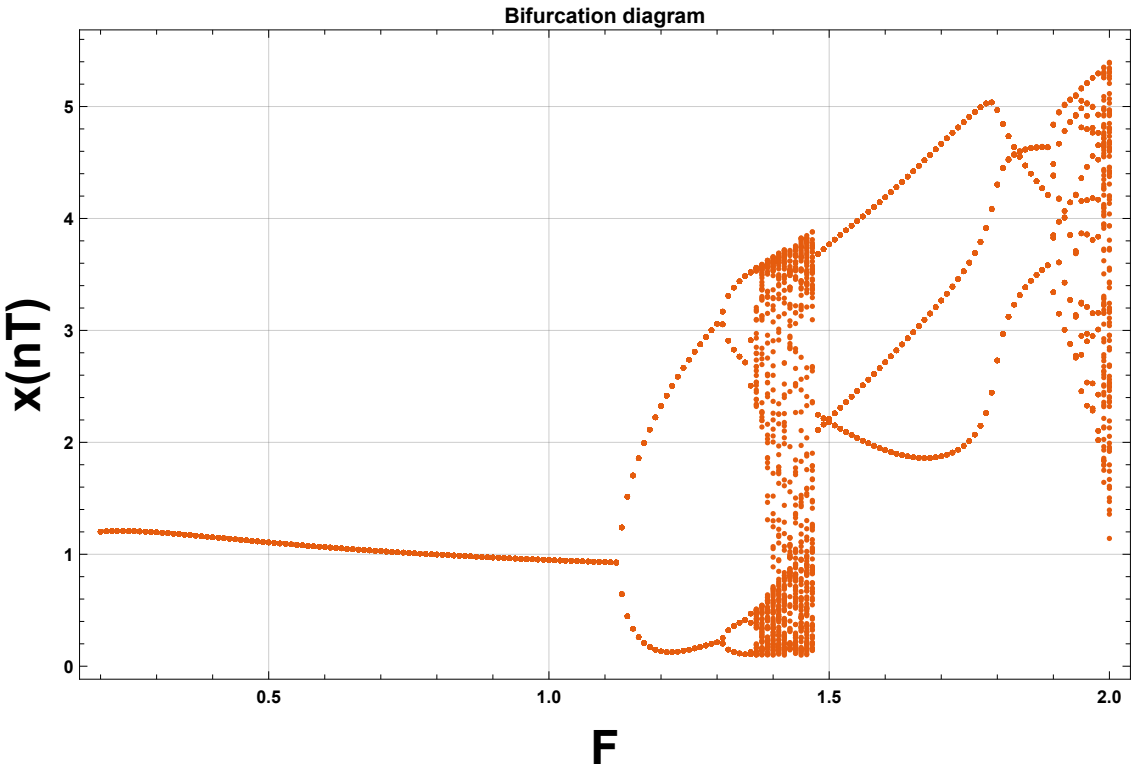
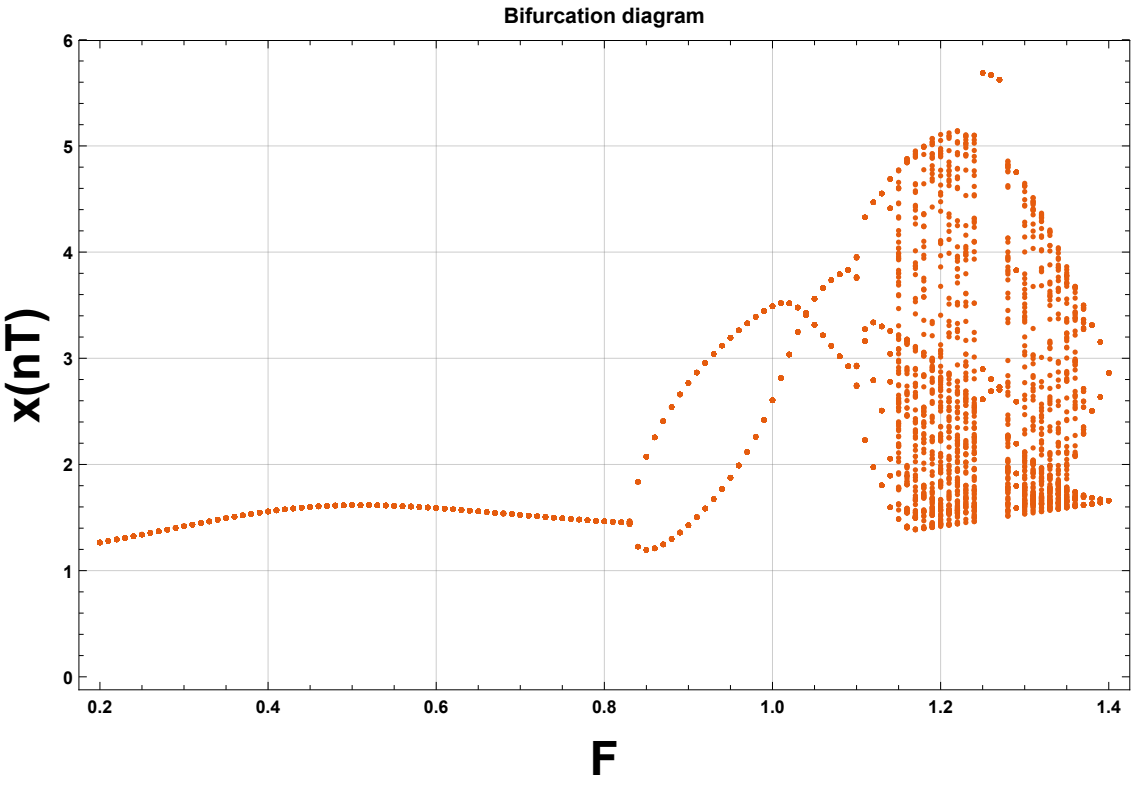


Figure 4.17: Bifurcation plot of the system when $\tilde{F}$ change it value: $\Phi = 0.1$, and $\Omega = 1$

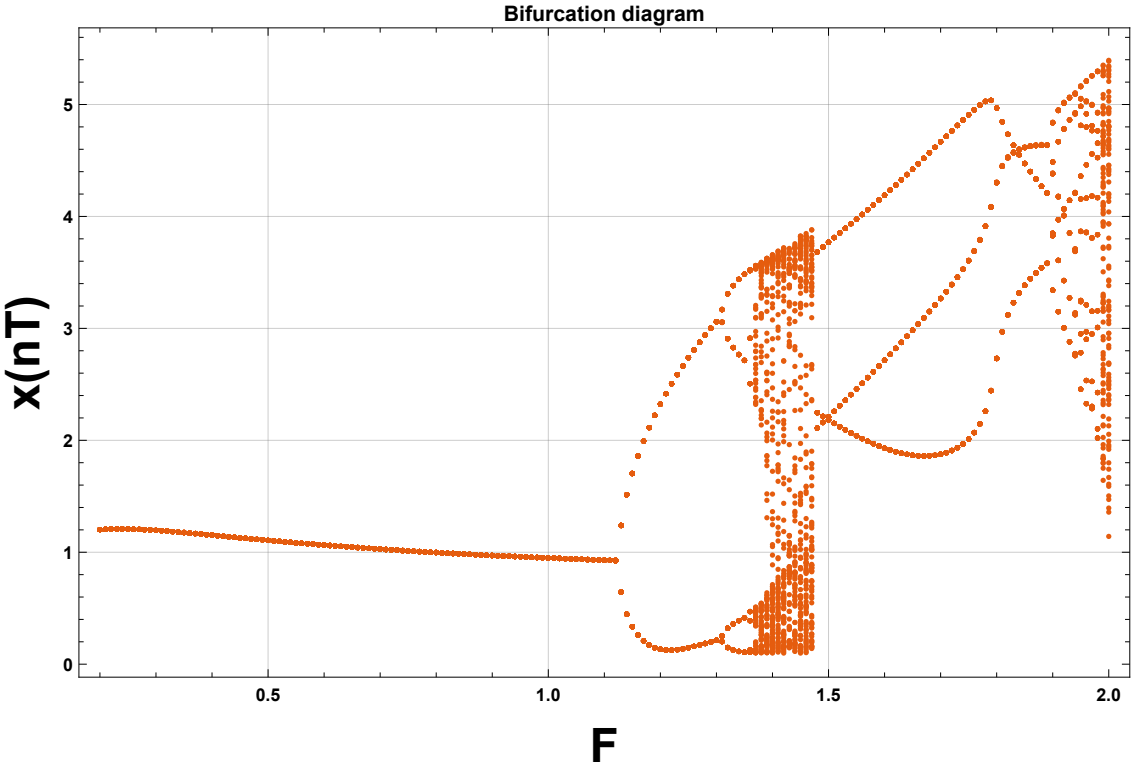
As it was shown in the figure 4.17 when the value of the force pass the axis ($\tilde{F} \approx 1.1$), the system begins chaotic, generating a bifurcation in our system.

Now, we can check how the system responds to different values in the friction number (Φ) and the frequency number (Ω). For bigger values of the friction number, we can speculate that the force number needs to increase to generate chaos, but with the frequency number, things change because it affects the magnitude of the force number.

Generating some bifurcation diagrams to see with which force number ($\tilde{F}$) the system Ω begins to be chaotic. This can be shown in the figures 4.18 and 4.19.

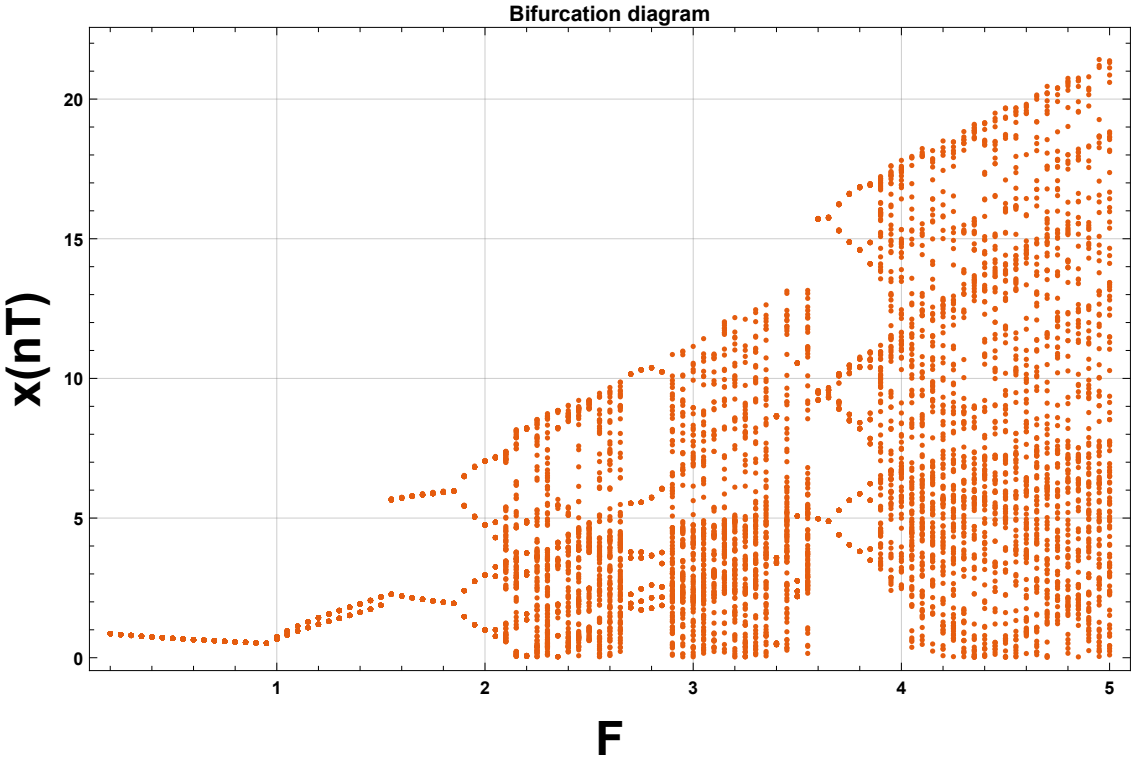


((a)) Bifurcation diagram of the position to different values of $\tilde{F}$ with $\Phi = 0.1$ and $\Omega = 0.5$

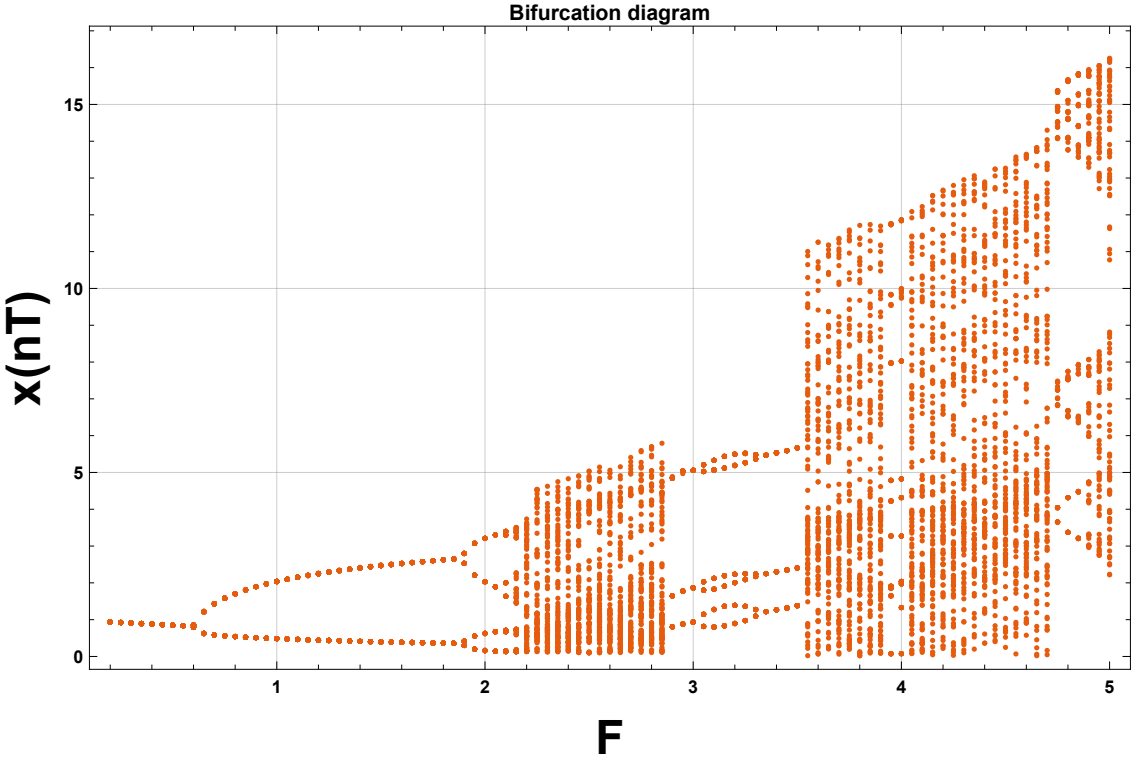


((b)) Bifurcation diagram of the position to different values of $\tilde{F}$ with $\Phi = 0.1$ and $\Omega = 1$

Figure 4.18: Numerical demonstration that the angle (Ω) plays a relevant role in the chaos of the system, and it defines the force value (F) that generates chaos in the system



((a)) Bifurcation diagram of the position to different values of $\tilde{F}$ with $\Phi = 0.1$ and $\Omega = 1.5$



((b)) Bifurcation diagram of the position to different values of $\tilde{F}$ with $\Phi = 0.1$ and $\Omega = 2$

Figure 4.19: Numerical demonstration that the angle (Ω) plays a relevant role in the chaos of the system, and it defines the force value (F) that generates chaos in the system

As we can see in the previous figures, the frequency number has an important role in the chaos, becuase when the value of the frequency increases, the needed force to generate chaos decreases. For this reason, it is important to know the frequency of our system.

Numerical analysis

For the nature of our differential equation, a nonhomogeneous second-order differential equation, we decided to use the Runge-Kutta method. Using the method, we can simulate the system. To know what values of the force are able to generate an stable oscillation, we firstly do the Lyapunov exponents plot. For this system, we are going to use the following values: $\Phi = 0.5$, and $\Omega = 0.9$.

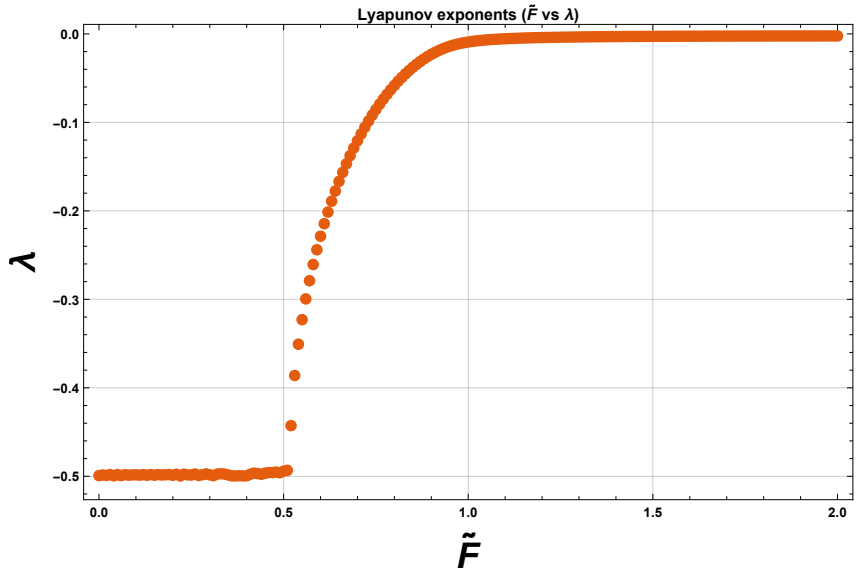


Figure 4.20: Value of Lyapunov exponents when $\tilde{F}$ change it value: $\Phi = 0.1$, and $\Omega = 1$

Now that we know the values of the force ($\tilde{F}$) that generates an stable oscillation, we can simulate our system using the following values: $\epsilon = 0.5$, $\tilde{F} = 1$, $\Phi = 0.5$, and $\Omega = 0.9$,they give us the following numerical solution.

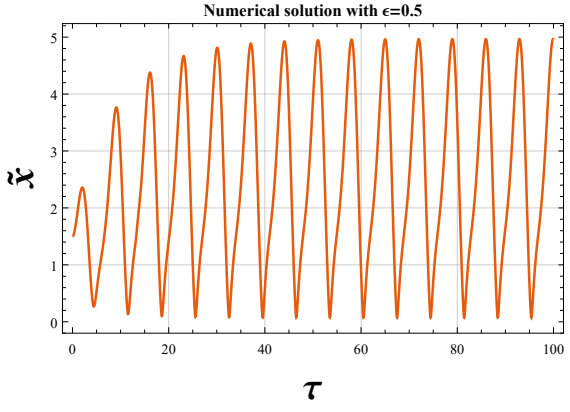


Figure 4.21: Numerical solution of our system with an external force, where the dimensionless parameters have the following values: $\tilde{F} = 1$, $\Phi = 0.5$, $\epsilon = 0.5$, and $\Omega = 0.9$

It is important to see that three dimensionless variables control the behavior of our system. These variables are the friction number (Φ), the magnitude of force ($\tilde{F}$), and the frequency (Ω).

The behavior of the system could be one of the following states, with a specific configuration of values of our system. These configurations can be shown in the table 4.3.

Condition	**System behavior**
$\Phi = 0$, $F = 0$	Conservative (harmonic) motion
$\Phi > 0$, $F = 0$	Damped oscillation to equilibrium
$\Phi > 0$, $F \neq 0$	Forced nonlinear response
$\Omega \approx \omega_0$	Resonance (maximum amplitude)

Table 4.3: Different behaviors of the system with different values of our dimensionless parameters

From the previous table (4.3) we can see that there are two variables that direct affect the amplitude of our system, they are the friction number (Φ) and the frequency (Ω). The principal reason that the dimensionless force is not consider like a direct parameter in the amplitude is because if the frequency of the external force is similar to the natural frequency, it can increase the action of the force until it will be a little.

To find the final amplitude of our forced system we decided to check the medium difference between the maximum amplitude value with the minimum amplitude value in the stable region. We can be obtain using the following equation.

$$A = \frac{max(x_{st}) - min(x_{st})}{2}, \tag{4.3.34}$$

where x_{st} is the position of our magnet in the stable region. Once we have the amplitude, we can contrast how the amplitude changes when we change the value of some dimensionless parameters.

Firstly, we are going to check how the amplitude changes when the frequency changes. To do that, we are going to consider a unitary force $\tilde{F} = 1$ and different values of the friction number (Φ).

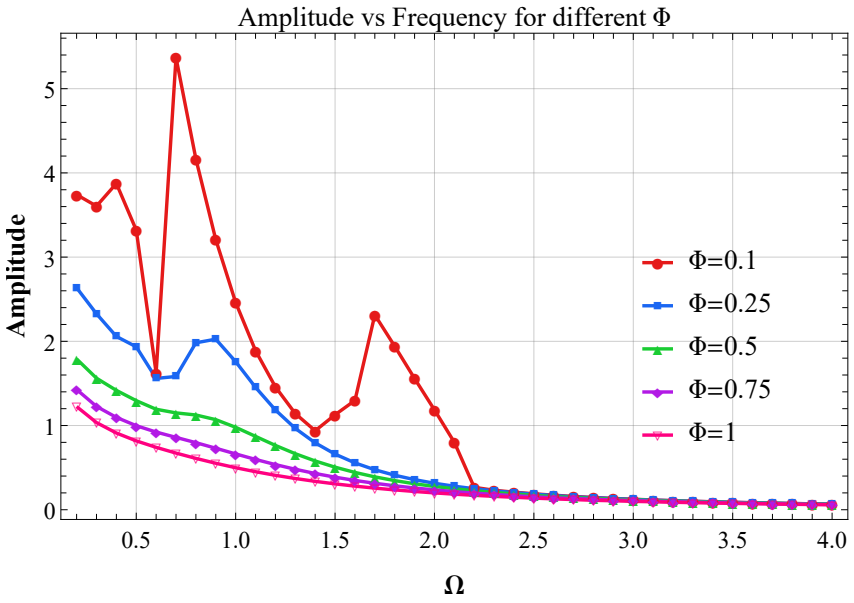


Figure 4.22: Change of the amplitude in terms of Φ, with different values of the frequency (Ω).

In the figure (4.22) we can see that the amplitude decreases when the frequency increases, and when we are close to 1, that is the natural frequency of the system, the amplitude increases a little, this is for the resonance.

Now, we can check how the amplitude changes as the friction number changes. As the same force is unitary, we try with a set of numbers of different frequencies.

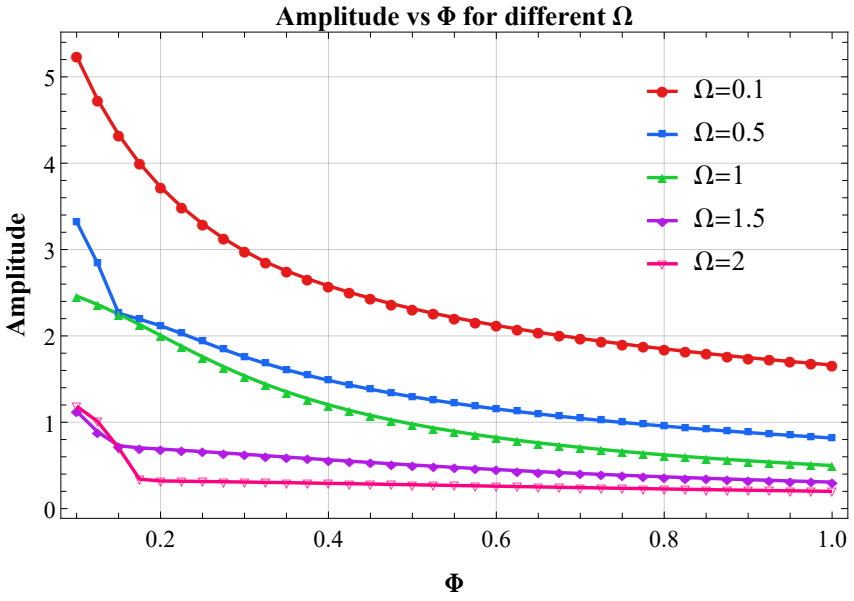


Figure 4.23: Change of the amplitude in terms of Ω, with different values of the friction number (Φ).

It is clear that as the friction number increases, the amplitude decreases. With the previous

analysis, we can generate a 3D plot that tells us what the amplitude of the stable region of our system is with a unitary force and with different frequencies and friction numbers.

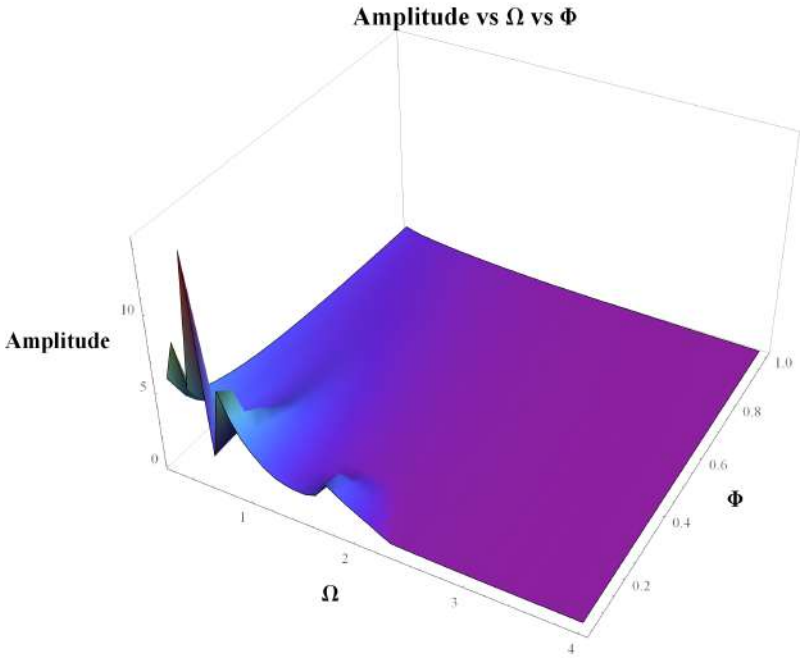


Figure 4.24: 3D graph of the amplitude of the system in terms of the frequency (Ω) and the friction number (Φ).

To this previous figure, we can see that our system needs a low friction number,, close to zero, and a frequency close to 1.

4.3.6 Fluid force

Once we see a general forced system, we are going to use a fluid force. This implies that everything in our previous treatment is right for our new system, with the difference in the force form.

We defined the Inertial–Gravitational Interaction Number Γ as the structural acceleration due to flow forcing relative to gravity acceleration, that is, $\Gamma = \frac{U}{\sqrt{gD}}$, where U is the speed of the fluid and D is the characteristic length, which in our system is $D = x_{eq}$.

The force of a fluid is :

$$F_f = \frac{1}{2}\rho U^2 dLC_L, \tag{4.3.35}$$

where L is the axial longitude of the free magnet, C_L is the lift coefficient, d is the width of our free magnet (diameter), and ρ is the density of our fluid. If the external force is the fluid force. Then, we can rewrite the force terms of the Gravity-inertia ratio as:

$$
\begin{aligned}
F_0 = \frac{1}{2}\rho U^2 dLC_L \implies \tilde{F}F_s = \frac{1}{2}\rho U^2 dLC_L \\
\tilde{F} = \frac{\rho U^2 dLC_L}{2F_s} \implies \tilde{F} = \frac{\rho U^2 dLC_L}{2mg} \\
\tilde{F} = \frac{\rho \Gamma^2 D dLC_L}{2m} \therefore \tilde{F} = \frac{1}{2m^*} C_L \Gamma^2.
\end{aligned}
\tag{4.3.36}
$$

We define m^* as the mass ratio between the mass of the structure and the mass of the fluid, that is, $m^* = \left(\frac{m}{\rho DS}\right)$. Something that is relevant to highlight is that the lift coefficient (C_L) could be a function of time, which would depend on the geometry, but in this specific example, we are going to work with a constant lift coefficient.

Then, substituting this in our dimensionless differential equation, we obtain

$$
\frac{d^2\tilde{x}}{d\tau^2} + 2\Phi\frac{d\tilde{x}}{d\tau} + 1 - \frac{1}{\tilde{x}} = \frac{1}{2m^*} C_L \Gamma^2 \cos(\Omega\tau) \tag{4.3.37}
$$

In this system, the frequency of the system is determined by the Strouhal number. Therefore, the shape of our magnet plays a crucial role in determining how the system responds to the force of the fluid. The Stouhal number definition is: $St = \frac{f_s D}{U}$. Then we can find a relation between the Strouhal and the frequency number. This relation is:

$$
\begin{aligned}
\omega = 2\pi f_s \implies \frac{\Omega}{t_s} = 2\pi\left(\frac{StU}{D}\right) \\
\implies \Omega = \frac{2\pi StU t_s}{D} \implies \Omega = \frac{2\pi StU}{Dg}\sqrt{\frac{M}{m}} \\
\implies \Omega = \frac{2\pi StU}{\sqrt{gD}}\sqrt{\frac{M}{mgD}} \implies \boxed{\Omega = 2\pi St\Gamma\sqrt{\frac{M}{mgD}} = \frac{2\pi StU t_s}{D}}.
\end{aligned}
\tag{4.3.38}
$$

Then, the frequency of the system is related to the Strouhal and Gravity-inertia ratios. If the characteristic length is equal to the equilibrium distance ($D = x_{eq} = \frac{M}{mg}$), the frequency number can be simplified as:

$$
\Omega = 2\pi St\Gamma. \tag{4.3.39}
$$

Then, we can rewrite our dimensionless differential equation as:

$$
\frac{d^2\tilde{x}}{d\tau^2} + 2\Phi\frac{d\tilde{x}}{d\tau} + 1 - \frac{1}{\tilde{x}} = \frac{1}{2m^*} C_L \Gamma^2 \cos(2\pi St\Gamma\tau). \tag{4.3.40}
$$

It is important to see that our system now depends on the values of the Strouhal (St) and Gravity-Inertia number (Γ) numbers and the friction number (Φ). In this system, the most important numbers are the Strouhal and Gravity-inertia ratios because they define the frequency of the oscillation.

New dimensionless equation

The equation of motion with a fluid force (equation (4.3.40)) is right, but the frequency of the vortices depends of two different values, but if we check the equation of the frequency (equation (4.3.38)), we can find that there is another possible dimensionless that help us to let the dimensionless frequency be only in terms of the Strouhal numbers. This new dimensionless uses fluid and structural properties. The new relation is.

$$\Omega = \frac{2\pi StUt_s}{D} = \Omega = 2\pi St \implies \frac{Ut_s}{D} = 1 \quad \therefore \quad \boxed{t_s = \frac{D}{U}}, \tag{4.3.41}$$

with this new time scale we are going to rewrite our differential equation. To rewrite it, we are going to use the dimensionless groups that we obtained previously. Starting the new dimensionless analysis in the equation of motion (4.3.13).

Applying the dimensionless groups in our differential equation, we can find the equation of motion.

$$\begin{aligned}
&\frac{d^2x}{dt^2} + 2\phi\frac{dx}{dt} + g - \frac{M}{mx} = \frac{F_0}{m}\cos(\omega t) \\
\implies &\frac{d^2(x_s\tilde{x})}{d(t_s\tau)^2} + 2(\phi_s\Phi)\frac{d(x_s\tilde{x})}{d(t_s\tau)} + g - \frac{M}{mx_s\tilde{x}} = \frac{(F_s\tilde{F})}{m}\cos((\omega_s\Omega)(t_s\tau)) \\
\implies &\frac{x_s}{t_s^2}\frac{d^2\tilde{x}}{d\tau^2} + \frac{2\Phi x_s}{t_s^2}\frac{d\tilde{x}}{d\tau} + g - \frac{M}{mx_s\tilde{x}} = \frac{F_s\tilde{F}}{m}\cos(\Omega\tau) \\
\implies &\frac{d^2\tilde{x}}{d\tau^2} + 2\Phi\frac{d\tilde{x}}{d\tau} + \frac{gt_s^2}{x_s} - \frac{Mt_s^2}{mx_s^2\tilde{x}} = \frac{t_s^2F_s\tilde{F}}{x_sm}\cos(\Omega\tau) \\
\implies &\frac{d^2\tilde{x}}{d\tau^2} + 2\Phi\frac{d\tilde{x}}{d\tau} + \frac{gD}{U^2} - \frac{M}{mU^2\tilde{x}} = \frac{M\tilde{F}}{U^2m}\cos(\Omega\tau)
\end{aligned} \tag{4.3.42}$$

Applying the definition of the equilibrium point ($x_{eq} = \frac{M}{mg}$), our differential equation is:

$$\frac{d^2\tilde{x}}{d\tau^2} + 2\Phi\frac{d\tilde{x}}{d\tau} + \frac{gD}{U^2} - \frac{x_{eq}g}{U^2\tilde{x}} = \frac{x_{eq}g\tilde{F}}{U^2}\cos(\Omega\tau). \tag{4.3.43}$$

If $D = x_{eq} = \frac{M}{mg}$ and using the definition of the Gravity-inertia ratio, then we can rewrite our equation:

$$\frac{d^2\tilde{x}}{d\tau^2} + 2\Phi\frac{d\tilde{x}}{d\tau} + \frac{1}{\Gamma^2} - \frac{1}{\Gamma^2\tilde{x}} = \frac{\tilde{F}}{\Gamma^2}\cos(\Omega\tau). \tag{4.3.44}$$

Now, using the dimensionless form of the fluid force (equation (4.3.36)), we can find the following equation.

$$\begin{aligned}
&\frac{d^2\tilde{x}}{d\tau^2} + 2\Phi\frac{d\tilde{x}}{d\tau} + \frac{1}{\Gamma^2} - \frac{1}{\Gamma^2\tilde{x}} = \frac{\tilde{F}}{\Gamma^2}\cos(\Omega\tau) \\
\implies &\frac{d^2\tilde{x}}{d\tau^2} + 2\Phi\frac{d\tilde{x}}{d\tau} + \frac{1}{\Gamma^2} - \frac{1}{\Gamma^2\tilde{x}} = \frac{\frac{1}{2m^*}C_L\Gamma^2}{\Gamma^2}\cos(\Omega\tau) \\
\therefore \quad &\frac{d^2\tilde{x}}{d\tau^2} + 2\Phi\frac{d\tilde{x}}{d\tau} + \frac{1}{\Gamma^2} - \frac{1}{\Gamma^2\tilde{x}} = \frac{1}{2m^*}C_L\cos(2\pi St\tau).
\end{aligned} \tag{4.3.45}$$

Now, the frequency of the external force only depends on the Strouhal number, or in other words, the vortex–shedding frequency of the von Kármán street. If we do some algebra, we can factorize the Gravity-inertia ratio. Obtaining the following equation.

$$\frac{d^2\tilde{x}}{d\tau^2}+2\Phi\frac{d\tilde{x}}{d\tau}+\frac{1}{\Gamma^2}-\frac{1}{\Gamma^2\tilde{x}}=\frac{1}{2m^*}C_L\cos\left(2\pi St\tau\right)\implies\frac{d^2\tilde{x}}{d\tau^2}+2\Phi\frac{d\tilde{x}}{d\tau}+\frac{1}{\Gamma^2}\left(1-\frac{1}{\tilde{x}}\right)=\frac{1}{2m^*}C_L\cos\left(2\pi St\tau\right) \tag{4.3.46}$$

Something important is that our system works when we know three dimensionless parameters, which are: the Strouhal (St), Gravity-Inertia ratio (Γ), and friction (Φ) numbers.

Something important to see is that if we work with small amplitudes, we can find an equation that has an analytical solution. Firstly, we are going to use an expansion at first order.

$$\tilde{x}(\tau)=1+\xi(\tau),\qquad|\xi|\ll 1. \tag{4.3.47}$$

Now, using a first order Taylor expanssion of the nonlinear term.

$$\frac{1}{\tilde{x}}=\frac{1}{1+\xi}=1-\xi+O(\xi^2). \tag{4.3.48}$$

Substituting $\tilde{x}=1+\xi$ in our differential equation.

$$\frac{d^2}{d\tau^2}(1+\xi)+2\Phi\frac{d}{d\tau}(1+\xi)+\frac{1}{\Gamma^2}-\frac{1}{\Gamma^2}(1-\xi)=\frac{1}{2m^*}C_L\cos(2\pi St\,\tau)+O(\xi^2). \tag{4.3.49}$$

We know that the constant term in $(1+\xi)$ is zero, and dropping $O(\xi^2)$ terms. Then, our differential equation is now.

$$\begin{aligned}&\frac{d^2\xi}{d\tau^2}+2\Phi\frac{d\xi}{d\tau}+\frac{1}{\Gamma^2}-\frac{1}{\Gamma^2}+\frac{1}{\Gamma^2}\xi=\frac{1}{2m^*}C_L\cos(2\pi St\,\tau)\\&\implies\frac{d^2\xi}{d\tau^2}+2\Phi\frac{d\xi}{d\tau}+\underbrace{\frac{1}{\Gamma^2}}_{\omega_0^2}\xi=\frac{1}{2m^*}C_L\cos(2\pi St\,\tau)\\&\implies\frac{d^2\xi}{d\tau^2}+2\Phi\frac{d\xi}{d\tau}+\omega_0^2\xi=\frac{1}{2m^*}C_L\cos(2\pi St\,\tau),\end{aligned} \tag{4.3.50}$$

where ω_0 is the natural frequency of the linearized system .Using the natural frequency of this small aplitudes system, we find the differential equation of a damped forced oscillator system.

The resonance of the system as we know, appears when the frequency of the external force is similar to the natural frequency ($\Omega\approx\omega_0=\frac{1}{\Gamma}$). But the frequency of the external force is defined by the Strouhal number, which reveals a clear relation between the fluid motion and the system's natural frequency. Then the Strouhal number that generates a resonance is:

$$2\pi St\approx\frac{1}{\Gamma}\quad\implies\quad\boxed{St\approx\frac{1}{2\pi\,\Gamma}.} \tag{4.3.51}$$

Another thing that we can obtain is the amplitude relation for our small-amplitude system, that is:

$$A(\Omega) \approx \frac{\frac{1}{2m^*}C_L}{\sqrt{(\omega_n^2 - \Omega^2)^2 + (2\Phi\Omega)^2}}. \tag{4.3.52}$$

4.4 Harnessing electric model

With the equation of motion of the system, we know the behavior of our dynamical system, and we are able to predict its motion. That lets us know where and when the system is near or far to a point. So, using this knowledge and the induction Faraday's law, we can induce a voltage in an open circuit. Then, introducing the Faraday's law:

$$\varepsilon = -n\frac{d\Phi_B}{dt}, \tag{4.4.1}$$

where n is the number of turns of a coil, Φ_B is the magnetic flux. Now, using the definition of the magnetic flux:

$$\Phi_B = BS\cos(\alpha), \tag{4.4.2}$$

where B is the magnetic field, S the surface area of the coil, and α is the angle between the magnetic field and the normal vector of the surface area. It is important to remark that in our system the magnet and the coil are always aligned with $\alpha = \pi$. Using this information our magnetic flux is:

$$\Phi_B = -BS. \tag{4.4.3}$$

If we use the magnetic field of a dipole magnet, we obtain that it depends of the distance. The magnetic field is.

$$B = \frac{\mu_0 \mathcal{M}}{2x^3}, \tag{4.4.4}$$

where μ_0 is the magnetic permeability of the fre space, $\mathcal{M}$ is the magnetic dipole moment,and x is the distance between the magnet and the coil. Applying the previous definition in the induction Faraday's law.

$$\varepsilon = n\frac{d}{dt}(BS) \implies \varepsilon = \frac{n\mu_0\mathcal{M}S}{2}\frac{d}{dt}\left(\frac{1}{x(t)^3}\right). \tag{4.4.5}$$

For simplicity we define $\zeta = \frac{n\mu_0\mathcal{M}S}{2}$. Then, our generated voltage is:

$$\varepsilon = \zeta\frac{-3x(t)^2\cdot\dot{x}(t)}{x(t)^6} = -\frac{3\zeta\dot{x}(t)}{x(t)^4}. \tag{4.4.6}$$

Now, we are going to dimensionless the position and time to relate to our previous solutions. Then, we obtain.

$$\varepsilon = -3\zeta\frac{\frac{d(x_s\tilde{x}(\tau))}{d(t_s\tau)}}{(x_s\tilde{x}(\tau))^4} = -\frac{3\zeta x_s}{t_s x_s^4}\cdot\frac{\dot{\tilde{x}}(\tau)}{\tilde{x}(\tau)^4} = -\frac{3\zeta}{t_s\tilde{x}(\tau)^3}\cdot\frac{\dot{\tilde{x}}(\tau)}{\tilde{x}(\tau)^4} \tag{4.4.7}$$

Using the definitions of the scale factors of our system, we obtain.

$$\varepsilon = -\frac{3\zeta U}{x_{eq}^4} \cdot \frac{\dot{\tilde{x}}(\tau)}{\tilde{x}(\tau)^4} = -\frac{3\zeta U m^4 g^4}{M^4} \cdot \frac{\dot{\tilde{x}}(\tau)}{\tilde{x}(\tau)^4} \tag{4.4.8}$$

.

Now, using the equation of the induced voltage and the numerical solution of the system, we can find the response of the voltage to the motion of our system. For an specific case we use in the system the following values: $\Phi = 0.5$, $x_0 = 1$, $\Omega_0 = 1$, $\tilde{F} = 1$. Then, the voltage generated in time is represented in the following figure.

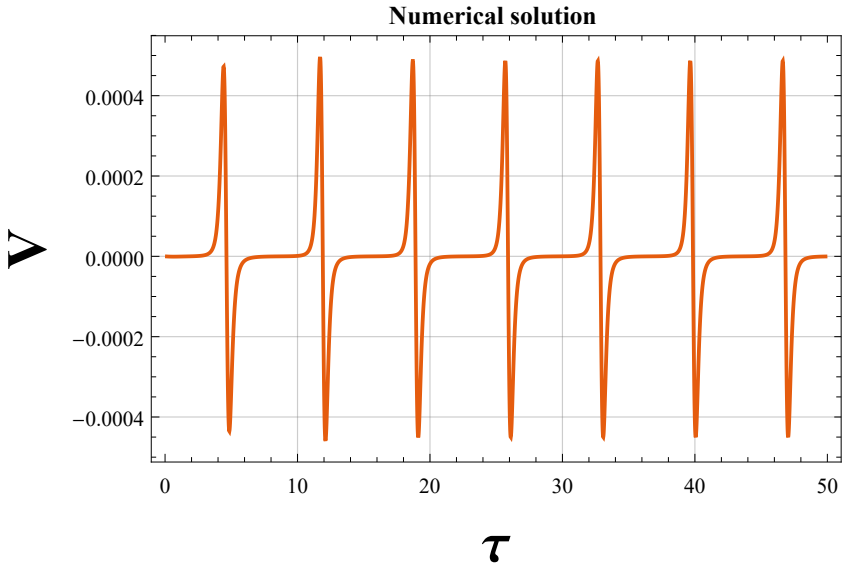


Figure 4.25: Voltage generated by the system with $\tilde{F}$, $\Omega_0 = 1$, $\Phi = 0.5$, $x_0 = 1$

Now, to see the dependence of the voltage on friction, we plot the voltage generated by different Φ values.

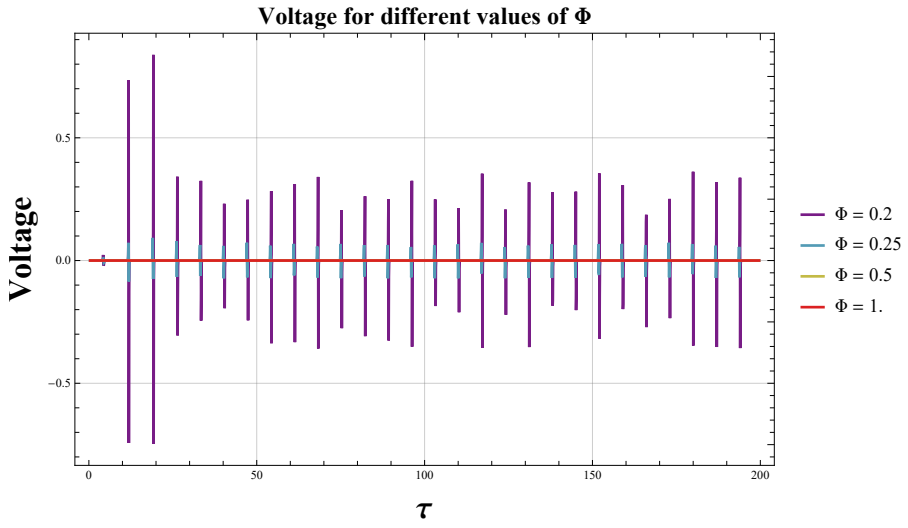


Figure 4.26: Reaction of the voltage generated to different values of damping.

For the previous figure, we can see that the voltage decreases when the damping increases, and this makes sense because more damping implies less movement, which directly affects the voltage's value. Another interesting case is how the voltage reacts to different frequency values. These frequencies are the frequencies of the vortices.

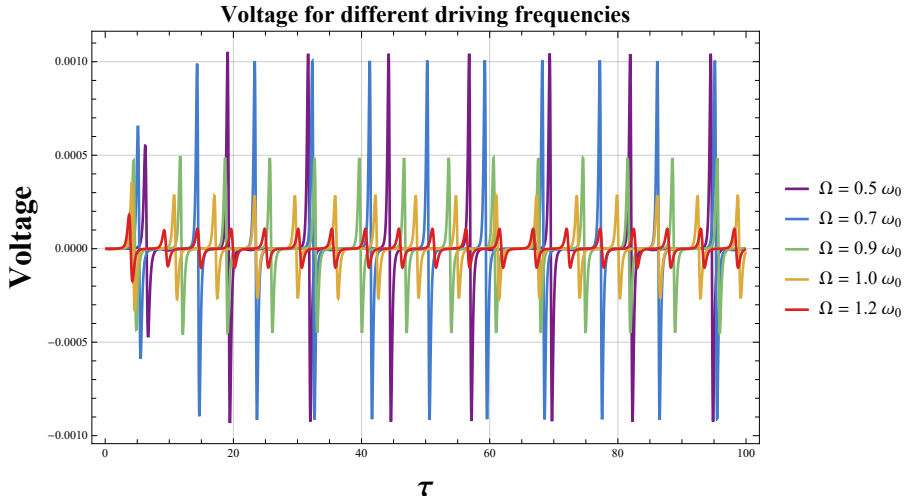


Figure 4.27: Reaction of the voltage generated to different frequencies.

As we can see in the previous figure, the magnitude of the voltage reduces when the frequency tends to the natural frequency of the system, but the voltage is more recurrent.

4.5 Conclusions

In this chapter, we confirm that our system can oscillate, and the interaction with a fluid in motion could be the source of energy that it needs to maintain the oscillation. The oscillation is asymmetric because in the lowest part, the system is repelled quickly, while in the highest part, the attraction is very slow.

In terms of the external force, we can see that the frequency of the vortices generated by the fluid-structure interaction are the responsable of the motion and the quantity of the voltage generated.

The amplitude of the oscillation depends mainly on the shape of the free magnet. For this reason, it becomes essential to study different geometries and choose the one that maximizes the lift coefficient. A better shape would increase the force that the fluid transfers to the magnet, allowing larger oscillations and a more efficient energy exchange.

We also observed that the fluid parameters have important roles. The Strouhal number sets the resonance condition that creates the oscillation, while the inertial gravity interaction number affects not only the resonance itself but also the stability and complexity of the dimensionless equation. This shows how sensitive the system is to the properties of the fluid and how these parameters can change the global behavior of the solution.

Finally, all the mathematical tools used throughout the chapter—Lagrangian and Hamiltonian formulations, dimensional analysis, and asymptotic approximations—were very useful to understand how the system reacts and why it behaves the way it does. These methods helped us find expressions that describe the dynamics more clearly and opened the possibility of using this kind of system in future projects related to clean energy generation.

The ideas developed here could be the base to design simple and stable oscillators powered only by natural fluid motion.

CHAPTER 5

Three magnets oscillation with gravity

> The energy of the magnetic field resides in the space between and around the magnets... This energy is constantly undergoing transformation, manifesting itself as a stress that exerts attractive or repulsive forces upon the bodies within its influence.
>
> James Clerk Maxwell, *"A Treatise on Electricity and Magnetism"*, 1873

The system of this problem is constituted by three magnets; two of them are fixed, one on the ground and the other is set some distance h over the first. The third magnet is free to move in the region between the two fixed magnets. The free magnet is initially displaced some distance A from the equilibrium position (x_0). The free magnet is affected by the gravitational force and the magnetic repulsion forces.

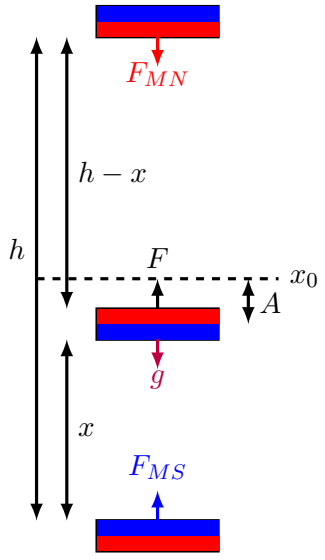


Figure 5.1: Graphic representation of the system, where F is the force, F_{rN} and F_{rS} are the magnetic repulsion force of each pole and g is the gravitational acceleration.

To simplify the system and the calculus we are going to consider that the two magnetic repulsion forces have the same form and are identically with the difference in the direction. The magnetic repulsion force F_M is expressed in this problem as a relation between the magnetic energy (M) and the distance between the magnets (x). So the mathematical form of the magnetic force is:

$$F_M = \frac{M}{x}. \tag{5.0.1}$$

As initial conditions for this problem we have that at the time $t = 0$ the distance between the magnets is a displacement from the equilibrium position and the speed of the free magnet in that initial position is cero. So our initial conditions mathematically are:

$$x(0) = x_{eq} + A \quad \wedge \quad \dot{x}(0) = 0. \tag{5.0.2}$$

The domain of the distance x goes from 0 to h, because the free magnet in the limit can be so close to any of the fixed magnets, but it can not touch any of them, so the domain is: $0 < x < h$. So the domain of the amplitude is:

$$\begin{aligned} 0 = x_{eq} + A &\implies A = -x_{eq} \\ h = x_{eq} + A &\implies A = h - x_{eq} \\ \therefore \quad &\boxed{-x_{eq} < A < h - x_{eq}} \end{aligned} \tag{5.0.3}$$

5.1 Ideal system

For the ideal case, the dissipative and external forces do not interact with the physical system. With this, the forces in the system are:

$$F = -F_g + F_M = -mg + \frac{M}{x} - \frac{M}{h-x}, \tag{5.1.1}$$

where F is the force, m is the mass of the free magnet, g the gravitational acceleration, x the distance between the two magnets and M is the magnetic energy constant. If the system is in equilibrium, then the net force is cero ($F = 0$) and we can find the equilibrium distance.

$$\begin{aligned} \frac{M}{x} - mg - \frac{M}{h-x} = 0 &\implies M(h-x) = mgx(h-x) + Mx \\ &\therefore \boxed{x_{eq} = \frac{2M + mgh \pm \sqrt{4M^2 + m^2g^2h^2}}{2mg}} \quad , \end{aligned} \tag{5.1.2}$$

for the previous solution, we have two options for the equilibrium distance, so we need to find what is the distance that belongs to the x domain. So we going to compare the positive solution to h.

$$\frac{2M+mgh+\sqrt{4M^2+m^2g^2h^2}}{2mg} < h \implies 2M+mgh+\sqrt{4M^2+m^2g^2h^2} < 2mgh$$
$$\implies 2M+\sqrt{4M^2+m^2g^2h^2} < mgh \implies \sqrt{4M^2+m^2g^2h^2} < mgh-2M$$
$$\implies 4M^2+m^2g^2h^2 < 4M^2+m^2g^2h^2-4mghM \boxed{\therefore 0 < -4mghM}. \tag{5.1.3}$$

This solution is a contradiction, so the positive root in the equation (5.1.2) is not in the x domain. Then, the equilibrium distance is:

$$\boxed{x_{eq} = \frac{2M+mgh-\sqrt{4M^2+m^2g^2h^2}}{2mg}} \tag{5.1.4}$$

as we can see, the equilibrium distance compares the magnetic energy with the gravitational force; then the increase or decrease in some of these magnitudes affects the equilibrium distance of the system, this lets us modulate the equilibrium distance.

5.1.1 Lagrangian analysis

The relevance of the lagrangian analysis is that we can study the dynamic of the system in terms of its coordinates and degrees of freedom. The lagrangian is described as the sum of the kinetic energies minus the potential energies present in the system.

$$L = T - V \implies L = \frac{1}{2}\sum_i m_i \dot{q_i}^2 - V(q_1, q_2, \ldots, q_i), \tag{5.1.5}$$

where m_i is the ith mass of the system, $\dot{q_i}$ is the ith speed of the system, V is the potential energy function and $q_1, q_2, \ldots, q_i$ are the coordinates of the system.

Our system only have a degree of freedom, so the dynamic of the system is described by one coordinate (x), then our lagrangian has the following expression.

$$L = \frac{1}{2}m\dot{x}^2 - V(x), \tag{5.1.6}$$

where $V(x)$ is the potential energy in our system. To find the potential energy we going to use the relation between the work and the potential energy when the system is conservative.

$$-\frac{d}{dx}V(x) = F \implies -V(x) = \int F dx \implies -V(x) = -\int mg dx + \int \frac{M}{x}dx - \int \frac{M}{h-x}dx$$
$$\implies -V(x) = -mg\int dx + M\int \frac{dx}{x} - M\int \frac{dx}{h-x} \implies -V(x) = -mgx + M\ln(x) + M\ln(h-x)$$
$$\implies -V(x) = -\left[mgx - M\ln(x) - M\ln(h-x)\right] \therefore \boxed{V(x) = mgx - M\ln(x) - M\ln(h-x)}. \tag{5.1.7}$$

Substituting this result in the our lagrangian function (equation (4.1.4)), we obtain that our lagrangian is:

$$\boxed{L = \frac{1}{2}m\dot{x}^2 - mgx + M\ln(x) + M\ln(h-x)}. \tag{5.1.8}$$

With our lagrangian, we can find the energy of the system and the equation of motion of the system.

First, we going to find the energy of the system. The energy of the syste is define as.

$$E = \sum_i \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L \implies E = \dot{x}\frac{\partial L}{\partial \dot{x}} - L \therefore \boxed{E = \frac{1}{2}m\dot{x}^2 + mgx - M\ln(x) - M\ln(h-x)}. \tag{5.1.9}$$

Now, using the Euler-Lagrange equation we can find the equation of motion of our system. As we know, the external forces are null ($Q = 0$). So the equation of motion for the coordinate x is.

$$\begin{gathered}\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = Q \implies m\ddot{x} + mg + \frac{M}{h-x} - \frac{M}{x} = 0 \\ \implies \ddot{x} + g + \frac{M}{m(h-x)} - \frac{M}{mx} = 0 \quad \therefore \quad \boxed{\frac{d^2x}{dt^2} + g + \frac{M}{m}\left[\frac{1}{h-x} - \frac{1}{x}\right] = 0}.\end{gathered} \tag{5.1.10}$$

This equation of motion is relevant because it lets us know the accelerations that generate the movement in our system. It's important to remark that the differential equation does not have an analytical solution.

5.1.2 Hamiltonian analysis

Now that we know the equation of motion, it is important to see if the system really oscillates. To do this we going to use the hamiltonian analysis. To use this analysis we need to construct the hamiltonian of our system.

$$\begin{gathered}H = E = \frac{1}{2}m\dot{x}^2 + mgx - M\ln(x) - M\ln(h-x), \\ \boxed{\therefore H = \frac{p^2}{2m} + mgx - M\ln(x) - M\ln(h-x)},\end{gathered} \tag{5.1.11}$$

where H is the hamiltonian of the system and p is the momentum.

Now, using the Hamilton's equations we can find the dynamic of the system. The Hamilton's equations are:

$$\dot{H} = \begin{pmatrix}\dot{x} \\ \dot{p}\end{pmatrix} = \begin{pmatrix}\dfrac{p}{m} \\ \dfrac{M}{x} - mg - \dfrac{M}{h-x}\end{pmatrix} \tag{5.1.12}$$

In the Hamilton's equations we can see that the change of momentum depends of position and vice versa. If we plot the solution of the Hamilton's equations we obtain a phase portrait of our system to see the behavior of it.

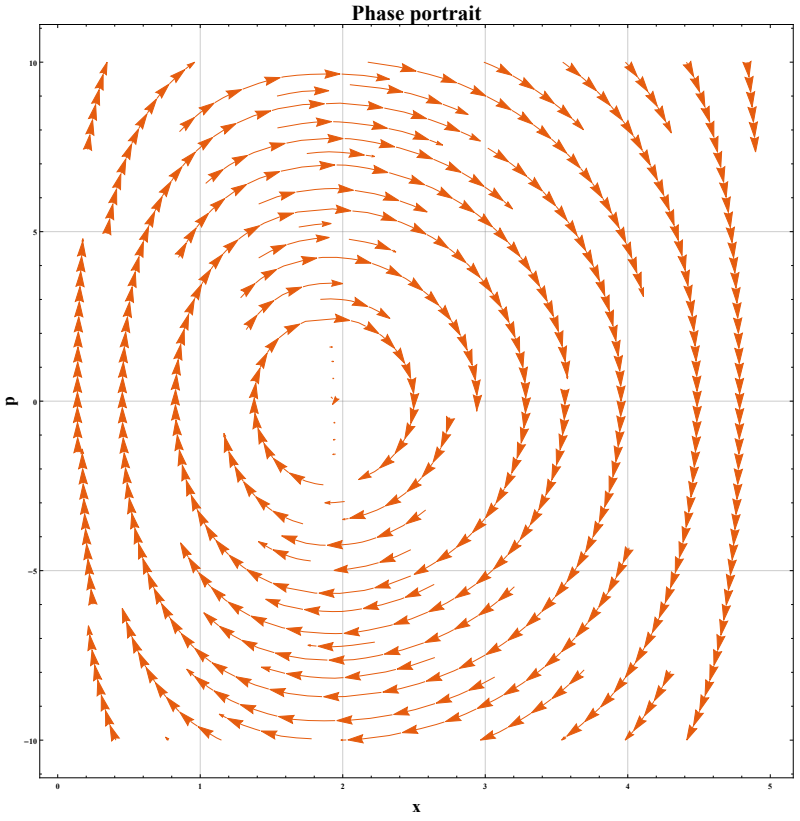


Figure 5.2: Phase portrait of the system with $M = 50$, $h = 5$, $g = 10$ and $m = 1$

As we can see in the figure (5.2), the system follows a closed cyclical trajectory, so we can confirm that the system oscillates.

5.1.3 Dimensional analysis

In our system we have seven variables ($n = 7$), this variables are the mass (m), the time (t), the position (x), the gravity (g), the oscillation amplitude (A), the magnetic energy (M) and the distance between the fixed magnets (h). For the nature of our problem, we only need to use three dimension ($j = 3$) to caracterize all the problem, the dimensions that we need to use is the matter, the longitude and the time. Using the Buckingham's theorem, we know that there are four dimensionless groups ($k = n - j = 4$) that describe our system.

Using a dimensional analysis over every variable:

$$[x] = L, [h] = L, [A] = L, [t] = T, [m] = M, [g] = LT^{-2}, [M] = ML^2T^{-2} \tag{5.1.13}$$

For this system we going to use h, g and M as dimensional variables, this because h is the geometric parameter of the system, g is the cinematic parameter and M is the dynamic parameter. As we saw in the previous chapter, every dimensinal group is the product of the dimensional variables, so we have that the ith group is expressed as:

$$\Pi_i = \delta h^{\alpha} g^{\beta} M^{\lambda}, \tag{5.1.14}$$

where δ is the variable that we want to dimensionless, in our system we have the variables x, t, m and A; α, β and λ are the powers of the distance h, the gravity and the magnetic energy that dimensionless the ith group.

To find the powers of our dimensional variables that dimensionless every group we can construct an equations system, this system is.

$$\begin{pmatrix} L \\ M \\ T \end{pmatrix} + \begin{pmatrix} 1 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & -2 & -2 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \lambda \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \implies \begin{pmatrix} \alpha \\ \beta \\ \lambda \end{pmatrix} = \begin{pmatrix} M - L - \frac{T}{2} \\ M + \frac{T}{2} \\ -M. \end{pmatrix} \tag{5.1.15}$$

It is important to remember that every dimensionless group can be expressed as a fraction. So the dimensionless groups can be rewrite as:

$$\Pi_i = \frac{\delta}{\delta_s} = \tilde{\delta}, \tag{5.1.16}$$

where δ_s is the scale factor of the variable δ y $\tilde{\delta}$ is the dimensionless form of the variable δ. Now, we can obtain the dimensionless groups for the variables x, t, m and A.

Dimensionless group of the position

The x's group is:

$$\begin{pmatrix} \alpha \\ \beta \\ \lambda \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} \quad \therefore \quad \boxed{\Pi_1 = \frac{x}{x_s} = \frac{x}{h} \implies x_s = h} \quad . \tag{5.1.17}$$

With the previous result we can se that the scale factor of the position (x_s) is the equilibrium distance, and the distance can be expressed as the product of the dimensionless distance and the scale factor. So, we obtain the following relation.

$$x = x_s\tilde{x} \implies dx = x_s d\tilde{x} \implies d^2x = x_s d^2\tilde{x} \tag{5.1.18}$$

Dimensionless group of the time

The group of t is.

$$\begin{pmatrix} \alpha \\ \beta \\ \lambda \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix} \quad \therefore \quad \boxed{\Pi_2 = \frac{t}{t_s} = t\sqrt{\frac{g}{h}} \implies t_s = \sqrt{\frac{h}{g}}} \quad . \tag{5.1.19}$$

From this result, we can find the expressions of the scale factors of the frequency and the period of oscillation. These expressions are:

$$\omega_s = \frac{1}{t_s} = \sqrt{\frac{g}{h}} \quad \wedge \quad T_s = \frac{2\pi}{\omega_s} = 2\pi\sqrt{\frac{h}{g}}. \tag{5.1.20}$$

Something important to see in the previous relations is that they are equivalent to the period and frequency of a simple pendulum. This lets us conclude that our system is analogous to a pendulum.

Now, using the relation between the variable and its scale factor, we can obtain.

$$t = t_s\tau \implies dt = t_s d\tau \implies dt^2 = t_s^2 d\tau^2 \tag{5.1.21}$$

Dimensionless group of the mass and the Energie's number

Now, the group of m:

$$\begin{pmatrix}\alpha\\ \beta\\ \lambda\end{pmatrix} = \begin{pmatrix}1\\ 1\\ -1\end{pmatrix} \quad \therefore \quad \boxed{\Pi_3 = \frac{m}{m_s} = \frac{mgh}{M} \implies m_s = \frac{M}{gh}} \quad . \tag{5.1.22}$$

This dimensionless group is very important because give us a relation between the magnetic energy and the mechanic potential energy. Using this dimensionless group we going to define Energie's number as:

$$\Lambda \equiv \frac{1}{\Pi_3} = \frac{M}{mgh}. \tag{5.1.23}$$

The Energie's number going to be so useful because it related all the variables that are not affected by time. An example of some quantity that is not affected by time is the equilibrium distance, so we can rewrite this number interm of the Energie's number as:

$$\begin{aligned} x_{eq} &= \frac{2M + mgh - \sqrt{4M^2 + m^2g^2h^2}}{2mg} = \frac{mgh\left(\frac{2M}{mgh} + 1\right) - \sqrt{m^2g^2h^2\left(\frac{4M^2}{m^2g^2h^2} + 1\right)}}{2mg} \\ &= \frac{mgh\left(\frac{2M}{mgh} + 1 - \sqrt{\frac{4M^2}{m^2g^2h^2} + 1}\right)}{2mg} = \frac{h}{2}\left(2\Lambda + 1 - \sqrt{4\Lambda^2 + 1}\right) \\ \therefore \boxed{x_{eq} = h\left(\frac{2\Lambda + 1 - \sqrt{4\Lambda^2 + 1}}{2}\right) \implies \tilde{x}_{eq} = \frac{x_{eq}}{h} = \frac{2\Lambda + 1 - \sqrt{4\Lambda^2 + 1}}{2}}. \end{aligned} \tag{5.1.24}$$

Dimensionless group of the amplitude

Obtaining the group of A.

$$\begin{pmatrix}\alpha\\ \beta\\ \lambda\end{pmatrix} = \begin{pmatrix}-1\\ 0\\ 0\end{pmatrix} \quad \therefore \quad \boxed{\Pi_4 = \frac{A}{A_s} = \frac{A}{h} \implies A_s = h} \quad . \tag{5.1.25}$$

This result lets us see that the amplitude of oscillation is a fraction of the distance between the fixed magnets. The amplitude does not have a role in the motion equation (equation 5.1.10) so we going to use this dimensionless group as the control number (ϵ) for our approximation.

$$\epsilon = \frac{A}{h} = \tilde{A}. \tag{5.1.26}$$

5.1.4 Dimensionless system

Now that we have our dimensionless groups, we going to use them to find the dimensioless equations for our system. First we going to find the dimensionless form of the equation of motion. So using the groups Π_1, Π_2 and Π_3 in the equation (5.1.10) we obtain.

$$\begin{aligned}
&\frac{d^2x}{dt^2} + g + \frac{M}{m}\left[\frac{1}{h-x} - \frac{1}{x}\right] = 0 \implies \frac{d^2(x_s\tilde{x})}{d(t_s\tau)^2} + g + \frac{M}{m}\left[\frac{1}{h - x_s\tilde{x}} - \frac{1}{x_s\tilde{x}}\right] = 0 \\
\implies &\frac{h}{\frac{h}{g}}\frac{d^2\tilde{x}}{d\tau^2} + g + \frac{M}{m}\left[\frac{1}{h-h\tilde{x}} - \frac{1}{h\tilde{x}}\right] = 0 \implies g\frac{d^2\tilde{x}}{d\tau^2} + g + \frac{M}{mh}\left[\frac{1}{1-\tilde{x}} - \frac{1}{\tilde{x}}\right] = 0 \\
\implies &\frac{d^2\tilde{x}}{d\tau^2} + 1 + \frac{M}{mgh}\left[\frac{1}{1-\tilde{x}} - \frac{1}{\tilde{x}}\right] = 0 \implies \boxed{\frac{d^2\tilde{x}}{d\tau^2} + 1 + \Lambda\left[\frac{1}{1-\tilde{x}} - \frac{1}{\tilde{x}}\right] = 0},
\end{aligned} \tag{5.1.27}$$

this differential equation is so relevant, because lets us see that the position of the free magnte only depends to the time and the Energie's number, then x is function of τ and Λ.

Now, doing some algebra we can reformulate the equation of motion as:

$$\begin{aligned}
&\frac{d^2\tilde{x}}{d\tau^2} + 1 + \Lambda\left[\frac{1}{1-\tilde{x}} - \frac{1}{\tilde{x}}\right] = 0 \implies \frac{d^2\tilde{x}}{d\tau^2} + 1 + \Lambda\left[\frac{2\tilde{x}-1}{\tilde{x}(1-\tilde{x})}\right] = 0 \\
\implies &\tilde{x}(1-\tilde{x})\left(\frac{d^2\tilde{x}}{d\tau^2} + 1 + \Lambda\left[\frac{2\tilde{x}-1}{\tilde{x}(1-\tilde{x})}\right] = 0\right) \implies \tilde{x}(1-\tilde{x})\frac{d^2\tilde{x}}{d\tau^2} + \tilde{x}(1-\tilde{x}) + \Lambda\left(2\tilde{x}-1\right) = 0 \\
&\therefore \boxed{\tilde{x}\frac{d^2\tilde{x}}{d\tau^2} - \tilde{x}^2\frac{d^2\tilde{x}}{d\tau^2} - \tilde{x}^2 + (2\Lambda+1)\,\tilde{x} - \Lambda = 0}.
\end{aligned} \tag{5.1.28}$$

As we can see in this result, the differential equation is non lineal and is impossible to solve it in an anallytical way, so we going to use a numerical solution and an approximation solution.

The next step is to dimensionless the initial conditions of the system, so using the dimensionless groups we obtain the following initial conditions.

$$\begin{aligned}
&x(0) = x_{eq} + A \implies h\tilde{x}(0) = x_{eq} + A \implies \tilde{x}(0) = \frac{x_{eq} + A}{h} \\
&\implies \boxed{\tilde{x}(0) = \frac{x_{eq}}{h} + \epsilon \therefore \tilde{x}(0) = \frac{2\Lambda + 1 - \sqrt{4\Lambda^2+1}}{2} + \epsilon.} \\
&\dot{x}(0) = 0 \implies h\dot{\tilde{x}}(0) = 0 \implies \boxed{\dot{\tilde{x}}(0) = 0}
\end{aligned} \tag{5.1.29}$$

Now, rescaling the domain of the dimensionless amplitude (ϵ) system.

$$0 = x_{eq} + A \implies A = -x_{eq} \implies \epsilon = -\frac{x_{eq}}{h} = -\frac{2\Lambda + 1 - \sqrt{4\Lambda^2 + 1}}{2}$$

$$h = x_{eq} + A \implies A = h - x_{eq} \implies \epsilon = 1 - \frac{x_{eq}}{h} = 1 - \frac{2\Lambda + 1 - \sqrt{4\Lambda^2 + 1}}{2} \tag{5.1.30}$$

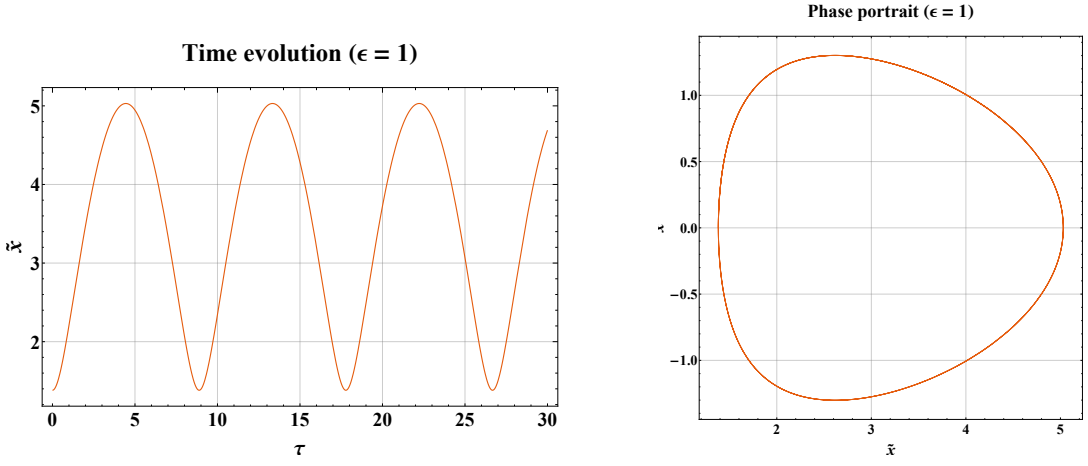
$$\therefore \quad \boxed{-\frac{x_{eq}}{h} < \epsilon < 1 - \frac{x_{eq}}{h} \implies -\frac{2\Lambda + 1 - \sqrt{4\Lambda^2 + 1}}{2} < \epsilon < \frac{1 - 2\Lambda + \sqrt{4\Lambda^2 + 1}}{2}}$$

5.1.5 Numerical solution

As in the previous problem, we need to find the numerical solution to contrast our series approximation with it. And to find the numerical solution, we need to use a numerical method. The numerical method that we are going to use is the Runge-Kutta method.

We are going to use the command ***NDSolve*** of the ***Wolfram Mathematica*** software. For this numerical simulation, we are going to use $\Lambda = 1$.

Suppose we use $\epsilon = 1$ in our approximated solution. In that case, we get the plot that expresses the evolution of the system by time, and the parametric plot communicates the behavior of the system (this can be seen in figure 5.3).



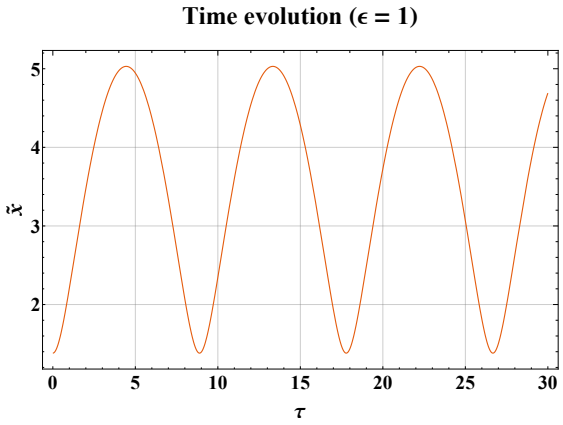
((a)) Time evolution ((b)) Phase portrait

Figure 5.3: Plot and parametric plot of the numerical solution with $\epsilon = 1$

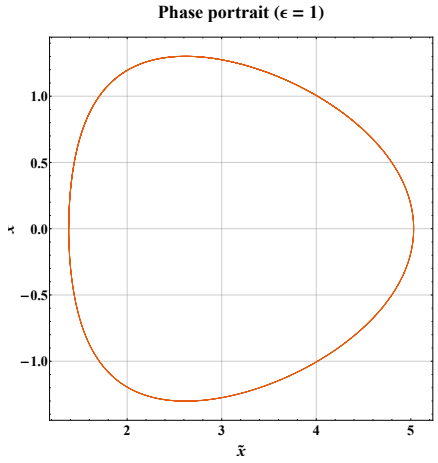
For the previous figure, we can see that the movement of our oscillator is similar to an elastic ball that is bouncing on the ground. So we can imagine that if the displacement from the equilibrium point is minimum, the behavior of the movement is more circular or cyclical. To discover if this is true, we need to simulate the system with different epsilon values.

Now, we simulated three different systems with the intention of contrasting different behaviors of our system. We are going to put these different systems into a figure. In our figure (5.4), we can see how the system acts under the epsilon values of 1, 0.5, and 0.01.

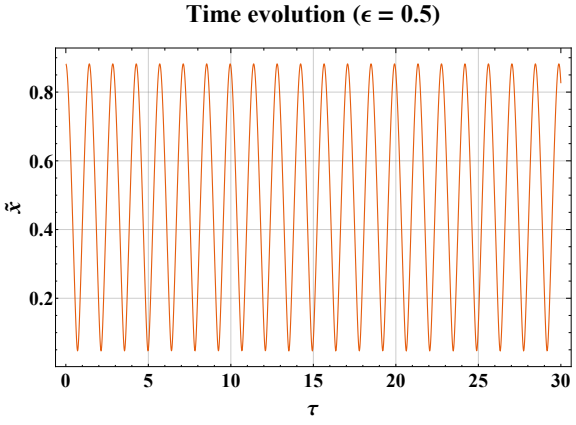
As we can see in the figure (5.4), the system's behavior is clearly similar to a circle when the epsilon value decreases. Then, the complexity of solving the system increases when epsilon increases.



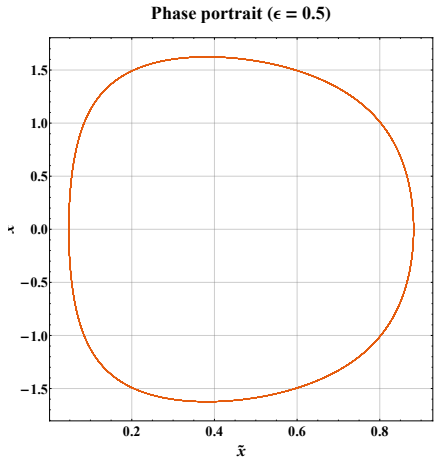
((a)) Time evolution ($\epsilon = 1$)



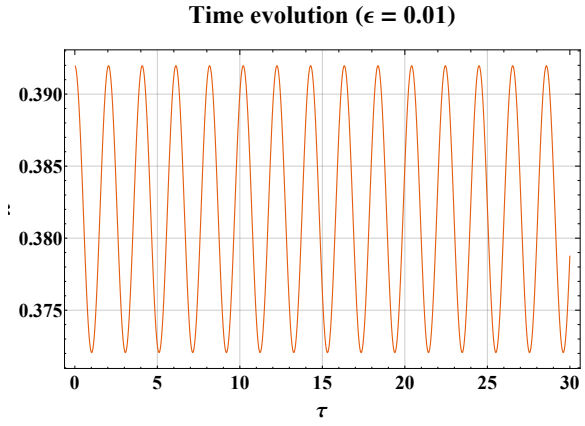
((b)) Phase portrait ($\epsilon = 1$)



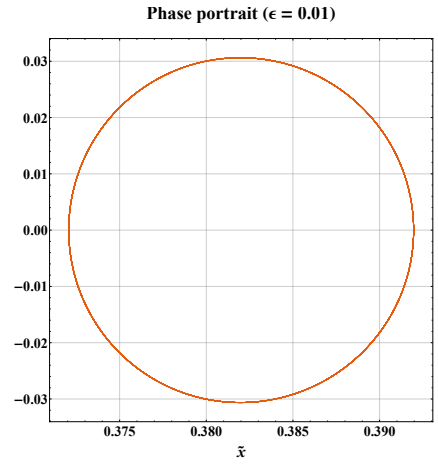
((c)) Time evolution ($\epsilon = 0.5$)



((d)) Phase portrait ($\epsilon = 0.5$)



((e)) Time evolution ($\epsilon = 0.01$)



((f)) Phase portrait ($\epsilon = 0.01$)

Figure 5.4: Time evolution and phase portraits of the system for $\epsilon = 1$ and $\epsilon = 0.5$.

In the figure previously named (figure (5.4)), we can see that when $\epsilon \approx 0$ is easier to fit a cyclical function, a cosine function. So, we can infer that the first-order approximation is a cosine function.

5.1.6 Asymptotic approximation

As we saw in the previous chapter, the asymptotic approximation consists of regulating the apport of every term in a polynomial with the intention to approximate that polynomial to the analytical solution. When the polynomial has infinite terms, the polynomial is equal to the solution. So our solution has the form:

$$x(t) \sim \sum_{i=0}^{N} \epsilon^i x_i(t) \implies x(t) = \lim_{N\to\infty} \sum_{i=0}^{N} \epsilon^i x_i(t). \tag{5.1.31}$$

Applying this to the dimensionless form, we obtain.

$$\tilde{x}(\tau) \sim \sum_{i=0}^{N} \epsilon^i \tilde{x}_i(\tau) \implies \tilde{x}(\tau) = \lim_{N\to\infty} \sum_{i=0}^{N} \epsilon^i \tilde{x}_i(\tau). \tag{5.1.32}$$

Applying to our dimensionless differential equation.

$$\begin{aligned}
\tilde{x}\frac{d^2\tilde{x}}{d\tau^2} - \tilde{x}^2\frac{d^2\tilde{x}}{d\tau^2} - \tilde{x}^2 + (2\Lambda+1)\,\tilde{x} - \Lambda = 0 \\
\sum_{i=0}^{N} \epsilon^i \tilde{x}_i \frac{d^2}{d\tau^2}\left(\sum_{j=0}^{N} \epsilon^j \tilde{x}_j\right) - \left(\sum_{k=0}^{N} \epsilon^k \tilde{x}_k\right)^2 \frac{d^2}{d\tau^2}\left(\sum_{\ell=0}^{N} \epsilon^\ell \tilde{x}_\ell\right) - \left(\sum_{m=0}^{N} \epsilon^m \tilde{x}_m\right)^2 \\
+ (2\Lambda+1)\left(\sum_{n=0}^{N} \epsilon^n \tilde{x}_n\right) - \Lambda = 0 \\
\sum_{i=0}^{N}\sum_{j=0}^{N} \epsilon^{i+j} \tilde{x}_i \frac{d^2\tilde{x}_j}{d\tau^2} - \left(\sum_{k=0}^{N} \epsilon^k \tilde{x}_k\right)^2 \left(\sum_{\ell=0}^{N} \epsilon^\ell \frac{d^2\tilde{x}_\ell}{d\tau^2}\right) - \left(\sum_{m=0}^{N} \epsilon^m \tilde{x}_m\right)^2 \\
+ (2\Lambda+1)\left(\sum_{n=0}^{N} \epsilon^n \tilde{x}_n\right) - \Lambda = 0.
\end{aligned} \tag{5.1.33}$$

5.1.7 First order approximation solution

Before to find the first order solution, we need to find the zero order solution. For that we need to balance all the terms that have ϵ^0 in our differential equation (equation 5.1.33) with the initial conditions. The equation tha we need to solve is:

$$\epsilon^{0+0}\tilde{x}_0\frac{d^2\tilde{x}_0}{d\tau^2} - \left(\epsilon^0\tilde{x}_0\right)^2\left(\epsilon^0\frac{d^2\tilde{x}_0}{d\tau^2}\right) - \left(\epsilon^0\tilde{x}_0\right)^2 + (2\Lambda+1)\left(\epsilon^0\tilde{x}_0\right) - \Lambda = 0. \tag{5.1.34}$$

It is important to highlight that x_0 is the constant term of the polynom, so the derivation of this term is zero and the equation is now:

$$-\tilde{x}_0^2 + (2\Lambda+1)\,\tilde{x}_0 - \Lambda = 0 \implies \boxed{\tilde{x}_0 = \frac{2\Lambda+1-\sqrt{4\Lambda^2+1}}{2} = \frac{x_{eq}}{h}}. \tag{5.1.35}$$

Now, we can continue to find the first order solution. For this order we need all the terms in our polinomic solution with ϵ^1. This terms are:

$$\begin{aligned}\tilde{x}_0\frac{d^2\tilde{x}_1}{d\tau^2}-\tilde{x}_0^2\frac{d^2\tilde{x}_1}{d\tau^2}-\underbrace{2\tilde{x}_0\tilde{x}_1\frac{d^2\tilde{x}_0}{d\tau^2}+\tilde{x}_1\frac{d^2\tilde{x}_0}{d\tau^2}}_{0}-2\tilde{x}_0\tilde{x}_1+2\Lambda\tilde{x}_1+\tilde{x}_1=0\\ (\tilde{x}_0-\tilde{x}_0^2)\frac{d^2\tilde{x}_1}{d\tau^2}+\tilde{x}_1(2\Lambda+1-2\tilde{x}_0)=0\\ \frac{d^2\tilde{x}_1}{d\tau^2}+\left(\frac{2\Lambda+1-2\tilde{x}_0}{\tilde{x}_0-\tilde{x}_0^2}\right)\tilde{x}_1=0.\end{aligned} \tag{5.1.36}$$

The last expression is the differential equation of a harmonic oscillator.

For simplify the calculus we define $\omega=\sqrt{\frac{2\Lambda+1-2\tilde{x}_0}{\tilde{x}_0-\tilde{x}_0^2}}$, which is the is the dimensionless frequency of the system. Using the value of $\tilde{x}_0$ the dimensionless frequency is:

$$\omega_0=\sqrt{4\Lambda+2\sqrt{1+4\Lambda^2}+\frac{1}{\Lambda}}. \tag{5.1.37}$$

Then, the differential equation is:

$$\frac{d^2\tilde{x}_1}{d\tau^2}+\omega^2\tilde{x}_1=0\implies\frac{d^2\tilde{x}_1}{d\tau^2}=-\omega^2\tilde{x}_1\implies\boxed{\tilde{x}_1(\tau)=C_1\cos(\omega\tau)+C_2\sin(\omega\tau)}. \tag{5.1.38}$$

As we can see for the last result is that ω is the dimensionless frequency of the system, so the frequency of oscillation is:

Now, using the solution of $\tilde{x}_1$ in the first order polinmic solution, we obtain that our solution has the next form:

$$\tilde{x}(\tau)\sim\tilde{x}_0+\tilde{x}_1(\tau)\epsilon=\tilde{x}_0+(C_1\cos(\omega\tau)+C_2\sin(\omega\tau))\,\epsilon \tag{5.1.39}$$

Using the initial conditions we can find the values of the constants C_1 and C_2. Starting with the position's initicial condition,

$$\begin{aligned}\tilde{x}(0)\sim\tilde{x}_0+(C_1\cos(0)+C_2\sin(0))\,\epsilon=\tilde{x}_0+C_1\epsilon\\ \frac{x_{eq}}{h}+\epsilon\sim\tilde{x}_0+C_1\epsilon=\frac{x_{eq}}{h}+C_1\epsilon\ \therefore\ \boxed{C_1=1}\end{aligned} \tag{5.1.40}$$

Continuing with the speed's initial condition.

$$\begin{aligned}\dot{\tilde{x}}(0)\sim\frac{d}{d\tau}\left[\tilde{x}_0+(C_1\cos(0)+C_2\sin(0))\,\epsilon\right]=0-(C_1\sin(0)-C_2\cos(0))\epsilon\\ 0\sim C_2\epsilon\ \therefore\ \boxed{C_2=0}\end{aligned} \tag{5.1.41}$$

Therefore, our solution at first order is:

$$\tilde{x}(\tau) \sim \tilde{x}_0 + \cos(\omega\tau)\epsilon \implies \boxed{\tilde{x}(\tau) \sim \frac{2\Lambda + 1 - \sqrt{4\Lambda^2+1}}{2} + \cos\left(\sqrt{4\Lambda + 2\sqrt{1+4\Lambda^2} + \frac{1}{\Lambda}}\tau\right)\epsilon}. \tag{5.1.42}$$

Now, plotting our approximation solution with the numerical solution, we can contrast the precision of our approximation. To contrast, we are going to plot three different epsilon values; they are 0.518034, 0.5, and 0.01.

We selected the number $\epsilon = 0.518034$ because we can not use a value where the distance between the free magnet and the lowest point ($\tilde{x} = 0$) will be bigger than 1, and remember that ϵ is the displacement that we pushed the free magnet out of the equilibrium point. Then, to obtain this specific epsilon value, we use $0.9 - \tilde{x}_0$.

If we see our figure (5.5), we can see that our approximation fits to the numerical solution when epsilon is very small.

5.1.8 Second order approximation solution

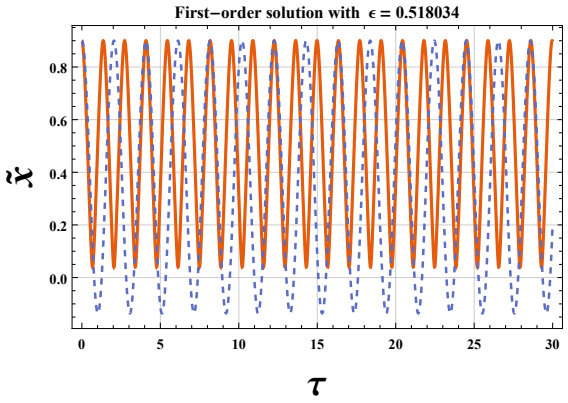
Now, for the second order we need only the terms with ϵ^2. The terms with power of 2 are:

$$\begin{aligned}\tilde{x}_0\frac{d^2\tilde{x}_2}{d\tau^2} - \tilde{x}_0^2\frac{d^2\tilde{x}_2}{d\tau^2} + \tilde{x}_1\frac{d^2\tilde{x}_1}{d\tau^2} - 2\tilde{x}_0\tilde{x}_1\frac{d^2\tilde{x}_1}{d\tau^2} - 2\tilde{x}_0\tilde{x}_2\frac{d^2\tilde{x}_0}{d\tau^2} \\ + \tilde{x}_2\frac{d^2\tilde{x}_0}{d\tau^2} + \tilde{x}_1^2\frac{d^2\tilde{x}_0}{d\tau^2} - 2\tilde{x}_0\tilde{x}_2 + 2\Lambda\tilde{x}_2 + \tilde{x}_2 - \tilde{x}_1^2 = 0\end{aligned} \tag{5.1.43}$$

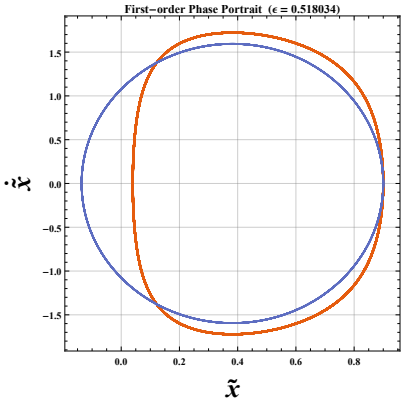
We know that $\tilde{x}_0$ is a constant, that lets us simplify the differential equation. Now, our equation has the next form:

$$\begin{aligned}\tilde{x}_0\frac{d^2\tilde{x}_2}{d\tau^2} - \tilde{x}_0^2\frac{d^2\tilde{x}_2}{d\tau^2} + \tilde{x}_1\frac{d^2\tilde{x}_1}{d\tau^2} - 2\tilde{x}_0\tilde{x}_1\frac{d^2\tilde{x}_1}{d\tau^2} - 2\tilde{x}_0\tilde{x}_2 + 2\Lambda\tilde{x}_2 + \tilde{x}_2 - \tilde{x}_1^2 &= 0 \\ \left(\tilde{x}_0 - \tilde{x}_0^2\right)\frac{d^2\tilde{x}_2}{d\tau^2} + \tilde{x}_1(-\omega^2\tilde{x}_1)\left(1 - 2\tilde{x}_0\right) + \tilde{x}_2\left(2\Lambda + 1 - 2\tilde{x}_0\right) - \tilde{x}_1^2 &= 0 \\ \left(\tilde{x}_0 - \tilde{x}_0^2\right)\frac{d^2\tilde{x}_2}{d\tau^2} - \tilde{x}_1^2\left(\omega_0^2 - 2\omega^2\tilde{x}_0 + 1\right) + \tilde{x}_2\left(2\Lambda + 1 - 2\tilde{x}_0\right) &= 0 \\ \frac{d^2\tilde{x}_2}{d\tau^2} - \tilde{x}_1^2\left(\frac{\omega^2 - 2\omega^2\tilde{x}_0 + 1}{\tilde{x}_0 - \tilde{x}_0^2}\right) + \tilde{x}_2\underbrace{\left(\frac{2\Lambda + 1 - 2\tilde{x}_0}{\tilde{x}_0 - \tilde{x}_0^2}\right)}_{\omega^2} &= 0 \\ \frac{d^2\tilde{x}_2}{d\tau^2} - \tilde{x}_1^2\left(\frac{\omega^2 - 2\omega^2\tilde{x}_0 + 1}{\tilde{x}_0 - \tilde{x}_0^2}\right) + \omega^2\tilde{x}_2 &= 0\end{aligned} \tag{5.1.44}$$

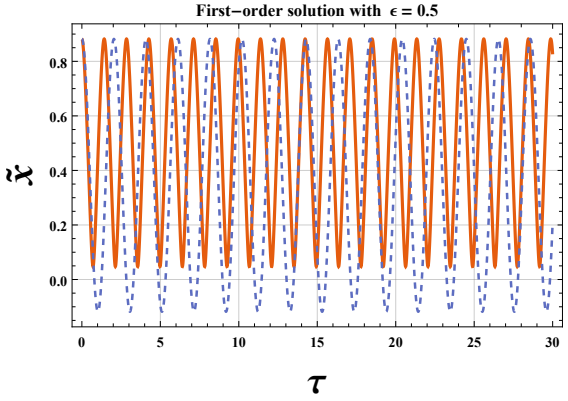
To simplify the differential equation, we define $\alpha^2 = \frac{\omega^2 - 2\omega^2\tilde{x}_0 + 1}{\tilde{x}_0 - \tilde{x}_0^2}$. Then our differential equation, using the definition of $\tilde{x}_1$ is:



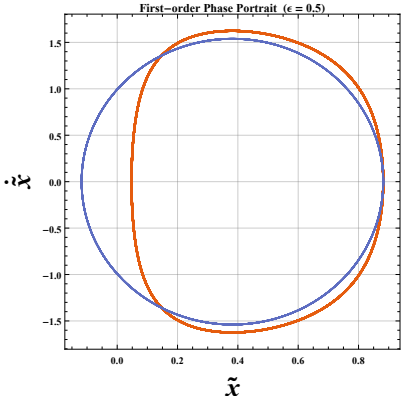
((a)) Time evolution contrast ($\epsilon = 0.518034$)



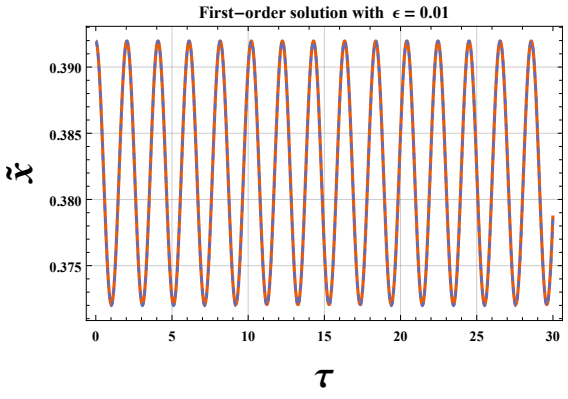
((b)) Phase portrait contrast ($\epsilon = 0.518034$)



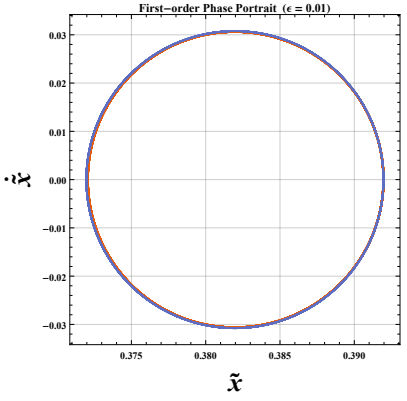
((c)) Time evolution ($\epsilon = 0.5$)



((d)) Phase portrait ($\epsilon = 0.5$)



((e)) Time evolution ($\epsilon = 0.01$)



((f)) Phase portrait ($\epsilon = 0.01$)

Figure 5.5: Approximation of the solution first order solution to many values of ϵ.

$$\frac{d^2\tilde{x}_2}{d\tau^2} - \alpha^2\tilde{x}_1^2 + \omega^2\tilde{x}_2 = 0$$

$$\frac{d^2\tilde{x}_2}{d\tau^2} - \alpha^2\cos^2(\omega\tau) + \omega^2\tilde{x}_2 = 0 \tag{5.1.45}$$

$$\therefore \boxed{\frac{d^2\tilde{x}_2}{d\tau^2} + \omega^2\tilde{x}_2 = \alpha^2\cos^2(\omega\tau)}$$

Now, we proceed to solve the differential equation. First we find the homogeneous solution.

$$\frac{d^2\tilde{x}_2}{d\tau^2} + \omega^2\tilde{x}_2 = 0 \implies \boxed{\tilde{x}_{2h}(\tau) = C_1\cos(\omega\tau) + C_2\sin(\omega\tau)} \tag{5.1.46}$$

The particular solution for our differential equation is.

$$\tilde{x}_{2p} = -\frac{\alpha^2\left[\cos(2\omega\tau) - 3\right]}{6\omega^2} \tag{5.1.47}$$

Therefore, the general solution for our differential equation.

$$\tilde{x}_2(\tau) = \tilde{x}_{2h} + \tilde{x}_{2p} = C_1\cos(\omega\tau) + C_2\sin(\omega\tau) - \frac{\alpha^2\left[\cos(2\omega\tau) - 3\right]}{6\omega^2} \tag{5.1.48}$$

Using the initial conditions to find the constants of the function. Applying the speed's initial condition.

$$\begin{aligned} \dot{\tilde{x}}(0) &= 0 \\ -C_1\omega_0\sin(0) + C_2\omega_0\cos(0) - \frac{\alpha^2\left[-2\omega\sin(0)\right]}{6\omega^2} &= 0 \\ \omega_0 C_2 &= 0 \\ \therefore \boxed{C_2 = 0}&. \end{aligned} \tag{5.1.49}$$

Exploiting the position's initial condition we obtain.

$$\begin{aligned} \tilde{x}(0) = \tilde{x}_{eq} + \epsilon + 0\epsilon^2 &\implies \tilde{x}_2 = 0 \\ \implies 0 = C_1\cos(0) - \frac{\alpha^2\left[\cos(0) - 3\right]}{6\omega^2} & \\ \implies 0 = C_1 + \frac{2\alpha^2}{6\omega^2} & \\ \therefore \boxed{C_1 = -\frac{\alpha^2}{3\omega^2}}&. \end{aligned} \tag{5.1.50}$$

Ergo our solution at second order now has the next form:

$$\tilde{x}_2(\tau) = -\frac{\alpha^2}{3\omega^2}\cos(\omega t) - \frac{\alpha^2\left[\cos(2\omega\tau) - 3\right]}{6\omega^2} = -\frac{\alpha^2}{3\omega_0^2}\left(\cos(\omega\tau) + \frac{1}{2}\left[\cos(2\omega\tau) - 3\right]\right), \tag{5.1.51}$$

to simplify our function we can use some trigonometric identities. Then the function for second-order is:

$$\tilde{x}_2(\tau) = \frac{2\alpha^2}{3\omega_0^2}\sin^2\left(\frac{\omega\tau}{2}\right)(2 + \cos(\omega\tau)) \tag{5.1.52}$$

Therefore, our approximated solution is:

$$\tilde{x}(\tau) \sim \tilde{x}_0 + \tilde{x}_1\epsilon + \tilde{x}_2\epsilon^2 = \tilde{x}_{eq} + \cos(\omega\tau)\epsilon + \frac{2\alpha^2\epsilon^2}{3\omega_0^2}\sin^2\left(\frac{\omega\tau}{2}\right)(2+\cos(\omega\tau)) \tag{5.1.53}$$

As the same as the first order approximation solution, we can plot our second order approximation series with the numerical solution. And it is not a surprise that this approximation is better than the first order. We can see it in the biggest epsilon value, because in the first order (5.5) the approximation passes the width of the numerical solution, but in our second order our approximation does not pass it.

5.2 System with friction

The friction system includes the dissipative forces, in the other hand the external forces does not interact with the physical system. With this, the forces in the system are:

$$F = -F_f - F_g + F_M = -\beta v - mg + \frac{M}{x} - \frac{M}{h-x}, \tag{5.2.1}$$

where F is the force, v is the speed of the free magnet, β is the friction coefficient, m is the mass of the free magnet, g the gravitational acceleration, x the distance between the two magnets and M is the magnetic energy constant. If the system is in equilibrium, then the net force and the speed movement of the free magnet are cero, then we can find the equilibrium distance.

$$\begin{aligned} \frac{M}{x} - mg - \frac{M}{h-x} = 0 \implies M(h-x) &= mgx(h-x) + Mx \\ \therefore \boxed{x_{eq} = \frac{2M + mgh \pm \sqrt{4M^2 + m^2g^2h^2}}{2mg}} &\quad . \end{aligned} \tag{5.2.2}$$

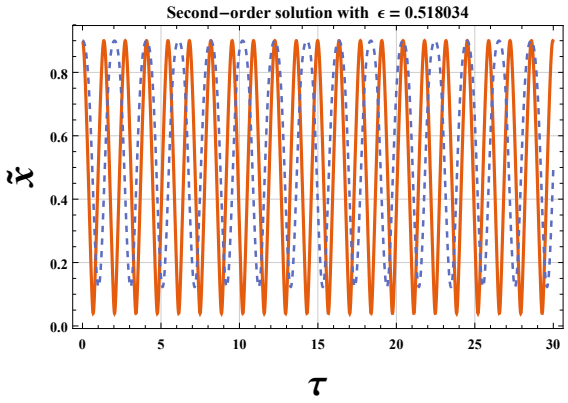
It is important to see that the equilibrium position requieres an static position, this is because our system is unstable for the presence of the friction. The equilibrium position that we obtained is the same that we found to the ideal case (5.1.4).

5.2.1 Lagrangian analysis

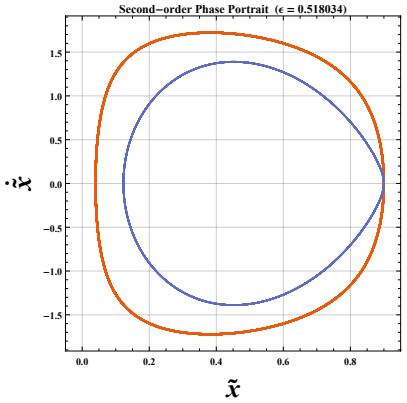
As we have shown, the lagrangian analysis let us know the equations of motion that we need to solve to described all the dynamic of the system. The lagrangian of our system only have a degree of freedom, so the dynamic of the system is described by one coordinate (x), then our lagrangian has the following expression.

$$L = \frac{1}{2}m\dot{x}^2 - V(x), \tag{5.2.3}$$

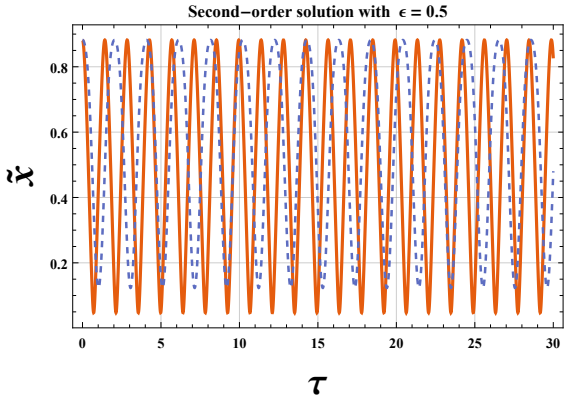
where $V(x)$ is the potential energy in our system. To find the potential energy we going to use the relation between the work and the potential energy when the system is conservative, as we calculated in the equation (5.1.7). Then, the potential energy is.



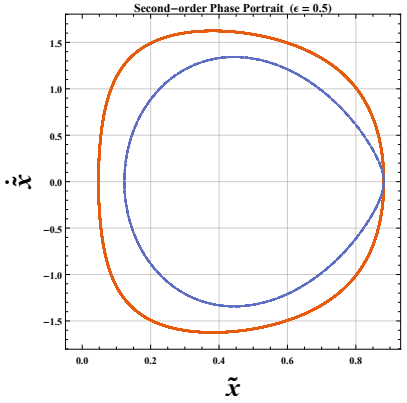
((a)) Time evolution contrast ($\epsilon = 0.518034$)



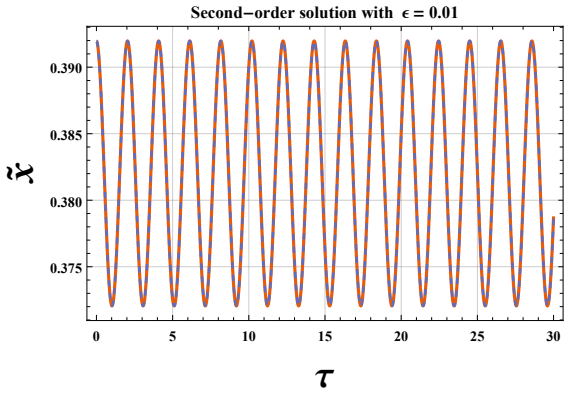
((b)) Phase portrait contrast ($\epsilon = 0.518034$)



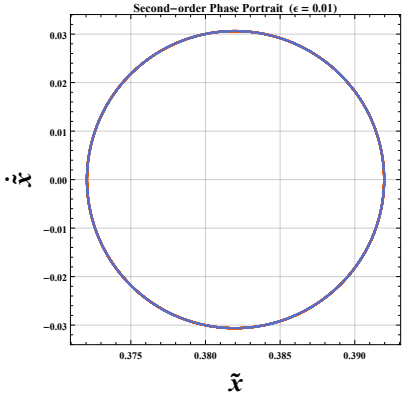
((c)) Time evolution ($\epsilon = 0.5$)



((d)) Phase portrait ($\epsilon = 0.5$)



((e)) Time evolution ($\epsilon = 0.01$)



((f)) Phase portrait ($\epsilon = 0.01$)

Figure 5.6: Approximation of the solution second order solution to many values of ϵ.

$$V(x) = mgx - M\ln(x) - M\ln(h - x). \tag{5.2.4}$$

Introducing our potential energy in the lagrangian, we obtain:

$$\boxed{L = \frac{1}{2}m\dot{x}^2 - mgx + M\ln(x) + M\ln(h-x)}. \tag{5.2.5}$$

With our lagrangian, we can applied the Euler-Lagrange equation to find the equation of motion of our system. As we know, our system present an external force that is the friction force ($Q = F_f$). So the equation of motion for the coordinate x is.

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} &= Q \\ \implies m\ddot{x} + mg + \frac{M}{h-x} - \frac{M}{x} &= -\beta v \\ \implies \ddot{x} + \frac{\beta}{m}\dot{x} + g + \frac{M}{m(h-x)} - \frac{M}{mx} &= 0 \\ \therefore \quad \boxed{\frac{d^2x}{dt^2} + \frac{\beta}{m}\frac{dx}{dt} + g + \frac{M}{m}\left[\frac{1}{h-x} - \frac{1}{x}\right] = 0}. & \end{aligned} \tag{5.2.6}$$

This equation of motion is relevant because it lets us know the accelerations that generate the movement in our system. It's important to remark that the differential equation does not have an analytical solution.

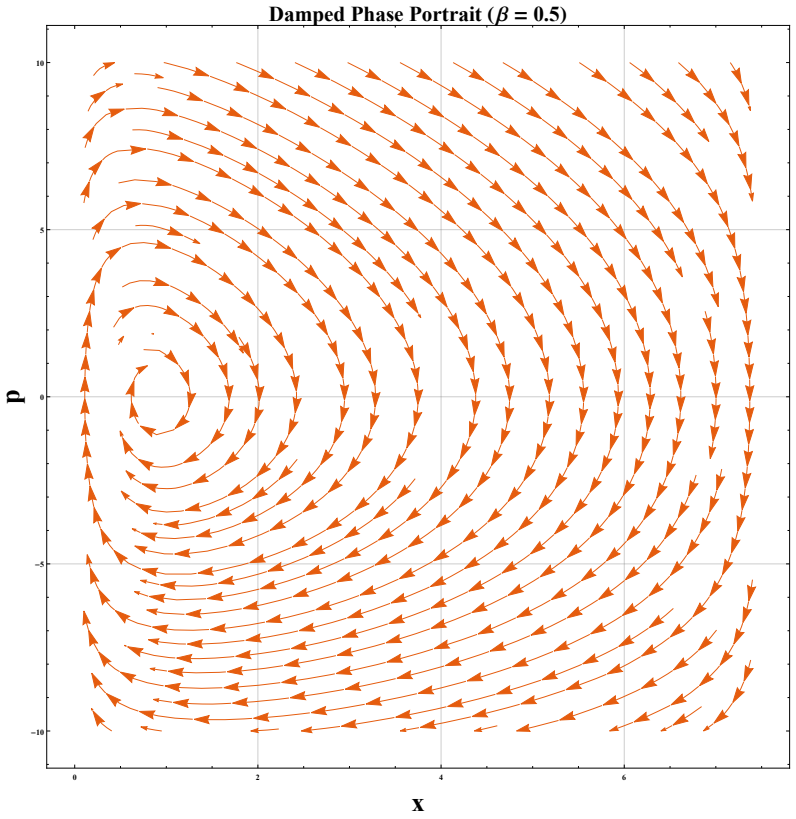


Figure 5.7: Phase portrait of the system with $M = 50$, $\beta = 0.5$, $h = 5$, $g = 10$ and $m = 1$

If we use our equation of motion, we can plot the Phase-portrait of this system to know the behavior of our system. We can see in the figure (5.2), the system follows a closed cyclical trajectory, so we can confirm that the system oscillates.

5.2.2 Dimensional analysis

In our system we have seven variables ($n = 8$), these variables are the mass (m), the time (t), the position (x), the gravity (g), the oscillation amplitude (A), the magnetic energy (M), (β) the friction coefficient and the distance between the fixed magnets (h). For the nature of our problem, we only need to use three dimension ($j = 3$) to caracterize all the problem, the dimensions that we need to use is the matter, the longitude and the time. Using the Buckingham's theorem, we know that there are four dimensionless groups ($k = n - j = 5$) that describe our system.

Using a dimensional analysis over every variable:

$$[x] = L, [h] = L, [A] = L, [t] = T, [m] = M, [g] = LT^{-2}, [M] = ML^2T^{-2}, [\beta] = MT^{-1} \tag{5.2.7}$$

For this system we going to use h, g and M as dimensional variables, this because h is the geometric parameter of the system, g is the cinematic parameter and M is the dynamic parameter. As we saw in the previous chapter, every dimensinal group is the product of the dimensional variables, so we have that the ith group is expressed as:

$$\Pi_i = \delta h^{\alpha} g^{\beta} M^{\lambda}, \tag{5.2.8}$$

where δ is the variable that we want to dimensionless, in our system we have the variables x, t, m and A; α, β and λ are the powers of the distance h, the gravity and the magnetic energy that dimensionless the ith group.

To find the powers of our dimensional variables that dimensionless every group we can construct an equations system, this system is.

$$\begin{pmatrix} L \\ M \\ T \end{pmatrix} + \begin{pmatrix} 1 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & -2 & -2 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \lambda \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \implies \begin{pmatrix} \alpha \\ \beta \\ \lambda \end{pmatrix} = \begin{pmatrix} M - L - \frac{T}{2} \\ M + \frac{T}{2} \\ -M. \end{pmatrix} \tag{5.2.9}$$

It is important to remember that every dimensionless group can be expressed as a fraction. So the dimensionless groups can be rewrite as:

$$\Pi_i = \frac{\delta}{\delta_s} = \tilde{\delta}, \tag{5.2.10}$$

where δ_s is the scale factor of the variable δ y $\tilde{\delta}$ is the dimensionless form of the variable δ. Now, we can obtain the dimensionless groups for the variables x, t, m and A.

Dimensionless group of the position

The x's group is:

$$\begin{pmatrix} \alpha \\ \beta \\ \lambda \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} \quad \therefore \quad \boxed{\Pi_1 = \frac{x}{x_s} = \frac{x}{h} \implies x_s = h} \quad . \tag{5.2.11}$$

Con el resultado anterior se puede observar que el factor de escala de posición es la distancia de equilibrio y la distancia se puede expresar como el productl de la distancia adimensional y el factor de escala, de tal manera que se obtiene.

$$x = x_s\tilde{x} \implies dx = x_s d\tilde{x} \implies d^2x = x_s d^2\tilde{x} \tag{5.2.12}$$

Dimensionless group of the time

The group of t is.

$$\begin{pmatrix}\alpha\\ \beta\\ \lambda\end{pmatrix} = \begin{pmatrix}-\frac{1}{2}\\ \frac{1}{2}\\ 0\end{pmatrix} \quad \therefore \quad \boxed{\Pi_2 = \frac{t}{t_s} = t\sqrt{\frac{g}{h}} \implies t_s = \sqrt{\frac{h}{g}}} \quad . \tag{5.2.13}$$

Del resultado anterior se puede encontrar la expresión de los factores de escala de la frecuencia y el periodo de oscilación, dichas expresiones son las siguientes.

$$\omega_s = \frac{1}{t_s} = \sqrt{\frac{g}{h}} \quad \wedge \quad T_s = \frac{2\pi}{\omega_s} = 2\pi\sqrt{\frac{h}{g}}. \tag{5.2.14}$$

Lo importante a observar de las expresiones anteriores es que son equivalentes a la frecuencia y periodo de un péndulo, por lo que podemos concluir que nuestro sistema es análogo a un movimiento pendular.

Ahora, utilizando la relación entre la variable del sistema y su factor de escala y la variable adimensional, se obtiene.

$$t = t_s\tau \implies dt = t_s d\tau \implies dt^2 = t_s^2 d\tau^2 \tag{5.2.15}$$

Dimensionless group of the mass and the Energie's number

Now, the group of m:

$$\begin{pmatrix}\alpha\\ \beta\\ \lambda\end{pmatrix} = \begin{pmatrix}1\\ 1\\ -1\end{pmatrix} \quad \therefore \quad \boxed{\Pi_3 = \frac{m}{m_s} = \frac{mgh}{M} \implies m_s = \frac{M}{gh}} \quad . \tag{5.2.16}$$

This dimensionless group is very important because give us a relation between the magnetic energy and the mechanic potential energy, Using this dimensionless group we going to define Energie's number as:

$$\Lambda \equiv \frac{1}{\Pi_3} = \frac{M}{mgh}. \tag{5.2.17}$$

The Energie's number going to be so useful because it related all the variables that are not affected by time. An example of some quantity that is not affected by time is the equilibrium distance, so we can rewrite this number interm of the Energie's number as:

$$
\begin{aligned}
x_{eq} &= \frac{2M + mgh - \sqrt{4M^2 + m^2g^2h^2}}{2mg} \\
&= \frac{mgh\left(\frac{2M}{mgh} + 1\right) - \sqrt{m^2g^2h^2\left(\frac{4M^2}{m^2g^2h^2} + 1\right)}}{2mg} \\
&= \frac{mgh\left(\frac{2M}{mgh} + 1 - \sqrt{\frac{4M^2}{m^2g^2h^2} + 1}\right)}{2mg} \\
&= \frac{h}{2}\left(2\Lambda + 1 - \sqrt{4\Lambda^2 + 1}\right) \\
\implies x_{eq} &= h\left(\frac{2\Lambda + 1 - \sqrt{4\Lambda^2 + 1}}{2}\right) \\
\therefore \boxed{\tilde{x}_{eq} = \frac{x_{eq}}{h} = \frac{2\Lambda + 1 - \sqrt{4\Lambda^2 + 1}}{2}}&.
\end{aligned}
\tag{5.2.18}
$$

Dimensionless group of the amplitude

Obtaining the group of A.

$$
\begin{pmatrix} \alpha \\ \beta \\ \lambda \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} \quad \therefore \quad \boxed{\Pi_4 = \frac{A}{A_s} = \frac{A}{h} \implies A_s = h} \quad . \tag{5.2.19}
$$

This result lets us see that the amplitude of oscillation is a fraction of the distance between the fixed magnets. The amplitude does not have a role in the motion equation (equation 5.2.6) so we going to use this dimensionless group as the control number (ϵ) for our approximation.

$$
\epsilon = \frac{A}{h} = \tilde{A}. \tag{5.2.20}
$$

5.2.3 Dimensionless system

Now that we have our dimensionless groups, we will use them to find the dimensionless equations for our system. First, we are going to find the dimensionless form of the equation of motion. So using the groups Π_1, Π_2 and Π_3 in the equation (5.2.6) we obtain.

$$
\begin{aligned}
\frac{d^2x}{dt^2} + \frac{\beta}{m}\frac{dx}{dt} + g + \frac{M}{m}\left[\frac{1}{h-x} - \frac{1}{x}\right] &= 0 \\
\implies \frac{d^2(x_s\tilde{x})}{d(t_s\tau)^2} + \frac{\beta}{m}\frac{d(x_s\tilde{x})}{d(t_s\tau)} + g + \frac{M}{m}\left[\frac{1}{h-x_s\tilde{x}} - \frac{1}{x_s\tilde{x}}\right] &= 0 \\
\implies \frac{h}{\frac{h}{g}}\frac{d^2\tilde{x}}{d\tau^2} + \frac{\beta h}{m\sqrt{\frac{h}{g}}}\frac{d\tilde{x}}{d\tau} + g + \frac{M}{m}\left[\frac{1}{h-h\tilde{x}} - \frac{1}{h\tilde{x}}\right] &= 0 \\
\implies g\frac{d^2\tilde{x}}{d\tau^2} + \frac{\beta}{m}\sqrt{hg}\frac{d\tilde{x}}{d\tau} + g + \frac{M}{mh}\left[\frac{1}{1-\tilde{x}} - \frac{1}{\tilde{x}}\right] &= 0 \\
\implies \frac{d^2\tilde{x}}{d\tau^2} + \frac{\beta}{m}\sqrt{\frac{h}{g}}\frac{d\tilde{x}}{d\tau} + 1 + \frac{M}{mgh}\left[\frac{1}{1-\tilde{x}} - \frac{1}{\tilde{x}}\right] &= 0 \\
\implies \boxed{\frac{d^2\tilde{x}}{d\tau^2} + 2\Phi\frac{d\tilde{x}}{d\tau} + 1 + \Lambda\left[\frac{1}{1-\tilde{x}} - \frac{1}{\tilde{x}}\right] = 0},
\end{aligned}
\tag{5.2.21}
$$

this differential equation is so relevant because it lets us see that the position of the free magnet only depends on the time and the Energie's number, then x is a function of τ and Λ.

Now, doing some algebra, we can reformulate the equation of motion as:

$$
\begin{aligned}
\frac{d^2\tilde{x}}{d\tau^2} + 2\Phi\frac{d\tilde{x}}{d\tau} + 1 + \Lambda\left[\frac{1}{1-\tilde{x}} - \frac{1}{\tilde{x}}\right] &= 0 \\
\implies \frac{d^2\tilde{x}}{d\tau^2} + 2\Phi\frac{d\tilde{x}}{d\tau} + 1 + \Lambda\left[\frac{2\tilde{x}-1}{\tilde{x}(1-\tilde{x})}\right] &= 0 \\
\implies \tilde{x}(1-\tilde{x})\left(\frac{d^2\tilde{x}}{d\tau^2} + 2\Phi\frac{d\tilde{x}}{d\tau} + 1 + \Lambda\left[\frac{2\tilde{x}-1}{\tilde{x}(1-\tilde{x})}\right]\right) &= 0 \\
\implies \tilde{x}(1-\tilde{x})\frac{d^2\tilde{x}}{d\tau^2} + 2\Phi\tilde{x}(1-\tilde{x})\frac{d\tilde{x}}{d\tau} + \tilde{x}(1-\tilde{x}) + \Lambda\left(2\tilde{x}-1\right) &= 0 \\
\therefore \boxed{\tilde{x}\frac{d^2\tilde{x}}{d\tau^2} - \tilde{x}^2\frac{d^2\tilde{x}}{d\tau^2} + 2\Phi\tilde{x}\frac{d\tilde{x}}{d\tau} - 2\Phi\tilde{x}^2\frac{d\tilde{x}}{d\tau} - \tilde{x}^2 + (2\Lambda+1)\,\tilde{x} - \Lambda = 0}&.
\end{aligned}
\tag{5.2.22}
$$

As we can see in this result, the differential equation is non lineal and is impossible to solve it in an anallytical way, so we going to use a numerical solution and an approximation solution.

The next step is to dimensionless the initial conditions of the system, so using the dimensionless groups we obtain the following initial conditions.

$$
\begin{aligned}
x(0) = x_{eq} + A \implies h\tilde{x}(0) = x_{eq} + A \implies \tilde{x}(0) &= \frac{x_{eq}+A}{h} \\
\implies \boxed{\tilde{x}(0) = \frac{x_{eq}}{h} + \epsilon \therefore \tilde{x}(0) = \frac{2\Lambda + 1 - \sqrt{4\Lambda^2+1}}{2} + \epsilon.} & \\
\dot{x}(0) = 0 \implies h\dot{\tilde{x}}(0) = 0 \implies \boxed{\dot{\tilde{x}}(0) = 0} &
\end{aligned}
\tag{5.2.23}
$$

Now, rescaling the domain of the dimensionless amplitude (ϵ) system.

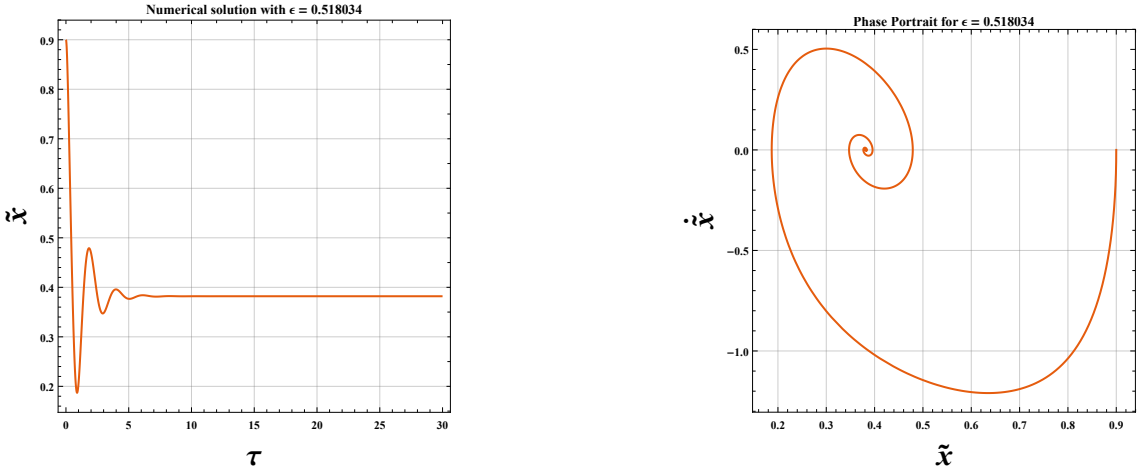
$$
\begin{aligned}
0 = x_{eq} + A \implies A = -x_{eq} \implies \epsilon = -\frac{x_{eq}}{h} = -\frac{2\Lambda + 1 - \sqrt{4\Lambda^2 + 1}}{2} \\
h = x_{eq} + A \implies A = h - x_{eq} \implies \epsilon = 1 - \frac{x_{eq}}{h} = 1 - \frac{2\Lambda + 1 - \sqrt{4\Lambda^2 + 1}}{2}
\end{aligned} \tag{5.2.24}
$$

$$
\therefore \quad \boxed{-\frac{x_{eq}}{h} < \epsilon < 1 - \frac{x_{eq}}{h} \implies -\frac{2\Lambda + 1 - \sqrt{4\Lambda^2 + 1}}{2} < \epsilon < \frac{1 - 2\Lambda + \sqrt{4\Lambda^2 + 1}}{2}}
$$

5.2.4 Numerical solution

Once again, we use the Runge-Kutta method to analyse the evolution of our system. In this numerical simulation, we use the following values: $\Lambda = 1$ and $\Phi = \frac{1}{2}$.

The presence of the friction number indicates to us that the amplitude and speed of our system will decrease over time. And this can be shown with the spiral movement that appears in the phase portrait.



((a)) Time evolution

((b)) Phase portrait

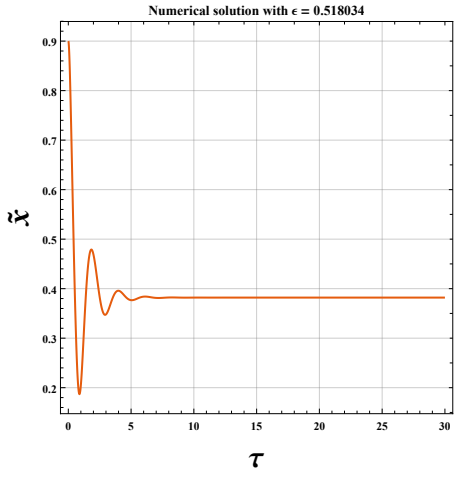
Figure 5.8: Plot and parametric plot of the numerical solution with $\epsilon = 0.518034$

We can see the characteristic spiral movement that a damped oscillator has. To see how the system changes when epsilon changes its values, we plot our system with three different values, those values are: 0.518034, 0.5, and 0.01.

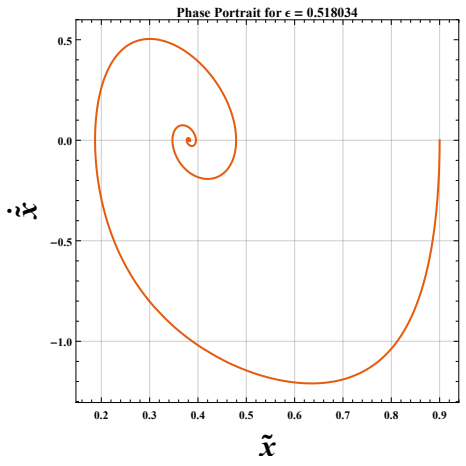
Observing three behaviors, we can see that all have the same behavior but with different amplitudes. This tells us that the friction number generates a homogeneous behavior without relevance to the epsilon value, with the only difference in the amplitude. This can be shown in the figure (5.9).

5.2.5 Asymptotic approximation

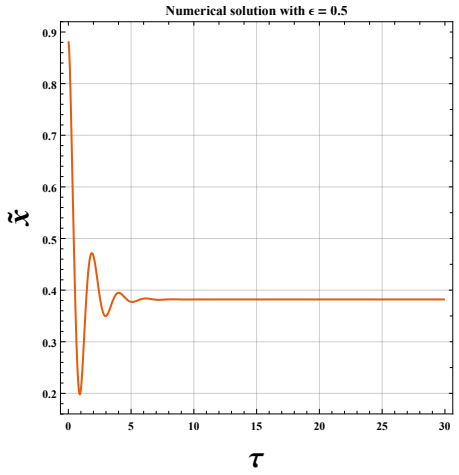
As we saw in the previous chapter, the asymptotic approximation consists in regulate the apport of every term in a polynomial with the intetion to approximate that polynomial to the anallytical solution. When the polynomial has infinity terms, the polynomial is equal to the solution.So our solution has the form:



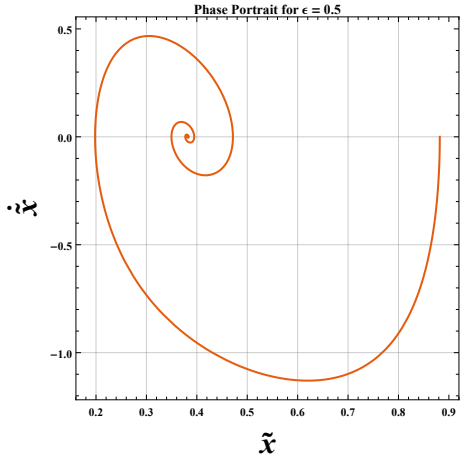
((a)) Time evolution ($\epsilon = 0.518034$)



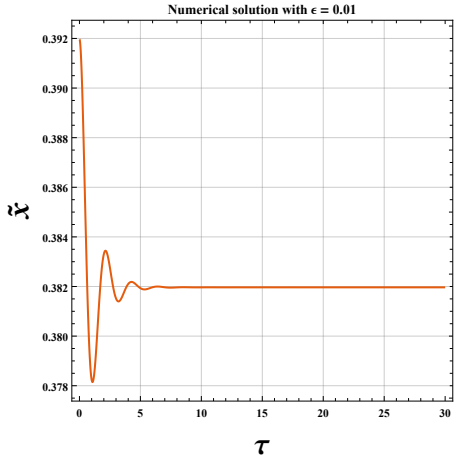
((b)) Phase portrait ($\epsilon = 0.518034$)



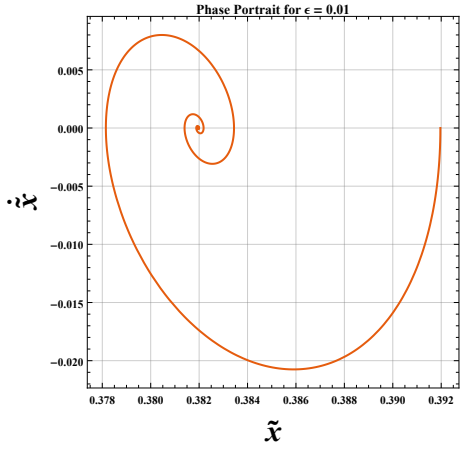
((c)) Time evolution ($\epsilon = 0.5$)



((d)) Phase portrait ($\epsilon = 0.5$)



((e)) Time evolution ($\epsilon = 0.01$)



((f)) Phase portrait ($\epsilon = 0.01$)

Figure 5.9: Time evolution and phase portraits of the system for $\epsilon = 0.518034$, $\epsilon = 0.5$, and $\epsilon = 0.01$.

$$x(t) \sim \sum_{i=0}^{N} \epsilon^i x_i(t) \implies x(t) = \lim_{N\to\infty} \sum_{i=0}^{N} \epsilon^i x_i(t). \tag{5.2.25}$$

Applying this to the dimensionless form we obtain.

$$\tilde{x}(\tau) \sim \sum_{i=0}^{N} \epsilon^i \tilde{x}_i(\tau) \implies \tilde{x}(\tau) = \lim_{N\to\infty} \sum_{i=0}^{N} \epsilon^i \tilde{x}_i(\tau). \tag{5.2.26}$$

Applying to our dimensionless differential equation.

$$\begin{aligned}
&\tilde{x}\frac{d^2\tilde{x}}{d\tau^2} - \tilde{x}^2\frac{d^2\tilde{x}}{d\tau^2} + 2\Phi\tilde{x}\frac{d\tilde{x}}{d\tau} - 2\Phi\tilde{x}^2\frac{d\tilde{x}}{d\tau} - \tilde{x}^2 + (2\Lambda+1)\,\tilde{x} - \Lambda = 0 \\
&\qquad \sum_{i=0}^{N}\sum_{j=0}^{N} \epsilon^{i+j}\tilde{x}_i\frac{d^2\tilde{x}_j}{d\tau^2} - \left(\sum_{k=0}^{N}\epsilon^k\tilde{x}_k\right)^2\left(\sum_{\ell=0}^{N}\epsilon^\ell\frac{d^2\tilde{x}_\ell}{d\tau^2}\right) \\
&+2\Phi\left[\sum_{m=0}^{N}\sum_{n=0}^{N}\epsilon^{m+n}\tilde{x}_m\frac{d\tilde{x}_n}{d\tau} - \left(\sum_{k=0}^{N}\epsilon^p\tilde{x}_p\right)^2\left(\sum_{q=0}^{N}\epsilon^q\frac{d\tilde{x}_q}{d\tau}\right)\right] \\
&\qquad - \left(\sum_{r=0}^{N}\epsilon^r\tilde{x}_r\right)^2 + (2\Lambda+1)\left(\sum_{s=0}^{N}\epsilon^s\tilde{x}_s\right) - \Lambda = 0
\end{aligned} \tag{5.2.27}$$

As we can see, our approximation series is long and complex. But we can work with it. Now, we are going to find the first and second order solutions.

To simplify, we do some algebra.

$$\begin{aligned}
\left(\tilde{x}-\tilde{x}^2\right)\left(\frac{d^2\tilde{x}}{d\tau^2} + 2\Phi\frac{d\tilde{x}}{d\tau} + 1\right) + \Lambda\left(2\tilde{x}-1\right) = 0 \\
\left(\sum_{i=0}^{N}\epsilon^i\tilde{x}_i - \left(\sum_{j=0}^{N}\epsilon^j\tilde{x}_j\right)^2\right)\left(\sum_{k=0}^{N}\epsilon^k\frac{d^2\tilde{x}_k}{d\tau^2} + 2\Phi\sum_{\ell=0}^{N}\epsilon^\ell\frac{d\tilde{x}_\ell}{d\tau} + 1\right) + \Lambda\left(2\sum_{m=0}^{N}\epsilon^m\tilde{x}_m - 1\right) = 0
\end{aligned} \tag{5.2.28}$$

5.2.6 First order approximation solution

Before to find the first order solution, we need to find the zero order solution. For that we need to balance all the terms that have ϵ^0 in our differential equation (equation 5.2.28) with the initial conditions. The equation tha we need to solve is:

$$\left(\epsilon^0\tilde{x}_0 - \left(\epsilon^0\tilde{x}_0\right)^2\right)\left(\epsilon^0\frac{d^2\tilde{x}_0}{d\tau^2} + 2\Phi\epsilon^0\frac{d\tilde{x}_0}{d\tau} + 1\right) + \Lambda\left(2\epsilon^0\tilde{x}_0 - 1\right) = 0 \tag{5.2.29}$$

It is important to highlight that x_0 is the constant term of the polynom, so the derivation of this term is zero and the equation is now:

$$-\tilde{x}_0^2 + (2\Lambda + 1)\,\tilde{x}_0 - \Lambda = 0 \implies \boxed{\tilde{x}_0 = \frac{2\Lambda + 1 - \sqrt{4\Lambda^2 + 1}}{2} = \frac{x_{eq}}{h}}. \tag{5.2.30}$$

Now, with the constant term, we can continue to find the first-order solution. For this order, we need all the terms in our polynomial solution with ϵ^1. These terms are:

$$\begin{aligned}
\left(\epsilon^0\tilde{x}_0 + \epsilon^1\tilde{x}_1 - \left(\epsilon^0\tilde{x}_0 + \epsilon^1\tilde{x}_1\right)^2\right)\left(\epsilon^0\frac{d^2\tilde{x}_0}{d\tau^2} + \epsilon^1\frac{d^2\tilde{x}_1}{d\tau^2} + 2\Phi\left(\epsilon^0\frac{d\tilde{x}_0}{d\tau} + \epsilon^1\frac{d\tilde{x}_1}{d\tau}\right) + 1\right)& \\
+\Lambda\left(2\epsilon^0\tilde{x}_0 + 2\epsilon^1\tilde{x}_1 - 1\right) = 0& \\
\left(\epsilon^0\tilde{x}_0 + \epsilon^1\tilde{x}_1 - \epsilon^0\tilde{x}_0^2 - 2\epsilon^1\tilde{x}_0\tilde{x}_1\right)\left(\underbrace{\epsilon^0\frac{d^2\tilde{x}_0}{d\tau^2}}_{0} + \epsilon^1\frac{d^2\tilde{x}_1}{d\tau^2} + 2\Phi\left(\underbrace{\epsilon^0\frac{d\tilde{x}_0}{d\tau}}_{0} + \epsilon^1\frac{d\tilde{x}_1}{d\tau}\right) + 1\right)& \\
+\Lambda\left(2\epsilon^1\tilde{x}_1\right) = 0& \\
\left(\epsilon^0\tilde{x}_0 + \epsilon^1\tilde{x}_1 - \epsilon^0\tilde{x}_0^2 - 2\epsilon^1\tilde{x}_0\tilde{x}_1\right)\left(\epsilon^1\frac{d^2\tilde{x}_1}{d\tau^2} + 2\Phi\epsilon^1\frac{d\tilde{x}_1}{d\tau} + 1\right) + \Lambda\left(2\epsilon^1\tilde{x}_1\right) = 0& \\
\left(\tilde{x}_0 - \tilde{x}_0^2\right)\left(\frac{d^2\tilde{x}_1}{d\tau^2} + 2\Phi\frac{d\tilde{x}_1}{d\tau}\right) + (2\Lambda + 1 - \tilde{x}_0)\,\tilde{x}_1 = 0& \\
\frac{d^2\tilde{x}_1}{d\tau^2} + 2\Phi\frac{d\tilde{x}_1}{d\tau} + \left(\frac{2\Lambda + 1 - \tilde{x}_0}{\tilde{x}_0 - \tilde{x}_0^2}\right)\tilde{x}_1 = 0&
\end{aligned} \tag{5.2.31}$$

The last expression is the differential equation of a damped oscillator, where the natural frequency is $\omega_0 = \sqrt{\frac{2\Lambda+1-2\tilde{x}_0}{\tilde{x}_0-\tilde{x}_0^2}}$, which is the dimensionless natural frequency of the system. Using the value of $\tilde{x}_0$ the dimensionless frequency is:

$$\omega_0 = \sqrt{4\Lambda + 2\sqrt{1 + 4\Lambda^2} + \frac{1}{\Lambda}}. \tag{5.2.32}$$

Then, the differential equation is:

$$\frac{d^2\tilde{x}_1}{d\tau^2} + 2\Phi\frac{d\tilde{x}_1}{d\tau} + \omega_0^2\tilde{x}_1 = 0 \implies \boxed{\tilde{x}_1(\tau) = e^{-\Phi\tau}\left(C_1 e^{\tau\sqrt{\Phi-\omega_0}} + C_2 e^{-\tau\sqrt{\Phi-\omega_0}}\right)}. \tag{5.2.33}$$

Something important to see is that the behavior depends of Φ and ω_0

If we use Euler's equation, we can transform our solution with exponentials into a trigonometric solution. Then, we obtain

$$\boxed{\tilde{x}_1 = e^{-\Phi t}\left[C_1\cos(\Delta t) + C_2\sin(\Delta t)\right]}. \tag{5.2.34}$$

, where Δ is $\Delta = \sqrt{\omega_0^2 - \Phi^2}$. Now, using the solution of $\tilde{x}_1$ in the first order polynomial solution, we obtain that our solution has the following form:

$$\tilde{x}(\tau) \sim \tilde{x}_0 + \tilde{x}_1(\tau)\epsilon = \tilde{x}_0 + e^{-\Phi t}\left[C_1\cos(\Delta t) + C_2\sin(\Delta t)\right]\epsilon. \tag{5.2.35}$$

Using the initial conditions, we can find the values of the constants C_1 and C_2. Starting with the position's initial condition,

$$\begin{aligned} \tilde{x}(0) &\sim \tilde{x}_0 + (C_1 \cos(0) + C_2 \sin(0))\,\epsilon = \tilde{x}_0 + C_1\epsilon \\ \frac{x_{eq}}{h} + \epsilon &\sim \tilde{x}_0 + C_1\epsilon = \frac{x_{eq}}{h} + C_1\epsilon \;\therefore\; \boxed{C_1 = 1} \end{aligned} \tag{5.2.36}$$

Continuing with the speed's initial condition.

$$\begin{aligned} \dot{\tilde{x}}(0) &\sim \frac{d}{d\tau}\left[\tilde{x}_0 + (C_1 \cos(0) + C_2 \sin(0))\,\epsilon\right] \\ 0 &\sim -e^{-\Phi\tau}\left(\Phi - \Delta C_2\right)\epsilon \;\therefore\; \boxed{C_2 = \frac{\Phi}{\Delta}} \end{aligned} \tag{5.2.37}$$

Therefore, our solution at first order is:

$$\begin{aligned} &\tilde{x}(\tau) \sim \tilde{x}_0 + \cos(\omega\tau)\epsilon \\ \Longrightarrow &\boxed{\tilde{x}(\tau) \sim \frac{2\Lambda + 1 - \sqrt{4\Lambda^2 + 1}}{2} + e^{-\Phi t}\left[\cos(\Delta t) + \frac{\Phi}{\Delta}\sin(\Delta t)\right]\epsilon}. \end{aligned} \tag{5.2.38}$$

Now, we plot our numerical solution with our first-order approximation solution to contrast them. We decide to use three different values to epsilon, they are $\epsilon = 0.518034$, $\epsilon = 0.5$ and $\epsilon = 0.01$.

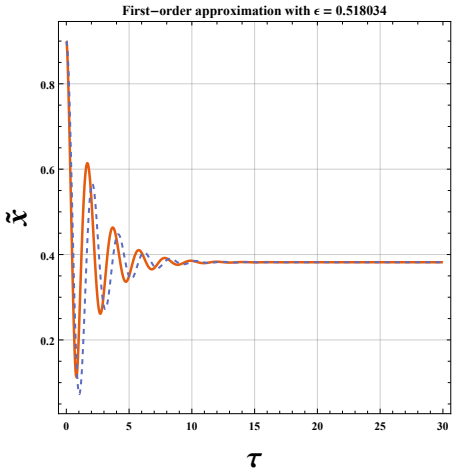
Observing the figure (5.10) we can see that our approximated solution increases its precision when epsilon is small, but is a good first approximation and has the same behavior than the numerical solution.

5.2.7 Second order approximation solution

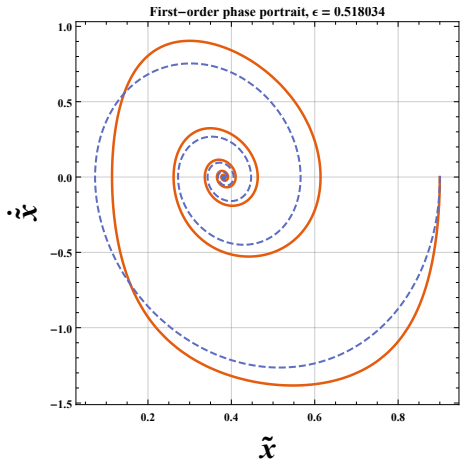
Now, for the second order we need only the terms with ϵ^2. The terms with the power of 2 are:

$$\begin{aligned} &\left(\tilde{x}_0 - \tilde{x}_0^2\right)\left(\frac{d^2\tilde{x}_2}{d\tau^2} + 2\Phi\frac{d\tilde{x}_2}{d\tau}\right) + \left(\tilde{x}_1 - 2\tilde{x}_0\tilde{x}_1\right)\left(\frac{d^2\tilde{x}_1}{d\tau^2} + 2\Phi\frac{d\tilde{x}_1}{d\tau}\right) \\ &+ \left(\tilde{x}_2 - 2\tilde{x}_0\tilde{x}_2 - \tilde{x}_1^2\right)\left(\frac{d^2\tilde{x}_0}{d\tau^2} + 2\Phi\frac{d\tilde{x}_0}{d\tau} + 1\right) + 2\Lambda\,\tilde{x}_2 \;=\; 0. \end{aligned} \tag{5.2.39}$$

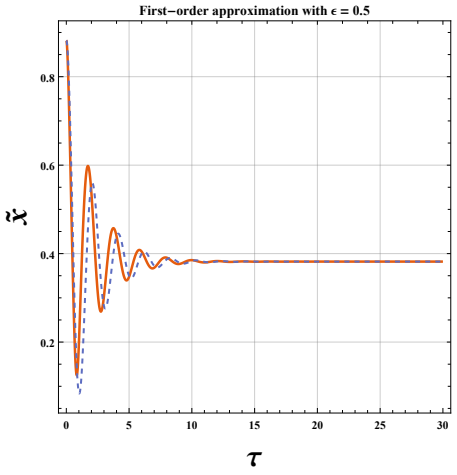
Doing some algebra, we can simplify our expression, the new form of our equation is:



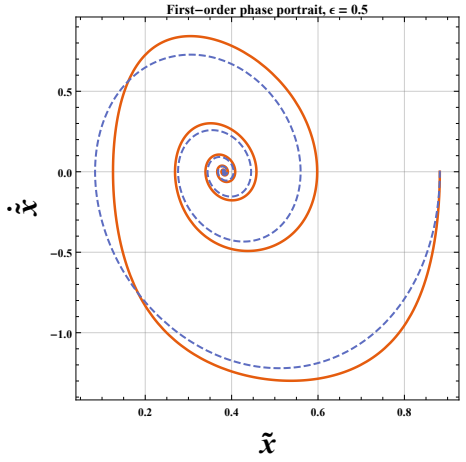
((a)) Time evolution contrast ($\epsilon = 0.518034$)



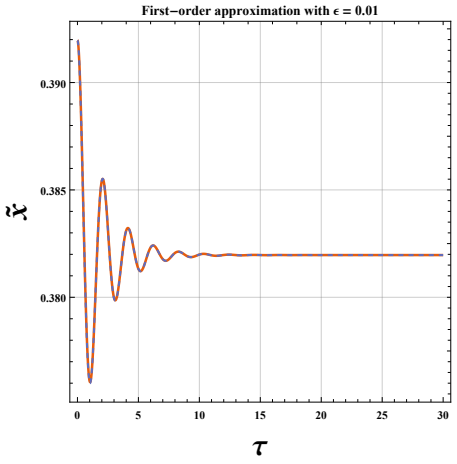
((b)) Phase portrait contrast ($\epsilon = 0.518034$)



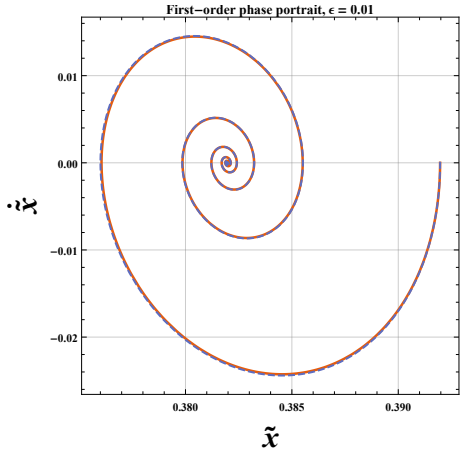
((c)) Time evolution ($\epsilon = 0.5$)



((d)) Phase portrait ($\epsilon = 0.5$)



((e)) Time evolution ($\epsilon = 0.01$)



((f)) Phase portrait ($\epsilon = 0.01$)

Figure 5.10: Approximation of the first-order solution to many values of ϵ with damping.

$$\frac{d^2\tilde{x}_2}{d\tau^2} + 2\Phi\,\frac{d\tilde{x}_2}{d\tau} + \frac{\tilde{x}_1 - 2\tilde{x}_0\tilde{x}_1}{\tilde{x}_0 - \tilde{x}_0^2}\left(\underbrace{\frac{d^2\tilde{x}_1}{d\tau^2} + 2\Phi\,\frac{d\tilde{x}_1}{d\tau}}_{-\omega_0^2\tilde{x}_1}\right)$$

$$+\frac{\tilde{x}_2 - 2\tilde{x}_0\tilde{x}_2 - \tilde{x}_1^2}{\tilde{x}_0 - \tilde{x}_0^2}\left(\underbrace{\frac{d^2\tilde{x}_0}{d\tau^2}}_{0} + 2\Phi\,\underbrace{\frac{d\tilde{x}_0}{d\tau}}_{0} + 1\right) + \frac{2\Lambda\,\tilde{x}_2}{\tilde{x}_0 - \tilde{x}_0^2} = 0$$

$$\frac{d^2\tilde{x}_2}{d\tau^2} + 2\Phi\,\frac{d\tilde{x}_2}{d\tau} + \frac{\tilde{x}_1 - 2\tilde{x}_0\tilde{x}_1}{\tilde{x}_0 - \tilde{x}_0^2}\left(-\omega_0^2\tilde{x}_1\right) + \frac{\tilde{x}_2 - 2\tilde{x}_0\tilde{x}_2 - \tilde{x}_1^2}{\tilde{x}_0 - \tilde{x}_0^2} + \frac{2\Lambda\,\tilde{x}_2}{\tilde{x}_0 - \tilde{x}_0^2} = 0$$

$$\frac{d^2\tilde{x}_2}{d\tau^2} + 2\Phi\,\frac{d\tilde{x}_2}{d\tau} + \left(\underbrace{\frac{2\Lambda + 1 - 2\tilde{x}_0}{\tilde{x}_0 - \tilde{x}_0^2}}_{\omega_0^2}\right)\tilde{x}_2 + \frac{\tilde{x}_1 - 2\tilde{x}_0\tilde{x}_1}{\tilde{x}_0 - \tilde{x}_0^2}\left(-\omega_0^2\tilde{x}_1\right) - \frac{\tilde{x}_1^2}{\tilde{x}_0 - \tilde{x}_0^2} = 0 \tag{5.2.40}$$

$$\frac{d^2\tilde{x}_2}{d\tau^2} + 2\Phi\,\frac{d\tilde{x}_2}{d\tau} + \omega_0^2\tilde{x}_2 + \frac{\tilde{x}_1 - 2\tilde{x}_0\tilde{x}_1}{\tilde{x}_0 - \tilde{x}_0^2}\left(-\omega_0^2\tilde{x}_1\right) - \frac{\tilde{x}_1^2}{\tilde{x}_0 - \tilde{x}_0^2} = 0$$

$$\frac{d^2\tilde{x}_2}{d\tau^2} + 2\Phi\,\frac{d\tilde{x}_2}{d\tau} + \omega_0^2\tilde{x}_2 + \frac{\left(-\omega_0^2\tilde{x}_1\right)\left(\tilde{x}_1 - 2\tilde{x}_0\tilde{x}_1\right) - \tilde{x}_1^2}{\tilde{x}_0 - \tilde{x}_0^2} = 0$$

$$\frac{d^2\tilde{x}_2}{d\tau^2} + 2\Phi\,\frac{d\tilde{x}_2}{d\tau} + \omega_0^2\tilde{x}_2 + \frac{\left(-\omega_0^2\right)\left(1 - 2\tilde{x}_0\right) - 1}{\tilde{x}_0 - \tilde{x}_0^2}\tilde{x}_1^2 = 0$$

$$\frac{d^2\tilde{x}_2}{d\tau^2} + 2\Phi\,\frac{d\tilde{x}_2}{d\tau} + \omega_0^2\tilde{x}_2 - \frac{\omega_0^2 - 2\omega_0^2\tilde{x}_0 + 1}{\tilde{x}_0 - \tilde{x}_0^2}\tilde{x}_1^2 = 0$$

To simplify the differential equation, we define $\alpha^2 = \frac{\omega_0^2 - 2\omega_0^2\tilde{x}_0 + 1}{\tilde{x}_0 - \tilde{x}_0^2}$. Then our differential equation, using the definition of $\tilde{x}_1$ is:

$$\frac{d^2\tilde{x}_2}{d\tau^2} + 2\Phi\frac{d\tilde{x}_2}{d\tau} - \alpha^2\tilde{x}_1^2 + \omega^2\tilde{x}_2 = 0$$

$$\frac{d^2\tilde{x}_2}{d\tau^2} + 2\Phi\frac{d\tilde{x}_2}{d\tau} - \alpha^2 e^{-2\Phi t}\left[\cos\left(\Delta t\right) + \frac{\Phi}{\Delta}\sin\left(\Delta t\right)\right]^2 + \omega^2\tilde{x}_2 = 0 \tag{5.2.41}$$

$$\therefore\ \boxed{\frac{d^2\tilde{x}_2}{d\tau^2} + 2\Phi\frac{d\tilde{x}_2}{d\tau} + \omega^2\tilde{x}_2 = \alpha^2 e^{-2\Phi t}\left[\cos\left(\Delta t\right) + \frac{\Phi}{\Delta}\sin\left(\Delta t\right)\right]^2}$$

Now, we proceed to solve the differential equation. We can see from the previous equation that our system is a forced damped oscillator. So, using the form to solve this kind of differential equation, and using the initial conditions, we find the following solution.

$$\begin{aligned}\tilde{x}_2(\tau) = &- \frac{e^{-2\Phi\tau}\alpha^2\Delta\left(8\Phi^2\omega^2 - 9\omega^4 + (16\Phi^2 - 18\omega^2\Phi^2 + 3\omega^4)\cos(2\Delta t)\right)}{2\omega^2\Delta(9\omega^4 - 17\omega^2\Phi^2 + 8\Phi^4)} \\ &- \frac{2e^{-2\Phi\tau}\alpha^2\Delta^2\left(\Phi(8\Phi^2 - 5\omega^2)\sin(2\Delta t) - (8\Phi^2 + 3\omega^2)\Delta e^{\Phi\tau}\cos(\Delta\tau)\right)}{2\omega^2\Delta(9\omega^4 - 17\omega^2\Phi^2 + 8\Phi^4)} \\ &+ \frac{2e^{-\Phi\tau}\alpha^2\Delta^2\Phi\left(8\Phi^2 - 13\omega^2\right)\sin(\Delta\tau)}{2\omega^2\Delta(9\omega^4 - 17\omega^2\Phi^2 + 8\Phi^4)}\end{aligned} \tag{5.2.42}$$

Therefore, our approximated solution is:

$$\begin{aligned}\tilde{x}(\tau) \sim &\tilde{x}_0 + \tilde{x}_1\epsilon + \tilde{x}_2\epsilon^2 = \tilde{x}_{eq} + e^{-\Phi t}\left[\cos(\Delta t) + \frac{\Phi}{\Delta}\sin(\Delta t)\right]\epsilon \\ &- \frac{e^{-2\Phi\tau}\alpha^2\Delta\left(8\Phi^2\omega^2 - 9\omega^4 + (16\Phi^2 - 18\omega^2\Phi^2 + 3\omega^4)\cos(2\Delta t)\right)}{2\omega^2\Delta(9\omega^4 - 17\omega^2\Phi^2 + 8\Phi^4)}\epsilon^2 \\ &- \frac{2e^{-2\Phi\tau}\alpha^2\Delta^2\left(\Phi(8\Phi^2 - 5\omega^2)\sin(2\Delta t) - (8\Phi^2 + 3\omega^2)\Delta e^{\Phi\tau}\cos(\Delta\tau)\right)}{2\omega^2\Delta(9\omega^4 - 17\omega^2\Phi^2 + 8\Phi^4)}\epsilon^2 \\ &+ \frac{2e^{-\Phi\tau}\alpha^2\Delta^2\Phi\left(8\Phi^2 - 13\omega^2\right)\sin(\Delta\tau)}{2\omega^2\Delta(9\omega^4 - 17\omega^2\Phi^2 + 8\Phi^4)}\epsilon^2\end{aligned} \tag{5.2.43}$$

If we plot the second-order solution of our system and compare it with the numerical solution, we obtain the figure (5.11).

5.3 Forced system

As we mentioned previously, an oscillator forced system is a system where an external force acts over an oscillator system changing the state of motion of the oscillator.

The forces that play a relevant role in the dynamics of the system are the dissipative force presented by the environment and which is proportional to the speed of movement ($F_f = \beta v$) and the external force.

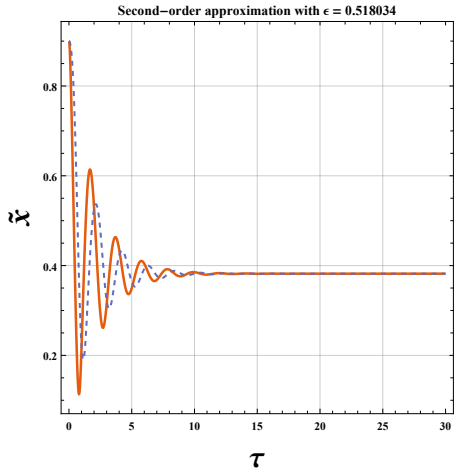
Using the equation of forced oscillations, we can write our system as:

$$\begin{aligned}F + F_f + F_R =& F\cos(\omega t) \\ m\frac{d^2x}{dt^2} + \beta\frac{dx}{dt} + mg - \frac{M}{x} + \frac{M}{h-x} =& F\cos(\omega t).\end{aligned} \tag{5.3.1}$$

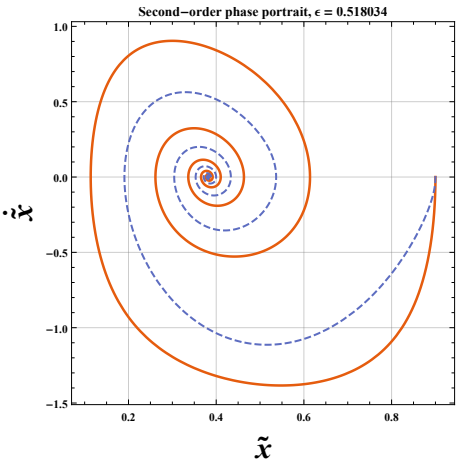
5.3.1 Lagrangian analysis

The Lagrangian that we use is the same Lagrangian of the damped system, then the Lagragian is.

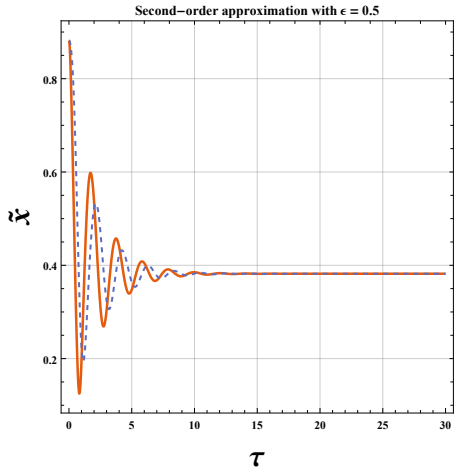
$$\boxed{L = \frac{1}{2}m\dot{x}^2 - mgx + M\ln(x) + M\ln(h-x)}. \tag{5.3.2}$$



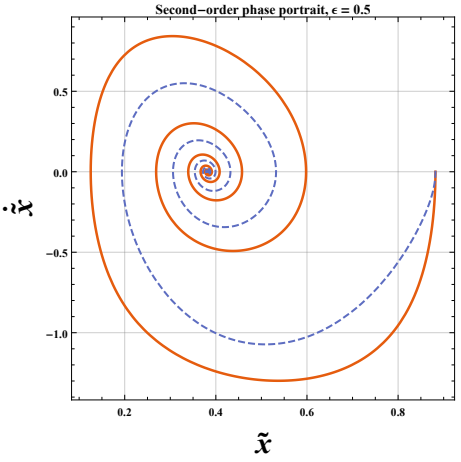
((a)) Time evolution contrast ($\epsilon = 0.518034$)



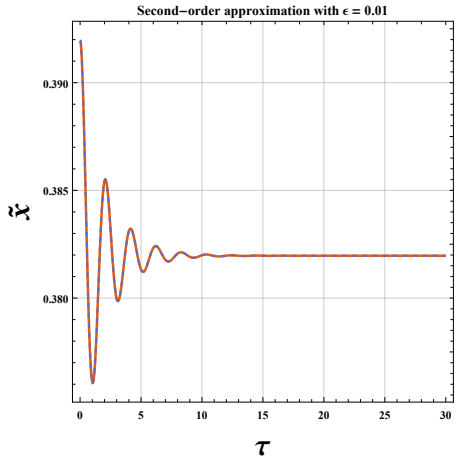
((b)) Phase portrait contrast ($\epsilon = 0.518034$)



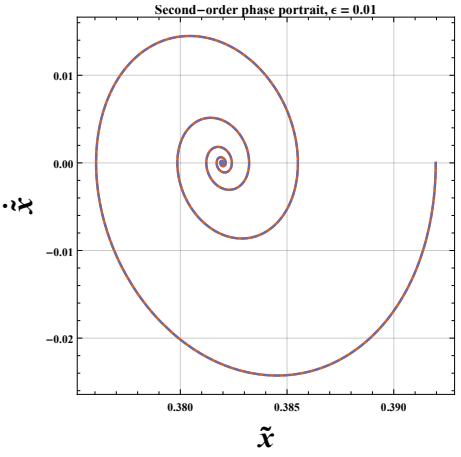
((c)) Time evolution ($\epsilon = 0.5$)



((d)) Phase portrait ($\epsilon = 0.5$)



((e)) Time evolution ($\epsilon = 0.01$)



((f)) Phase portrait ($\epsilon = 0.01$)

Figure 5.11: Approximation of the second-order solution to many values of ϵ with damping.

Applying the Euler-Lagrange equation, we find the equation of motion of our system. As we know, our system presents two external forces that are the friction force and the periodic external force ($Q = F_f + F_E$). So the equation of motion for the coordinate x is.

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} &= Q \\ \implies m\ddot{x} + mg + \frac{M}{h-x} - \frac{M}{x} &= -\beta v + F\cos(\omega t) \\ \implies \ddot{x} + \frac{\beta}{m}\dot{x} + g + \frac{M}{m(h-x)} - \frac{M}{mx} &= \frac{F}{m}\cos(\omega t) \\ \therefore \quad \boxed{\frac{d^2x}{dt^2} + \frac{\beta}{m}\frac{dx}{dt} + g + \frac{M}{m}\left[\frac{1}{h-x} - \frac{1}{x}\right] = \frac{F}{m}\cos(\omega t)}. & \end{aligned} \tag{5.3.3}$$

The previous equation lets us know the different accelerations that act on the system and how they govern the motion of the oscillator.

We can not plot the phase portrait diagram for this system, because the time plays an explicit role in the system, but we can plot it only for a specific time.

5.3.2 Dimensional analysis

For the forced system, we have ten variables ($n = 10$), these variables are the mass (m), the time (t), the position (x), the gravity (g), the oscillation amplitude (A), the magnetic energy (M), (β) the friction coefficient and the distance between the fixed magnets (h), the frequency of the external force (ω) and the external force (F). For the nature of our problem, we only need to use three dimensions ($j = 3$) to characterize the problem; the dimensions that we need to use are the matter, the longitude, and the time. Using Buckingham's theorem, we know that there are four dimensionless groups ($k = n - j = 7$) that describe our system.

Using a dimensional analysis over every variable:

$$\begin{aligned} &[x] = L, [h] = L, [A] = L, [t] = T, [m] = M, [g] = LT^{-2}, \\ &[M] = ML^2T^{-2}, [\beta] = MT^{-1}, [\omega] = T^{-1}, [F] = MLT^{-2}, \end{aligned} \tag{5.3.4}$$

For this system, we are going to use h, g, and M as dimensional variables. This is because h is the geometric parameter of the system, g is the cinematic parameter, and M is the dynamic parameter. As we saw in the previous chapter, every dimensional group is the product of the dimensional variables, so we have that the ith group is expressed as:

$$\Pi_i = \delta h^{\alpha} g^{\beta} M^{\lambda}, \tag{5.3.5}$$

where δ is the variable that we want to dimensionless, in our system we have the variables x, t, m and A; α, β and λ are the powers of the distance h, the gravity and the magnetic energy that dimensionless the ith group.

To find the powers of our dimensional variables that dimensionless every group we can construct an equations system, this system is.

$$\begin{pmatrix} L \\ M \\ T \end{pmatrix} + \begin{pmatrix} 1 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & -2 & -2 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \lambda \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \implies \begin{pmatrix} \alpha \\ \beta \\ \lambda \end{pmatrix} = \begin{pmatrix} M - L - \frac{T}{2} \\ M + \frac{T}{2} \\ -M. \end{pmatrix} \tag{5.3.6}$$

It is important to remember that every dimensionless group can be expressed as a fraction. So the dimensionless groups can be rewrite as:

$$\Pi_i = \frac{\delta}{\delta_s} = \tilde{\delta}, \tag{5.3.7}$$

where δ_s is the scale factor of the variable δ y $\tilde{\delta}$ is the dimensionless form of the variable δ. Now, we can obtain the dimensionless groups for the variables x, t, m and A.

Dimensionless group of the position

The x's group is:

$$\begin{pmatrix} \alpha \\ \beta \\ \lambda \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} \quad \therefore \quad \boxed{\Pi_1 = \frac{x}{x_s} = \frac{x}{h} \implies x_s = h} \quad . \tag{5.3.8}$$

By the previous result, the scale factor of the position is the distance between the two fixed magnets. The position can be expressed as the product of the scale factor of the position and the dimensionless position.

$$x = x_s \tilde{x} \implies dx = x_s d\tilde{x} \implies d^2x = x_s d^2\tilde{x} \tag{5.3.9}$$

Dimensionless group of the time

The group of t is.

$$\begin{pmatrix} \alpha \\ \beta \\ \lambda \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix} \quad \therefore \quad \boxed{\Pi_2 = \frac{t}{t_s} = t\sqrt{\frac{g}{h}} \implies t_s = \sqrt{\frac{h}{g}}} \quad . \tag{5.3.10}$$

Using the previous result and the definitions of the frequency and the oscillation period, we obtain.

$$\omega_s = \frac{1}{t_s} = \sqrt{\frac{g}{h}} \quad \wedge \quad T_s = \frac{2\pi}{\omega_s} = 2\pi\sqrt{\frac{h}{g}}. \tag{5.3.11}$$

If we analyze these expression, we can see that they are the frequency and the period of a pendulum. Then, we can conclude that our system is analogous to a pendulum.

Now, using the relation between the scale factor and the dimensionless variable, we obtain.

$$t = t_s \tau \implies dt = t_s d\tau \implies dt^2 = t_s^2 d\tau^2 \tag{5.3.12}$$

Dimensionless group of the mass and the Energie's number

Now, the group of m:

$$\begin{pmatrix}\alpha\\ \beta\\ \lambda\end{pmatrix} = \begin{pmatrix}1\\ 1\\ -1\end{pmatrix} \quad \therefore \quad \boxed{\Pi_3 = \frac{m}{m_s} = \frac{mgh}{M} \implies m_s = \frac{M}{gh}} \quad . \tag{5.3.13}$$

This dimensionless group is very important because give us a relation between the magnetic energy and the mechanic potential energy, Using this dimensionless group we going to define Energie's number as:

$$\Lambda \equiv \frac{1}{\Pi_3} = \frac{M}{mgh}. \tag{5.3.14}$$

The Energie's number going to be so useful because it related all the variables that are not affected by time. An example of some quantity that is not affected by time is the equilibrium distance, so we can rewrite this number interm of the Energie's number as:

$$\begin{aligned} x_{eq} &= \frac{2M + mgh - \sqrt{4M^2 + m^2g^2h^2}}{2mg} = \frac{mgh\left(\frac{2M}{mgh} + 1\right) - \sqrt{m^2g^2h^2\left(\frac{4M^2}{m^2g^2h^2} + 1\right)}}{2mg} \\ &= \frac{mgh\left(\frac{2M}{mgh} + 1 - \sqrt{\frac{4M^2}{m^2g^2h^2} + 1}\right)}{2mg} = \frac{h}{2}\left(2\Lambda + 1 - \sqrt{4\Lambda^2 + 1}\right) \\ \therefore \boxed{x_{eq} = h\left(\frac{2\Lambda + 1 - \sqrt{4\Lambda^2 + 1}}{2}\right) \implies \tilde{x}_{eq} = \frac{x_{eq}}{h} = \frac{2\Lambda + 1 - \sqrt{4\Lambda^2 + 1}}{2}}&. \end{aligned} \tag{5.3.15}$$

Dimensionless group of the amplitude

Obtaining the group of A.

$$\begin{pmatrix}\alpha\\ \beta\\ \lambda\end{pmatrix} = \begin{pmatrix}-1\\ 0\\ 0\end{pmatrix} \quad \therefore \quad \boxed{\Pi_4 = \frac{A}{A_s} = \frac{A}{h} \implies A_s = h} \quad . \tag{5.3.16}$$

This result lets us see that the amplitude of oscillation is a fraction of the distance between the fixed magnets. The amplitude does not have a role in the motion equation (equation 5.3.3) so we going to use this dimensionless group as the control number (ϵ) for our approximation.

$$\epsilon = \frac{A}{h} = \tilde{A}. \tag{5.3.17}$$

Dimensionless group of the external force

The dimensionless group of F is.

$$\begin{pmatrix}\alpha\\ \beta\\ \lambda\end{pmatrix} = \begin{pmatrix}1\\ 0\\ -1\end{pmatrix} \quad \therefore \quad \boxed{\Pi_5 = \frac{F}{F_s} = \frac{Fh}{M} \implies F_s = \frac{M}{h}} \quad . \tag{5.3.18}$$

This result is so important because the force scale is expressed by the energy provided by the magnets and the distance to the magnets.

5.3.3 Dimensionless system

Applying the dimensionless groups in our differential equation, we can find the equation of motion

$$\begin{aligned}
&\frac{d^2x}{dt^2} + \frac{\beta}{m}\frac{dx}{dt} + g + \frac{M}{m}\left[\frac{1}{h-x} - \frac{1}{x}\right] = \frac{F}{m}\cos(\omega t)\\
\implies &\frac{d^2(x_s\tilde{x})}{d(t_s\tau)^2} + \frac{\beta}{m}\frac{d(x_s\tilde{x})}{d(t_s\tau)} + g + \frac{M}{m}\left[\frac{1}{h-(x_s\tilde{x})} - \frac{1}{(x_s\tilde{x})}\right] = \frac{F}{m}\cos(\omega_s\Omega(t_s\tau))\\
\implies &\frac{x_s}{t_s^2}\frac{d^2\tilde{x}}{d\tau^2} + \frac{\beta}{m}\frac{x_s}{t_s}\frac{d\tilde{x}}{d\tau} + g + \frac{M}{m}\left[\frac{1}{h-x_s\tilde{x}} - \frac{1}{x_s\tilde{x}}\right] = \frac{F}{m}\cos\left(\frac{1}{t_s}\Omega(t_s\tau)\right)\\
\implies &\frac{d^2\tilde{x}}{d\tau^2} + \frac{\beta t_s}{m}\frac{d\tilde{x}}{d\tau} + \frac{gt_s^2}{x_s} + \frac{Mt_s^2}{mx_s}\left[\frac{1}{h-x_s\tilde{x}} - \frac{1}{x_s\tilde{x}}\right] = \frac{Ft_s^2}{mx_s}\cos(\Omega\tau).
\end{aligned} \tag{5.3.19}$$

Now, using the definitions of the scale factors of position and time, we can dimensionless our equation of motion.

$$\begin{aligned}
&\frac{d^2\tilde{x}}{d\tau^2} + \frac{\beta}{m}\sqrt{\frac{h}{g}}\frac{d\tilde{x}}{d\tau} + \frac{g\left(\frac{h}{g}\right)}{h} + \frac{M\left(\frac{h}{g}\right)}{mh}\left[\frac{1}{h-h\tilde{x}} - \frac{1}{h\tilde{x}}\right] = \frac{\left(\frac{h}{g}\right)F}{mh}\cos(\Omega\tau)\\
\implies &\frac{d^2\tilde{x}}{d\tau^2} + \frac{\beta}{m}\sqrt{\frac{h}{g}}\frac{d\tilde{x}}{d\tau} + 1 + \frac{M}{mgh}\left[\frac{1}{1-\tilde{x}} - \frac{1}{\tilde{x}}\right] = \frac{F}{mg}\cos(\Omega\tau)
\end{aligned} \tag{5.3.20}$$

This last equation has three different dimensionless magnitudes that can be defined as dimensionless parameters that characterize the dynamics of the system. Now, using the dimensionless numbers, we obtain the following equation.

$$\frac{d^2\tilde{x}}{d\tau^2} + 2\Phi\frac{d\tilde{x}}{d\tau} + 1 + \Lambda\left[\frac{1}{1-\tilde{x}} - \frac{1}{\tilde{x}}\right] = \tilde{F}\cos(\Omega\tau), \tag{5.3.21}$$

the previous equation is a dimensionless differential equation. The three dimensionless numbers are the ***Energie's number*** (Λ), the ***Friction number*** (Φ) and the dimensionless external force ($\tilde{F}$). Where Φ and $\tilde{F}$ are defined as :

$$\Phi = \frac{\beta}{2m}\sqrt{\frac{h}{g}}, \quad \tilde{F} = \frac{F}{mg} \tag{5.3.22}$$

Now, doing some algebra, we can simplify our differential equation as follows.

$$
\begin{aligned}
\frac{d^2\tilde{x}}{d\tau^2} + 2\Phi\frac{d\tilde{x}}{d\tau} + 1 + \Lambda\left[\frac{\tilde{x}-(1-\tilde{x})}{\tilde{x}(1-\tilde{x})}\right] &= \tilde{F}\cos(\Omega\tau) \\
\implies \frac{d^2\tilde{x}}{d\tau^2} + 2\Phi\frac{d\tilde{x}}{d\tau} + 1 + \Lambda\left[\frac{2\tilde{x}-1}{\tilde{x}(1-\tilde{x})}\right] &= \tilde{F}\cos(\Omega\tau) \\
\implies \tilde{x}(1-\tilde{x})\frac{d^2\tilde{x}}{d\tau^2} + 2\Phi\tilde{x}(1-\tilde{x})\frac{d\tilde{x}}{d\tau} + \tilde{x}(1-\tilde{x}) + \Lambda(2\tilde{x}-1) &= \tilde{x}(1-\tilde{x})\tilde{F}\cos(\Omega\tau) \\
\implies (\tilde{x}-\tilde{x}^2)\left(\frac{d^2\tilde{x}}{d\tau^2} + 2\Phi\frac{d\tilde{x}}{d\tau} + 1 - \tilde{F}\cos(\Omega\tau)\right) + \Lambda(2\tilde{x}-1) &= 0,
\end{aligned}
\tag{5.3.23}
$$

this last equation is an unsolvable differential equation, so we need to use another method to analyze the evolution of the system.

5.3.4 Numerical solution

As we have seen previously, the numerical solution is fundamental to knowing the behavior of our differential equation, which it will follow.

So, we require a numerical method to plot the evolution of the system. There are many numerical methods in the literature, but the complexity and properties of our differential equation forced us to use the more powerful methods, such as the Runge-Kutta method.

Nowadays, there are different software that have the ability to apply the most efficient numerical method to some differential equations, one of them is the Wolfram Mathematica software. So we are going to use the ***NDSolve*** function to find the numerical solution of our system.

Dynamical system stability

To analyze the behavior of our system, we can use our techniques for differential equations. We can not use the phase portrait technique because our dynamical system depends directly on time. Then we are going to use the Poincaré map to find the stability of our system. It is important to highlight that the Poincaré map is very useful in the stroboscopy space, where $\tau = T = \frac{2\pi}{\Omega}$

Firstly, we need to reduce our second-order differential equation to a set of first-order differential equations. The forcing phase relation is $\tilde{\theta} = \Omega\tau$. These equations are:

$$\tilde{v} = \frac{d\tilde{x}}{d\tau} \tag{5.3.24}$$

$$\dot{\tilde{v}} = \tilde{F}\cos(\tilde{\theta}) - 2\Phi\tilde{v} - 1 - \Lambda\left(\frac{1}{1-\tilde{x}} - \frac{1}{\tilde{x}}\right) \tag{5.3.25}$$

$$\Omega = \frac{d\tilde{\theta}}{d\tau} \tag{5.3.26}$$

These equations can be expressed as a vectorial system as:

$$\tilde{r} = \frac{d}{d\tau}\begin{pmatrix}\tilde{x}\\ \tilde{v}\\ \tilde{\theta}\end{pmatrix} = \begin{pmatrix}\tilde{v}\\ \tilde{F}\cos(\tilde{\theta}) - 2\Phi\tilde{v} - 1 - \Lambda\left(\frac{1}{1-\tilde{x}} - \frac{1}{\tilde{x}}\right)\\ \Omega\end{pmatrix}, \tag{5.3.27}$$

where $\tilde{r}$ is the vector that models our system. If we want to know the stability of our system, we need the non-autonomous system, which implies that we do not need to attend to the time. Then our vector $\tilde{r}$ is really formed only by the first and second terms. To find the stability we need the Jacobian matrix of our non-autonomous system. Which is:

$$\mathbf{J}(\tilde{x},\tilde{v}) = \begin{pmatrix}\frac{\partial\tilde{r}_1}{\partial\tilde{x}} & \frac{\partial\tilde{r}_1}{\partial\tilde{v}}\\ \frac{\partial\tilde{r}_2}{\partial\tilde{x}} & \frac{\partial\tilde{r}_2}{\partial\tilde{v}}\end{pmatrix} \implies \mathbf{J}(\tilde{x},\tilde{v}) = \begin{pmatrix}0 & 1\\ -\Lambda\left(\frac{1}{(1-\tilde{x}(\tau))^2} + \frac{1}{\tilde{x}^2(\tau)}\right) & -2\Phi\end{pmatrix} = \mathbf{J}(\tau). \tag{5.3.28}$$

With the Jacobian of the system, we can find the Monodromy matrix (that is, the transformation of our Jacobian matrix to the Poincaré discrete time space), and this matrix follows the next:

$$\dot{\mathbf{M}}(\tau) = \mathbf{J}(\tau)\mathbf{M}(\tau), \quad \mathbf{M}(0) = \mathbf{I}. \tag{5.3.29}$$

This system is so complicated to solve analytically. Then we are going to use Liouville's formula:

$$\det(\mathbf{M}(\tau)) = \det\mathbf{M}(t_0)e^{\int_{t_0}^{\tau}\operatorname{tr}(\mathbf{J}(\tau))d\tau}. \tag{5.3.30}$$

Now, integrating from 0 to T, we obtain:

$$\begin{aligned}\det(\mathbf{M}(T)) &= \det(\mathbf{M}(0))e^{\int_0^T \operatorname{tr}(\mathbf{J}(\tau))d\tau}\\ \det(\mathbf{M}(T)) &= \det(\mathbf{I})e^{\int_0^T -2\Phi d\tau}\\ \therefore \boxed{\det(\mathbf{M}(T)) = e^{-2\Phi T}}&.\end{aligned} \tag{5.3.31}$$

With this result, we can find the Floquet multipliers, which are the eigenvalues of the Monodromy matrix ($\mathbf{M}(T)$) and they are fundamental to determine the stability of the system. Using the definition of Floquet multipliers, we obtain.

$$\rho 1 \cdot \rho 2 = \det(\mathbf{M}(T)) = e^{-2\Phi T}, \tag{5.3.32}$$

where ρ_1 and ρ_2 are the Floquet multipliers. The Floque multipliers let us know if the trajectory that the system follows is stable or unstable. To know this, we have the following table:

Condition	**How many?**	**System behavior**
$\rho < 1$	All	Stable limit cycle (perturbations decay)
$\rho = 1$	At least one (others < 1)	Marginal stability (bifurcation point)
$\rho > 1$	At least one	Unstable limit cycle (perturbations grow)

Table 5.1: Different behaviors of the system in terms of the Floquet multipliers

If $\Phi > 0$, that is the case of a damped system, the product of the Floquet multipliers is always equal or less than one. That implies that the system has a limit cycle. To show that a limit cycle exists, we now make a Poincaré map.

To make our Poincaré map, we are going to use the Runge-Kutta method in our dynamical system, and we are going to save the values of $(\tilde{x}, \tilde{v})$ when we pass for a multiple of T. Doing this, we obtain the following Poincaré map.

Lyapunov exponents

While Floquet multipliers characterize the stability of a periodic orbit, many systems, in particular the forced nonlinear oscillators, can exhibit chaotic behavior where no periodic orbit exists. In such cases, a more general tool is needed: the ***Lyapunov exponents***. The behavior of the system in terms of the Lyapunov exponents follows the next table:

Condition	**How many?**	**System behavior**
$\lambda < 0$	All	Stable dynamics
$\lambda = 0$	At least one (others < 0)	Marginal stability
$\lambda > 0$	At least one	Chaotic dynamics

Table 5.2: Different behaviors of the system in terms of the Lyapunov exponents

When a system is stable, there exists a relation between the Floquet multipliers and the Lyapunov exponents. That relation is:

$$\lambda_i = \frac{1}{T} \ln(\rho_i) \tag{5.3.33}$$

Now, using the Lyapunov exponents equation:

$$\lambda_1 + \lambda_2 = \frac{1}{T} \ln(\rho_1) + \frac{1}{T} \ln(\rho_2) = \frac{1}{T} \ln(\rho_1 \cdot \rho_2) \tag{5.3.34}$$

Using the value of our Floquet product (5.3.32)

$$\lambda_1 + \lambda_2 = \frac{1}{T} \ln\left(e^{-2\Phi T}\right) \implies \lambda_1 + \lambda_2 = -2\Phi \tag{5.3.35}$$

Using the condition of stability:

$$\lambda_1 + \lambda_2 = -2\Phi < 0 \implies \Phi > 0, \tag{5.3.36}$$

this result implies that if the friction number (Φ) increases, the stability increases too. But the amplitude of the oscillation decreases. Another important thing that we can find is the values of Lyapunov exponents that give us stable system

$$\lambda_1 + \lambda_2 = -2\Phi \implies \lambda_1 = -(2\Phi + \lambda_2), \tag{5.3.37}$$

we know that a stable system needs $\lambda_i < 0$, then:

$$\lambda_1 < 0 \implies -(2\Phi + \lambda_2) < 0 \implies 2\Phi + \lambda_2 > 0 \implies \lambda_2 > -2\Phi, \tag{5.3.38}$$

but both Lyapunov exponents need to be less than one; therefore:

$$-2\Phi < \lambda_2 < 0. \tag{5.3.39}$$

Now, we can find the limit for the first Lyapunov exponent domain for a stable system:

$$\begin{aligned} \lambda_1 &= -(2\Phi + 0) = -2\Phi \\ \lambda_1 &= -(2\Phi - 2\Phi) = 0 \\ \therefore &-2\Phi < \lambda_1 < 0. \end{aligned} \tag{5.3.40}$$

We can see that both Lyapunov exponents have the same domain if the system is stable, but it is important to highlight that they can have different values, the only thing that they need to follow is that the sum of both is -2Φ as is presented in the stability condition equation 5.3.36

The previous result indicates that chaos in the system is completely related to the force ($\tilde{F}$) and frequency ($\tilde{\Omega}$) parameters.

Now, that we have our domain for the Lyapunov exponentes, we can start to plot them an see under what kind of parameters the system is stable. It is important to remember that we use a numerical method to find the Lyapunov exponents, and they change as time increases.

For a particular value of $\tilde{F}$, Φ and Ω, we can find the value of time where the system begins to be stable or not. An example of this is the figure 5.12

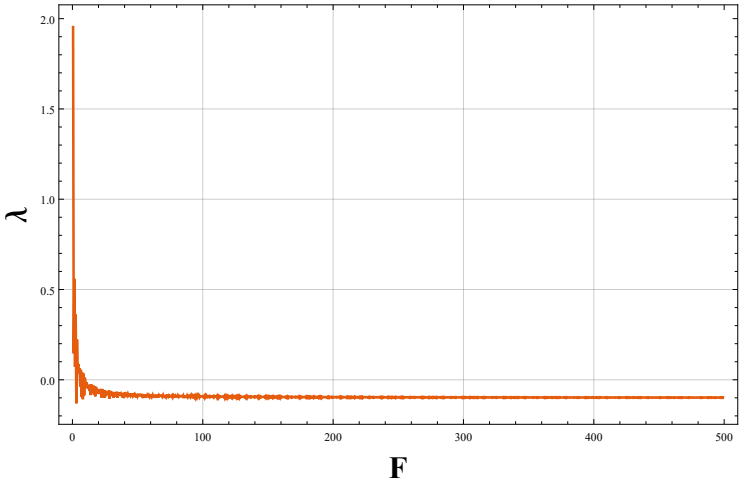


Figure 5.12: Change of the Lyupanov exponents in terms of time for: $\tilde{F} = 1$, $\Phi = 0.1$, $\Lambda = 1$, and $\Omega = 1$

In the previous figure we can see how the system starts acting like a chaotic system, but with the pass of the time the system begins to be stable. This tell us that our system with a that specific parameters is closed to the parameters that generate chaos in the system. So we need to see what kind of values of the force number can generate chaos.

So we decide to run our numerical method to see how Lyapunov exponents change when the value of the force increases. This can be shown in a plot as follows:

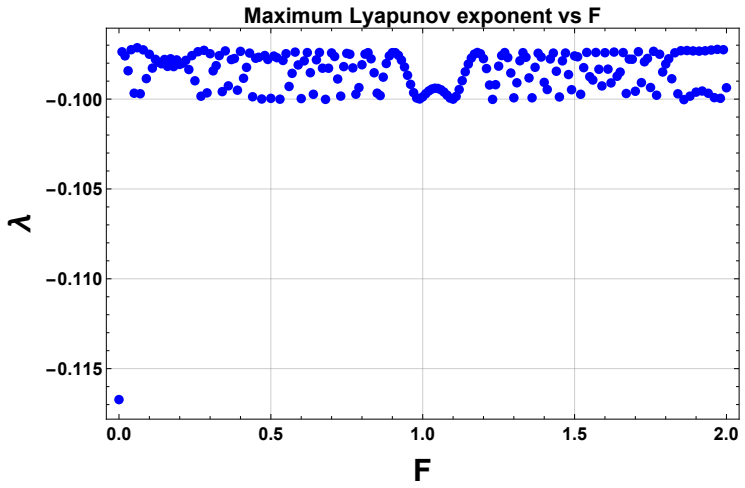


Figure 5.13: Value of Lyupanov exponents when $\tilde{F}$ change it value: $\Phi = 0.1$, and $\Omega = 1$

As we can see in the previous figure, for our system with $\Phi = 0.1$ and $\Omega = 1$, the force does not cross the horizontal axis, so the system in that range of forces will never be chaotic and is always stable. This system is very stable because of the presence of the two magnets that return the free magnet to the equilibrium position, then the presence of the Energies number generates more stability for the system. To see how the value of the Energies number impacts in the dynamics of the system, we plot a bifurcation plot in terms of the force. Firstly, we are going to use the system with $\Lambda = 1$, $\Phi = 0.1$ and $\Omega = 1$.

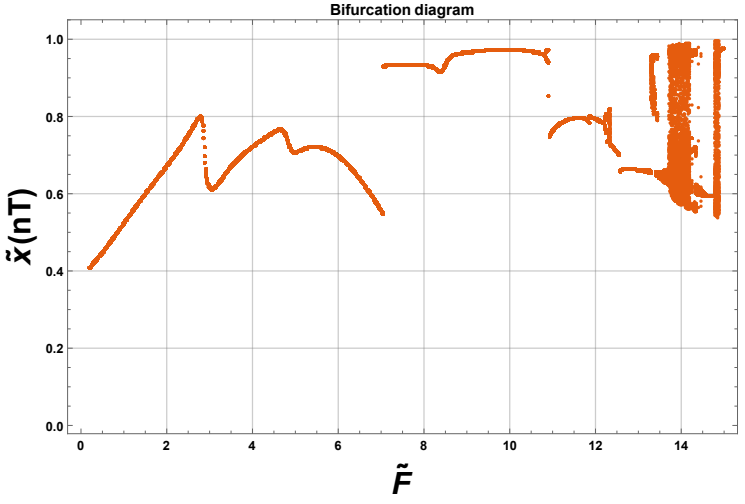


Figure 5.14: Bifurcation diagram for different values of the force with $\Lambda = 1$, $\Phi = 0.1$, and $\Omega = 1$

As we can see in the figure, the force needed to generate chaos in the system is $\tilde{F} \approx 12.5$. Now, we are going to generate another bifurcation diagram with a different Λ value and see how the system reacts to that change. For the following bifurcation diagram, we are going to use $\Lambda = 10$.

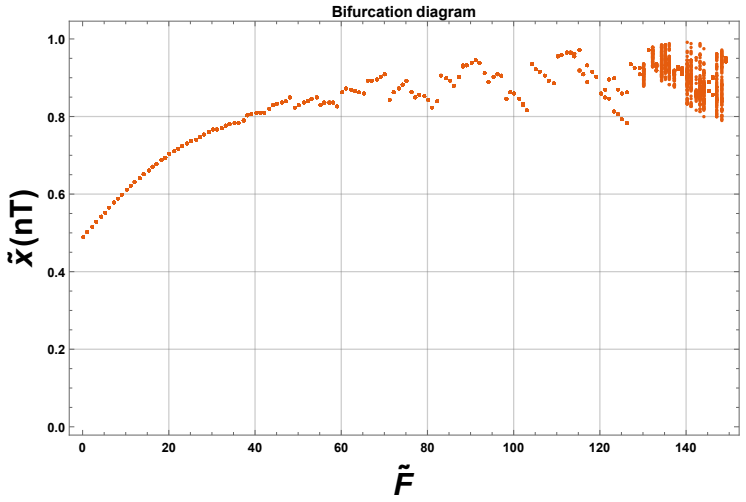


Figure 5.15: Bifurcation diagram for different values of the force with $\Lambda = 1$, $\Phi = 0.1$, and $\Omega = 1$

With the previous figure, we can see that when Λ increases, the force needed to generate chaos increases too. The next bifurcation diagram is with a low value of Λ. For the following plot our Energies number is $\Lambda = 0.1$

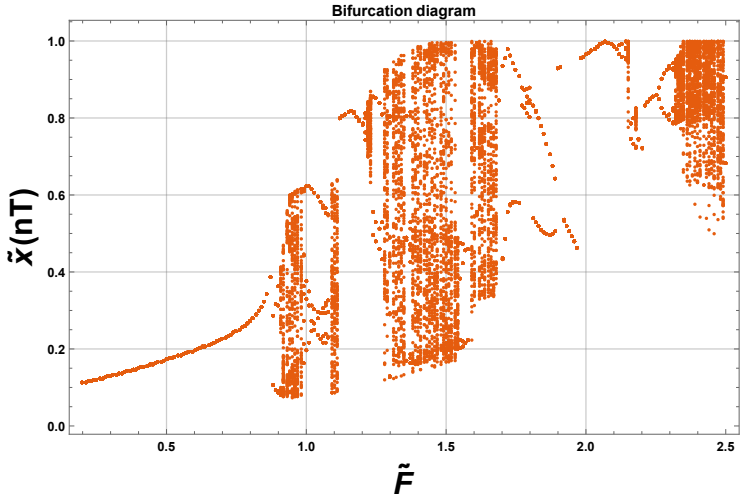
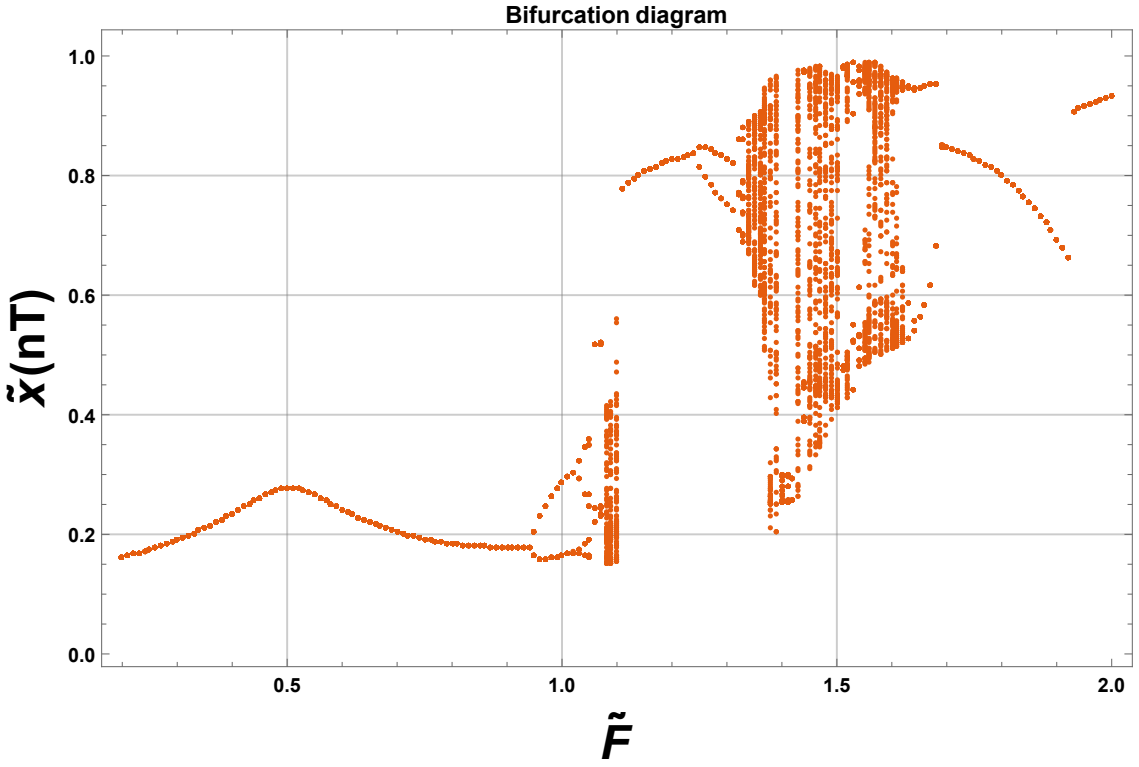
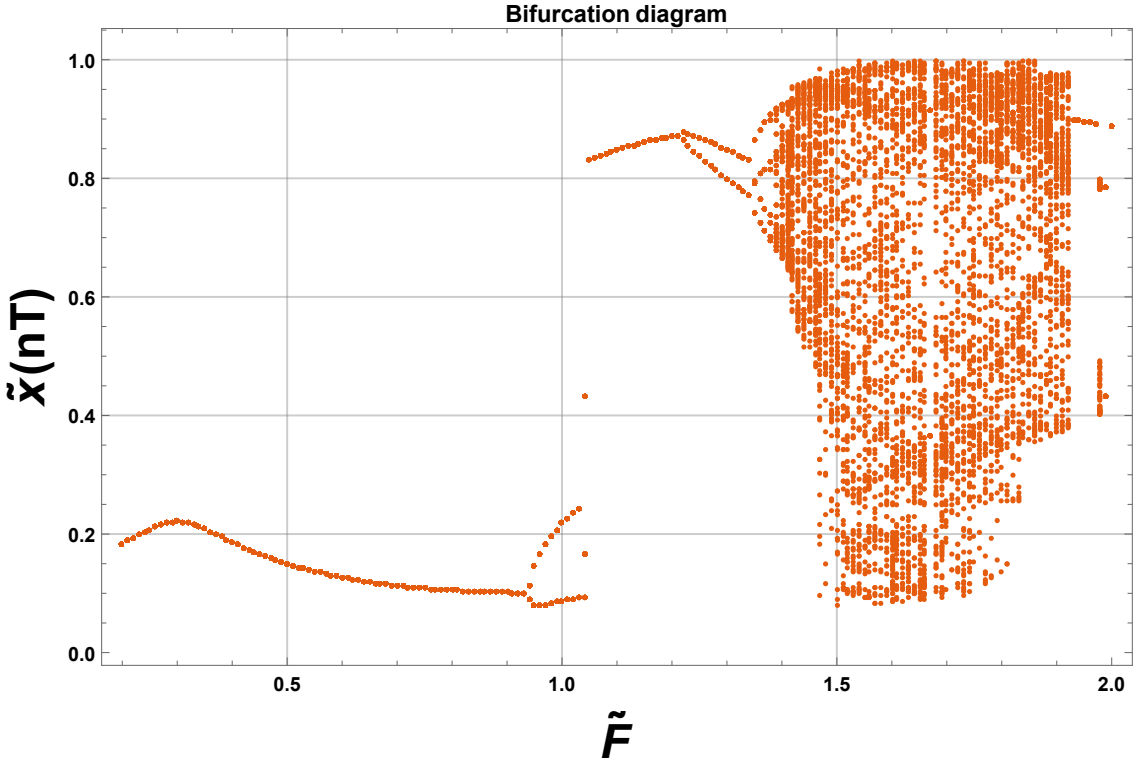


Figure 5.16: Bifurcation diagram for different values of the force with $\Lambda = 0.1$, $\Phi = 0.1$, and $\Omega = 1$

Now, we can see that for a low Energies number, the Force number is low too. So, there is a direct relation between the Energies number and the force number in terms of chaos. Then, to generate chaos in this system, we need to have a very low Energies number value, or a very large value of the force number. For the next step, we are going to see how the frequency number affects the force needed to generate chaos in the system. This can be shown in the figures 5.17 and 5.18.

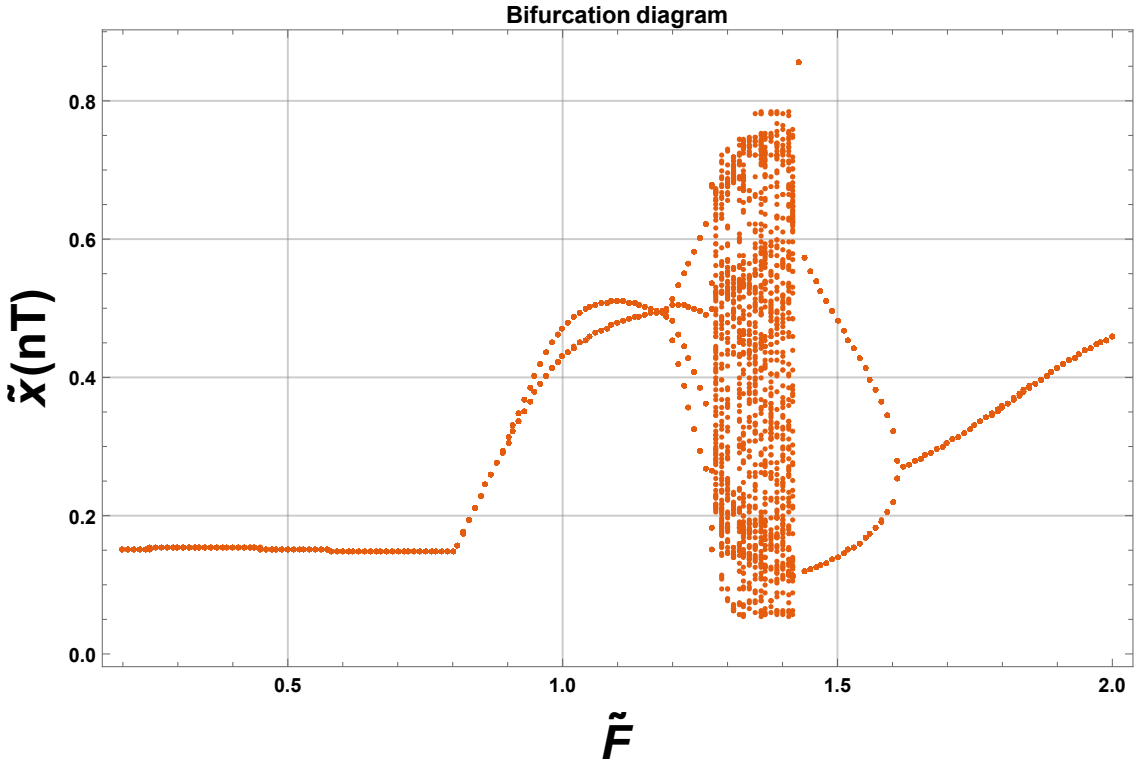


((a)) Bifurcation diagram of the position to different values of $\tilde{F}$ with $\Lambda = 0.15$, $\Phi = 0.1$ and $\Omega = 1$

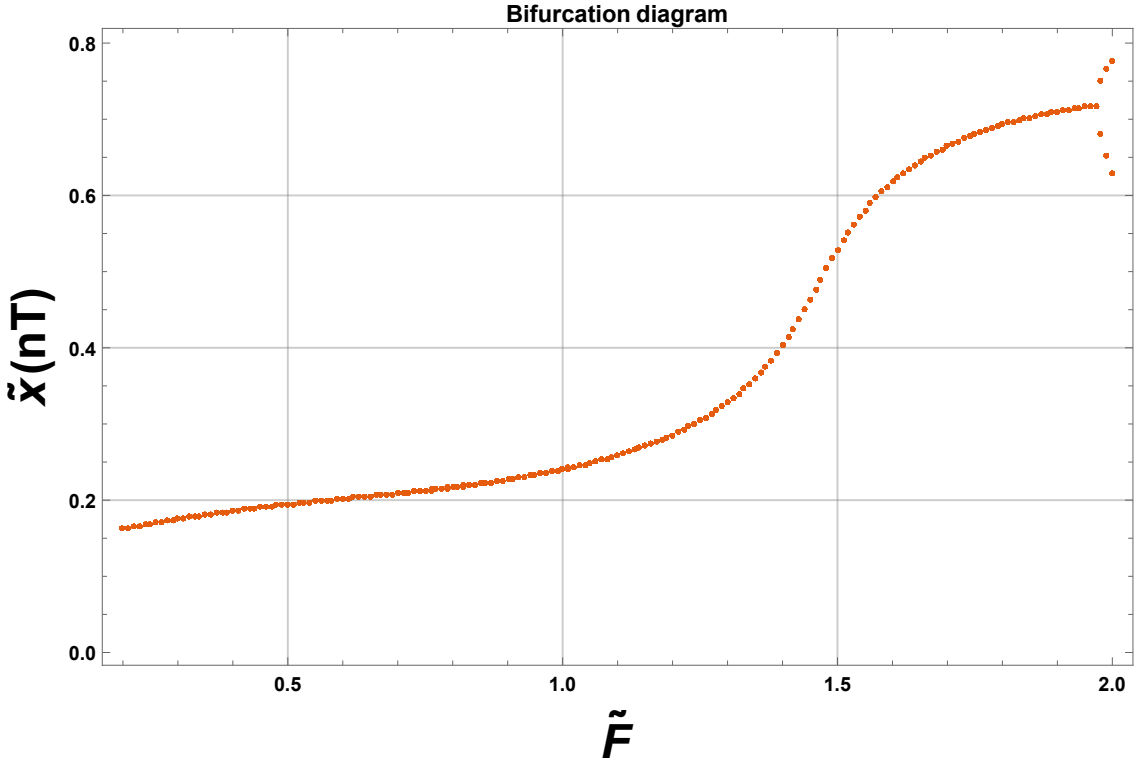


((b)) Bifurcation diagram of the position to different values of $\tilde{F}$ with $\Lambda = 0.15$, $\Phi = 0.1$ and $\Omega = 1.5$

Figure 5.17: Numerical demonstration that the angle (Ω) plays a relevant role in the chaos of the system, and it defines the force value (F) that generates chaos in the system



((a)) Bifurcation diagram of the position to different values of $\tilde{F}$ with $\Lambda = 0.15$, $\Phi = 0.1$ and $\Omega = 1.75$



((b)) Bifurcation diagram of the position to different values of $\tilde{F}$ with $\Lambda = 0.15$, $\Phi = 0.1$ and $\Omega = 2$

Figure 5.18: Numerical demonstration that the angle (Ω) plays a relevant role in the chaos of the system, and it defines the force value (F) that generates chaos in the system

As we can see in the previous figures, the frequency number plays an important role in chaos, because in every case that we check the force needed to generate chaos changes. For this reason, it is important to know the frequency of our system.

Numerical analysis

Something we can appreciate from our differential equation is that it is analytically unsolvable, so the only way to explore the dynamics of the system is to check how the system changes when our dimensionless parameters (Λ, Φ, Ω, and $\tilde{F}$) change. With that, we can make a table to understand the system behavior.

Condition	**System behavior**
$\Lambda < 1$	Linear oscillations
$\Lambda \geq 1$	Nonlinear regime, possible limit cycles
$\Phi = 0$, $F = 0$	Conservative (harmonic) motion
$\Phi > 0$, $F = 0$	Damped oscillation to equilibrium
$\Phi > 0$, $F \neq 0$	Forced nonlinear response
$\Omega \approx \omega_0$	Resonance (maximum amplitude)

Table 5.3: Different behaviors of the system with different values of our dimensionless parameters

As we know, the energy of our system is proportional to its amplitude, so we need to find the set of numbers that maximizes the amplitude. To see the amplitude of a numerical solution, we need to obtain the maximum and minimum values of the steady-state part and compute the mid-point difference between them.

$$A = \frac{max(x_{st}) - min(x_{st})}{2}, \tag{5.3.41}$$

where x_{st} is the steady state position. With this, we can create a code that allows us to compare how the amplitude of the steady-state changes when another dimensionless parameter is altered.

The proportionality of the amplitude to the external force is indicative that the other parameters play a more important role in the dynamics of the system. The second parameter with a less relevant role is the friction number, because the increase in its values reduces the amplitude, so we know that we want a system with the least possible value. The previous can be shown in the following figure (figure 5.19).

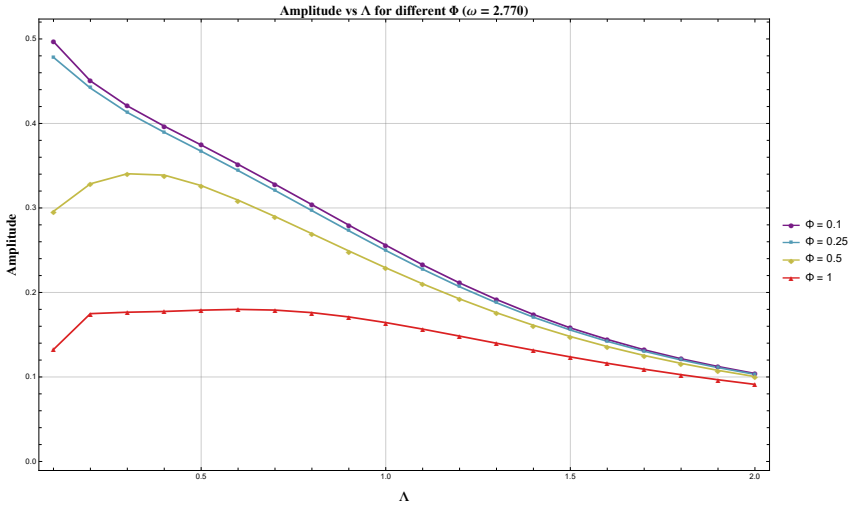


Figure 5.19: Change of the amplitude in terms of Λ with different values of Φ with Ω.

The two principal parameters are the Energie's number (Λ) and the frequency of the external force (Ω). This is because the Energie's number defines the natural frequency of the system, and the external force brings us the resonance, when its value is approximately the same as the natural frequency, increasing the amplitude.

Firstly, we are going to show the behavior of the amplitude with different Λ values at a specific frequency, in a medium with a fixed friction number (Φ), this can be shown in the following figure (5.20).

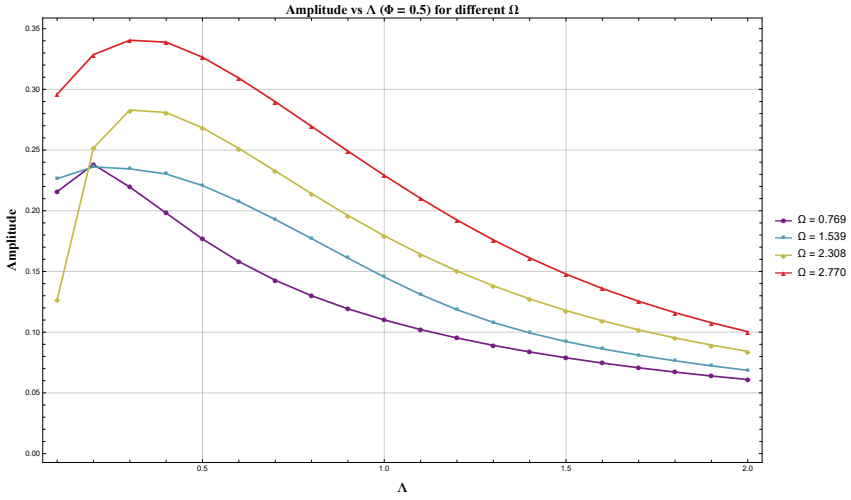


Figure 5.20: Change of the amplitude in terms of Λ with different values of Φ with Ω.

We know that the frequency (Ω) plays an important role, because its values figure (5.21)

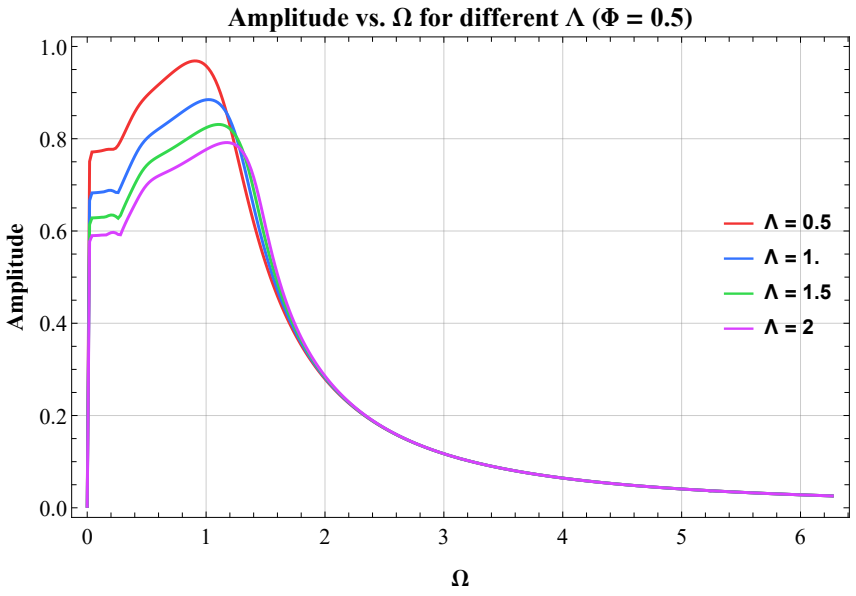


Figure 5.21: Change of the amplitude in terms of Λ with different values of Φ with Ω.

5.3.5 Fluid force

As we mentioned in the fluid chapter, there are a lot of dimensionless numbers that help us to describe the behavior of the fluid, but for the VIV nature, the most important number is the Strouhal number. This is because it describes the vortex shedding behind a body.

Another important dimensionless number is the inertial–gravitational interaction number Γ ($\Gamma = \frac{U}{\sqrt{gD}}$, where U is the speed of the fluid and D is the characteristic length), because it compares the structure acceleration due to flow forcing relative to gravity acceleration, and we know that gravity plays an important role in this system, in particular in the definition of the Energie's number.

Now, we are going to rewrite our differential equation using the fluid expressions. Using fluid dimensionless numbers, like in the previous problem, our external force has the following expression.

$$F_f = \frac{1}{2}\rho U^2 dLC_L, \tag{5.3.42}$$

where L is the axial longitude of the free magnet, C_L is the lift coefficient, d is the width of our free magnet (diameter), and ρ is the density of our fluid. If the external force is the fluid force. Then, we can rewrite the force terms of the Gravity-inertial number as:

$$\begin{aligned} F_0 = \frac{1}{2}\rho U^2 dLC_L \implies \tilde{F}F_s = \frac{1}{2}\rho U^2 dLC_L \\ \tilde{F} = \frac{\rho U^2 dLC_L}{2F_s} \implies \tilde{F} = \frac{\rho U^2 dLC_L}{2\Lambda mg} \\ \tilde{F} = \frac{\rho \Gamma^2 DdLC_L}{2m\Lambda} \therefore \tilde{F} = \frac{1}{2m^*}\frac{C_L\Gamma^2}{\Lambda}. \end{aligned} \tag{5.3.43}$$

We define m^* as the mass ratio between the mass of the structure and the mass of the fluid, that is, $m^* = \left(\frac{m}{\rho DS}\right)$. Something that is relevant to highlight is that the lift coefficient

(C_L) could be a function of time, which would depend on the geometry, but in this specific example, we are going to work with a constant lift coefficient.

$$\frac{d^2\tilde{x}}{d\tau^2} + 2\Phi\frac{d\tilde{x}}{d\tau} + 1 + \Lambda\left[\frac{1}{1-\tilde{x}} - \frac{1}{\tilde{x}}\right] = \frac{1}{2m^*}C_L\,\Gamma^2\cos(\Omega\tau), \tag{5.3.44}$$

as in the general case of a forced system, the amplitude of our oscillation will depend on the energy's number (Λ) and the frequency of the vortices, which in this case is the Strouhal number (St).

Another thing that we need to consider when we are working with a fluid force is the geometry of our structure, cause its shape will define the lift coefficient and the Strouhal number (St). The Stouhal number definition is: $St = \frac{f_s D}{U}$. Then we can find a relation between the Strouhal and the frequency number. This relation is:

$$\begin{aligned} \omega = 2\pi f_s \implies \frac{\Omega}{t_s} &= 2\pi\left(\frac{StU}{D}\right) \\ \implies \Omega = \frac{2\pi StUt_s}{D} \implies \Omega &= \frac{2\pi StU}{D}\sqrt{\frac{h}{g}} \\ \implies \Omega = \frac{2\pi StU}{\sqrt{gD}}\sqrt{\frac{h}{D}} \implies &\boxed{\Omega = 2\pi St\Gamma\sqrt{\frac{h}{D}} = \frac{2\pi StUt_s}{D}}. \end{aligned} \tag{5.3.45}$$

Then, the frequency of the system is related to the Strouhal and Gravity-inertial numbers. If the characteristic length is equal to the equilibrium distance ($D = h$), the frequency number can be simplified as:

$$\Omega = 2\pi St\Gamma. \tag{5.3.46}$$

Then, we can rewrite our dimensionless differential equation as:

$$\frac{d^2\tilde{x}}{d\tau^2} + 2\Phi\frac{d\tilde{x}}{d\tau} + 1 + \Lambda\left[\frac{1}{1-\tilde{x}} - \frac{1}{\tilde{x}}\right] = \frac{1}{2m^*}C_L\,\frac{\Gamma^2}{\Lambda}\cos(2\pi St\Gamma\tau), \tag{5.3.47}$$

It is important to see that our system now depends on the values of the Strouhal (St), Gravity-inertia(Γ) and Energie's (Λ) numbers and the friction number (Φ). In this system, the most important numbers are the Strouhal and Gravity-inertial numbers because they define the frequency of the oscillation.

Something interesting that we can find in our equation is the relation $R = \frac{\Gamma^2}{\Lambda}$, because it relates the amplitude of the system to the properties of the fluid and the energy of the magnets. So, it is important to analyse how the system reacts to this variable, for this reason, we made the following table.

As observed, the optimal operating regime for our system occurs when $R \approx 1$, as this allows the resonance to maintain the oscillations within the optimal operatin range. In contrast, if $R \gg$, the fluid force dominates the magnetic forces, leading to excessive oscillation amplitudes that may cause the free magnet to interact undesirably with the structure. On the other hand, if $R \ll 1$, the fluid forcing is too weak, and the system remains under-excited.

Regime	R	Physical Interpretation
Gravity–dominated	$R \ll 1$	Fluid inertial forces are much weaker than magnetic forces. Motion is primarily governed by magnetic attraction/repulsion and gravity. The system behaves like a nonlinear magnetic oscillator with weak fluid forcing.
Balanced regime	$R \approx 1$	Fluid inertial effects and magnetic forces are comparable in magnitude. Possible resonant responses depending on forcing. System sensitivity to initial conditions increases.
Fluid–dominated	$R \gg 1$	Fluid inertial forces dominate the dynamics over magnetic forces. Motion resembles a fluid–structure interaction problem, where drag/lift control the oscillations.

Table 5.4: Different physical regimes determined by the ratio $R = \Gamma^2/\Lambda$.

New dimensionless equation

If we pay attention to the Strouhal number-frequency relation (equation (5.3.45)), we find that there is another possible dimensionless form that could help us to let the dimensionless frequency be only in terms of the Strouhal numbers. This new dimensionless use fluid and structure properties. The new relation is:

$$\Omega = \frac{2\pi StUt_s}{D} = \Omega = 2\pi St \implies \frac{Ut_s}{D} = 1 \quad \therefore \quad \boxed{t_s = \frac{D}{U} = \frac{h}{U}}, \tag{5.3.48}$$

this new time scale lets us to rewrite our differential equation. To rewrite it, we are going to use the dimensionless groups that we obtained previously. Starting the new dimensionless analysis in the equation of motion (5.3.3).

Applying the dimensionless groups in our differential equation, we can find the equation of motion

$$\begin{aligned}
&\frac{d^2x}{dt^2} + \frac{\beta}{m}\frac{dx}{dt} + g + \frac{M}{m}\left[\frac{1}{h-x} - \frac{1}{x}\right] = \frac{F}{m}\cos(\omega t) \\
\implies &\frac{d^2\tilde{x}}{d\tau^2} + \frac{\beta t_s}{m}\frac{d\tilde{x}}{d\tau} + \frac{gt_s^2}{x_s} + \frac{Mt_s^2}{mx_s}\left[\frac{1}{h - x_s\tilde{x}} - \frac{1}{x_s\tilde{x}}\right] = \frac{Ft_s^2}{mx_s}\cos(2\pi St\tau) \\
\implies &\frac{d^2\tilde{x}}{d\tau^2} + \frac{\beta h}{mU}\frac{d\tilde{x}}{d\tau} + \underbrace{\frac{gh^2}{hU^2}}_{1/\Gamma^2} + \frac{Mh^2}{mhU^2}\left[\frac{1}{h - h\tilde{x}} - \frac{1}{h\tilde{x}}\right] = \frac{Fh^2}{mhU^2}\cos(2\pi St\tau) \\
&\implies \frac{d^2\tilde{x}}{d\tau^2} + \frac{\beta h}{mU}\frac{d\tilde{x}}{d\tau} + \frac{1}{\Gamma^2} + \frac{M}{mU^2}\left[\frac{1}{1-\tilde{x}} - \frac{1}{\tilde{x}}\right] = \frac{Fh}{mU^2}\cos(2\pi St\tau)
\end{aligned} \tag{5.3.49}$$

Now, using the definitions of the Energie's number ($\Lambda = \frac{M}{mgh}$), and the dimensionless form of the force ($\tilde{F} = \frac{Fh}{M}$); we can dimensionless our equation of motion.

$$
\begin{aligned}
&\frac{d^2\tilde{x}}{d\tau^2} + \frac{\beta h}{mU}\frac{d\tilde{x}}{d\tau} + \frac{1}{\Gamma^2} + \frac{\Lambda g h}{U^2}\left[\frac{1}{1-\tilde{x}} - \frac{1}{\tilde{x}}\right] = \frac{F_s \tilde{F} h}{mU^2}\cos(2\pi St\tau) \\
\implies &\frac{d^2\tilde{x}}{d\tau^2} + \frac{\beta h}{mU}\frac{d\tilde{x}}{d\tau} + \frac{1}{\Gamma^2} + \frac{\Lambda}{\Gamma^2}\left[\frac{1}{1-\tilde{x}} - \frac{1}{\tilde{x}}\right] = \frac{\tilde{F} M}{mU^2}\cos(2\pi St\tau) \\
\implies &\frac{d^2\tilde{x}}{d\tau^2} + \frac{\beta h}{mU}\frac{d\tilde{x}}{d\tau} + \frac{1}{\Gamma^2} + \frac{\Lambda}{\Gamma^2}\left[\frac{1}{1-\tilde{x}} - \frac{1}{\tilde{x}}\right] = \frac{\tilde{F}\Lambda m g h}{mU^2}\cos(2\pi St\tau) \\
\implies &\frac{d^2\tilde{x}}{d\tau^2} + \frac{\beta h}{mU}\frac{d\tilde{x}}{d\tau} + \frac{1}{\Gamma^2} + \frac{\Lambda}{\Gamma^2}\left[\frac{1}{1-\tilde{x}} - \frac{1}{\tilde{x}}\right] = \frac{\tilde{F}\Lambda g h}{U^2}\cos(2\pi St\tau) \\
\implies &\frac{d^2\tilde{x}}{d\tau^2} + \frac{\beta h}{mU}\frac{d\tilde{x}}{d\tau} + \frac{1}{\Gamma^2} + \frac{\Lambda}{\Gamma^2}\left[\frac{1}{1-\tilde{x}} - \frac{1}{\tilde{x}}\right] = \frac{\tilde{F}\Lambda}{\Gamma^2}\cos(2\pi St\tau).
\end{aligned}
\tag{5.3.50}
$$

Now, we can define the Friction number in terms of the new time scale, this new friction number is:

$$
\Phi = \frac{\beta h}{2mU}. \tag{5.3.51}
$$

Then using this new definition, the magneto-fluid ratio ($R = \frac{Fr^2}{\Lambda}$) and the fluid force (equation (5.3.43))definitions, our differential equation is:

$$
\begin{aligned}
\implies &\frac{d^2\tilde{x}}{d\tau^2} + \frac{\beta h}{mU}\frac{d\tilde{x}}{d\tau} + \frac{1}{\Gamma^2} + \frac{\Lambda}{\Gamma^2}\left[\frac{1}{1-\tilde{x}} - \frac{1}{\tilde{x}}\right] = \frac{\tilde{F}\Lambda}{\Gamma^2}\cos(2\pi St\tau) \\
\implies &\frac{d^2\tilde{x}}{d\tau^2} + 2\Phi\frac{d\tilde{x}}{d\tau} + \frac{1}{\Gamma^2} + \frac{1}{R}\left[\frac{1}{1-\tilde{x}} - \frac{1}{\tilde{x}}\right] = \frac{\tilde{F}}{R}\cos(2\pi St\tau) \\
\implies &\frac{d^2\tilde{x}}{d\tau^2} + 2\Phi\frac{d\tilde{x}}{d\tau} + \frac{1}{\Gamma^2} + \frac{1}{R}\left[\frac{1}{1-\tilde{x}} - \frac{1}{\tilde{x}}\right] = \frac{\frac{1}{2m^*}C_L R}{R}\cos(2\pi St\tau) \\
\therefore \quad &\frac{d^2\tilde{x}}{d\tau^2} + 2\Phi\frac{d\tilde{x}}{d\tau} + \frac{1}{\Gamma^2} + \frac{1}{R}\left[\frac{1}{1-\tilde{x}} - \frac{1}{\tilde{x}}\right] = \frac{1}{2m^*}C_L\cos(2\pi St\tau).
\end{aligned}
\tag{5.3.52}
$$

This last result is important because it lets us the equation of motion in terms of four dimensionless parameters that are: the Strouhal (St) and Gravity-inertia(Γ) numbers, the friction number (Φ) and the magneto-fluid ratio (R), where two are directly obtained by the fluid, one is by the environment and the last is the relation between the movement of the fluid and the energy of the free magnet. And it is very important because the frequency of the external force only depends in terms of the Strouhal number, or in other words, the vortex–shedding frequency of the von Kármán street.

If we use the optimal regime of R in our system, where is when $R \approx 1$, then our differential equation can be re expressed in terms of only three dimensionless parametes (St, Γ, Φ), like we show in the following equation.

$$
\frac{d^2\tilde{x}}{d\tau^2} + 2\Phi\frac{d\tilde{x}}{d\tau} + \frac{1}{\Gamma^2} + \frac{1}{1-\tilde{x}} - \frac{1}{\tilde{x}} = \frac{1}{2m^*}C_L\cos(2\pi St\tau) \tag{5.3.53}
$$

5.4 Harnessing energy model

With the equation of motion of the system, we know the behavior of our dynamical system, and we are able to predict its motion. That lets us know where and when the system is near or far from a point. So, using this knowledge and Faraday's law, we can induce a voltage in an open circuit. Then, introducing Faraday's law:

$$\varepsilon = -n\frac{d\Phi_B}{dt}, \tag{5.4.1}$$

where n is the number of turns of a coil, Φ_B is the magnetic flux. Now, using the definition of the magnetic flux:

$$\Phi_B = BS\cos(\alpha), \tag{5.4.2}$$

where B is the magnetic field, S the surface area of the coil, and α is the angle between the magnetic field and the normal vector of the surface area. It is important to remark that in our system the magnet and the coil are always aligned with $\alpha = \pi$. Using this information, our magnetic flux is:

$$\Phi_B = -BS. \tag{5.4.3}$$

If we use the magnetic field of a dipole magnet, we obtain that it depends on the distance. The magnetic field is.

$$B = \frac{\mu_0 \mathcal{M}_D}{2x^3}, \tag{5.4.4}$$

where μ_0 is the magnetic premeability of the free space, $\mathcal{M}_D$ is the magnetic dipole moment,and x is the distance between the magnet and the coil. Applying the previous definition to the induction Faraday's law.

$$\varepsilon = n\frac{d}{dt}(BS) \implies \varepsilon = \frac{n\mu_0\mathcal{M}_D S}{2}\frac{d}{dt}\left(\frac{1}{x(t)^3}\right). \tag{5.4.5}$$

For simplicity we define $\zeta = \frac{n\mu_0\mathcal{M}_D S}{2}$. Then, our generated voltage is:

$$\varepsilon = \zeta\frac{-3x(t)^2\cdot\dot{x}(t)}{x(t)^6} = -\frac{3\zeta\dot{x}(t)}{x(t)^4}. \tag{5.4.6}$$

Now, using the dimensionless parameters and solutions.

$$\varepsilon = -3\zeta\frac{\frac{d(x_s\tilde{x})}{d(t_s\tau)}}{(x_s\tilde{x})^4} = -3\zeta\left(\frac{x_s}{t_s x_s^4}\right)\frac{\dot{\tilde{x}}(\tau)}{\tilde{x}(\tau)^4} = -\frac{3\zeta}{t_s x_s^3}\left(\frac{\dot{\tilde{x}}(\tau)}{\tilde{x}(\tau)^4}\right) \quad \therefore \quad \boxed{\varepsilon = -\frac{3\zeta U}{h^4}\left(\frac{\dot{\tilde{x}}(\tau)}{\tilde{x}(\tau)^4}\right)} \tag{5.4.7}$$

Now, using the equation of the induced voltage and the numerical solution of the system, we can find the response of the voltage to the motion of our system. For a specific case, we use in the system the following values: $\Lambda = 1$, $\Phi = 0.5$, $\epsilon = 0.5$, $\Omega = 0.9\Omega_0$, $F = 1$; and

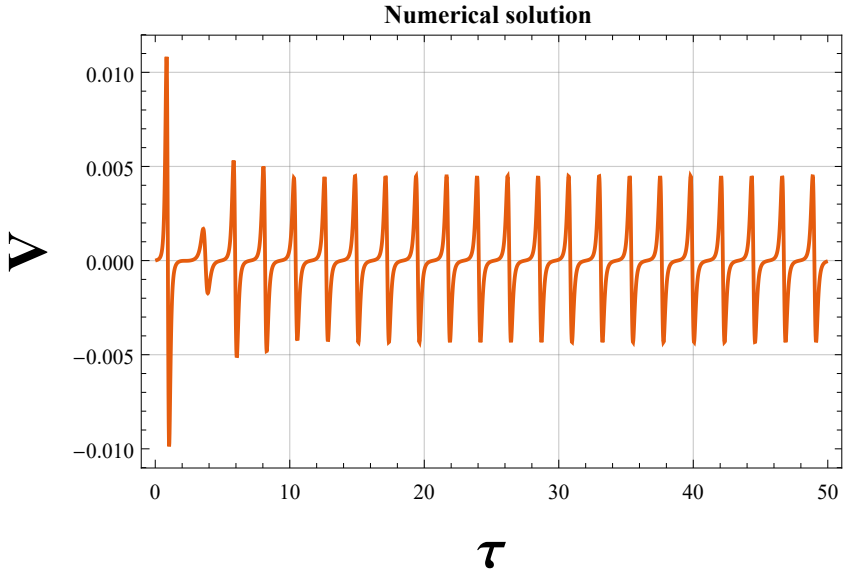


Figure 5.22: Voltage generated by the system

the coil has the following parameters: $n = 10$, $h = 1$, $S = 1$, $\mathcal{M} = 1$, $U = 1$. Then, the voltage generated over time is represented in the following figure.

Now, to see the dependence of the voltage to the friction, we plot the voltage generated by different Φ values

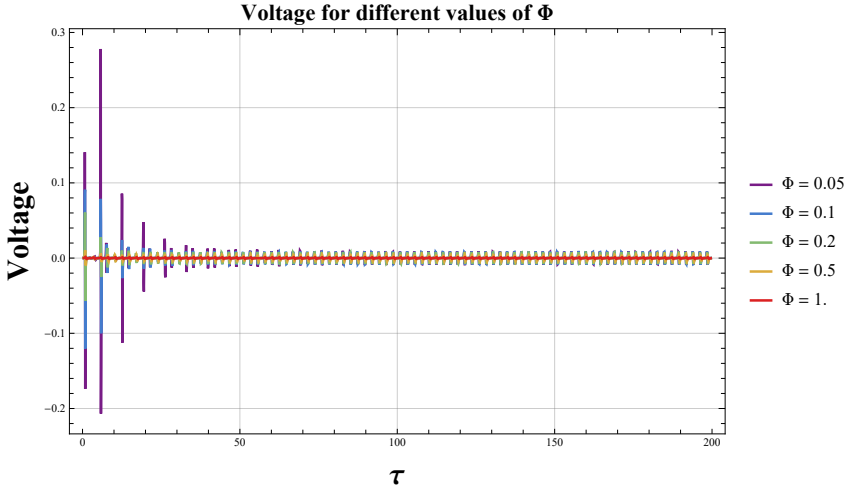


Figure 5.23: Voltage generated by different values of friction.

For the previous figure, we can see that the voltage decreases when the damping increases. Another interesting case is how the voltage reacts to different frequency values. These frequencies are the frequencies of the vortices.

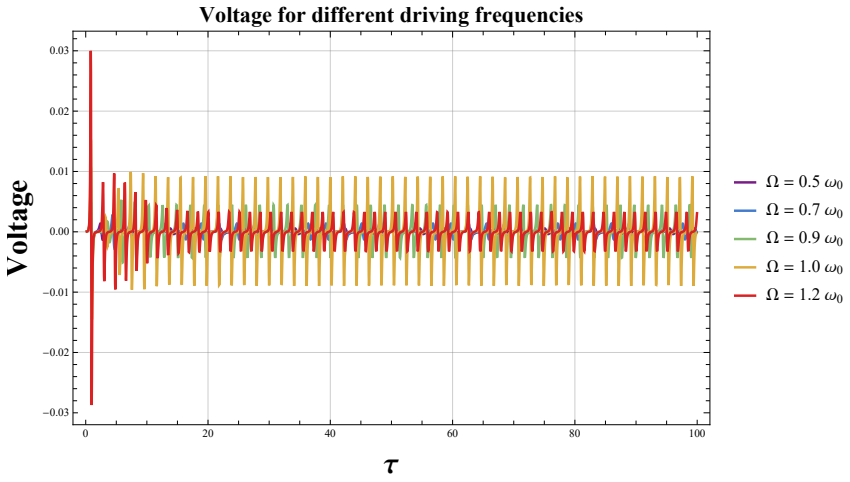


Figure 5.24: Voltage generated by different frequencies, multiples of the natural frequency Ω_0

As we can see in the previous figure, the magnitude of the voltage reduces when the frequency tends to the natural frequency of the system, but the voltage is more recurrent.

5.5 Conclusions

In this chapter, we confirm that our system can oscillate, and the interaction with a fluid in motion could be the source of energy that it needs to maintain the oscillation. In contrast to the first system (magneto-gravitational system), this system has a symmetric oscillation, which is caused by the structural design that establishes a region where the system moves.

The presence of the two fixed magnets indicates that the magnetic energy plays an important role. This can be shown in the expressions of the natural frequency and the equilibrium position, which are written in terms of the Energie's number (Λ).

$$\omega_0 = \sqrt{4\Lambda + 2\sqrt{1+4\Lambda^2} + \frac{1}{\Lambda}}, \tag{5.5.1}$$

$$\tilde{x}_{eq} = \frac{x_{eq}}{h} = \frac{2\Lambda + 1 - \sqrt{4\Lambda^2+1}}{2} \tag{5.5.2}$$

In terms of the external force, we can see that the frequency of the vortices generated by the fluid-structure interaction is responsible for the motion and the quantity of the voltage generated.

For this system, the amplitude of the oscillation depends on some dimensionless numbers, which are the Energie's number (Λ), the inertial-gravitational interaction number (Γ), the friction number (Φ), the lift coefficient (C_L), the mass ratio (m^*), and the Strouhal number (St). One of the most important parameters is the lift coefficient, because it changes with the shape of the body. For this reason, it becomes essential to study different geometries and choose the one that maximizes the lift coefficient. A better shape would increase the force that the fluid transfers to the system, and it can increase the voltage that we generate.

We also observed that the fluid parameters play an important role. The Strouhal number sets the resonance condition that creates the oscillation, while the inertial gravity interaction number affects not only the resonance itself but also the stability and complexity of the dimensionless equation. This shows how sensitive the system is to the properties of the fluid and how these parameters can change the global behavior of the solution.

As in the previous chapter, the mathematical tools used throughout the chapter were very useful to understand how the system reacts and why it behaves the way it does. These methods helped us find expressions that describe the dynamics more clearly and opened the possibility of using this kind of system in future projects related to clean energy generation. The ideas developed here could be the base to design simple and stable oscillators powered only by natural fluid motion.

CHAPTER 6

Rigid pendulum with magnets

> The production of electricity from magnetism is a result of motion... when a magnet passes before a conductor, or a conductor before a magnet, a power is awakened that flows as a current, manifesting the beautiful unity of force and motion.
>
> Michael Faraday, *"Experimental Researches in Electricity"*, 1831

In this chapter, we analyze a system composed of a body that oscillates like a pendulum, with magnets at the extremum points that introduce repulsive forces. To see the system, we present the following figure.

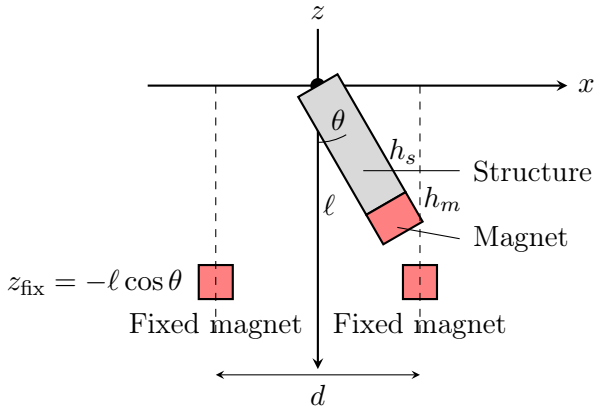


Figure 6.1: Graphic representation of our system

Before starting the analysis of this system, we need to describe it and find its center. The system consists of two coupled bodies and a set of magnets that will be used to create repulsive forces.

The bodies are quadrangular prisms, but for simplicity, we will assume that they are square prisms. The smaller body (a magnet) is positioned beneath the larger one (the oscillating structure). Both bodies are aligned along the vertical symmetry axis, so the centers of both bodies are aligned.

If both bodies are homogeneous, their centers of mass (COM) will coincide with their geometric centers, meaning they are aligned. Now, if the structure has a mass m_s and a height h_s, and the magnet has a mass m_m and a height h_m, we can determine the COM of the entire system. It is important to note that the bodies undergo pendular motion, so it is better to use polar coordinates. Thus, we have:

$$\vec{R}_{COM} = \frac{1}{M}\sum_{i=1}^{N} m_i \vec{r}_i = \frac{m_s \vec{r}_{COMs} + m_m \vec{r}_{COMm}}{m_m + m_s}, \tag{6.0.1}$$

where $\vec{r}_{COMs}$ is the position vector of the COM of the structure and $\vec{r}_{COMm}$ is the position vector of the COM of the magnet. Next, we will calculate the position vector of the COM for the structural body. To do that, we suppose that the structure has a homogeneous, and then the density is:

$$\rho(\vec{r}) = \frac{m_s}{V} = \frac{m_s}{LWH}, \tag{6.0.2}$$

where V is the volume, L is the length, W is the width, and H is the height of the structure body. Now, using the previous result and the formula of the position vector of the COM, we obtain:

$$\begin{aligned}
\vec{r}_{COMs} &= \frac{1}{m_s}\int_V \vec{r}\rho(\vec{r})dV \\
&= \frac{1}{m_s}\int_V \frac{m_s\vec{r}}{LWH}dV \\
&= \frac{1}{LWH}\int_V \vec{r}dV \\
&= \frac{1}{LWH}\int_V x\hat{x} + y\hat{y} + z\hat{z}dV \\
&= \frac{1}{LWH}\int_{z_0}^{z_1}\int_{y_0}^{y_1}\int_{x_0}^{x_1} x\hat{x} + y\hat{y} + z\hat{z}dxdydz,
\end{aligned} \tag{6.0.3}$$

To simplify the calculus, we are going to do the integrals using Einstein's sum convention. Hence, we obtain:

$$\begin{aligned}
\int_{\alpha_0}^{\alpha_1}\int_{\beta_0}^{\beta_1}\int_{\gamma_0}^{\gamma_1} x_i\hat{x}_i dx_i dx_j dx_k &= (x_j)\Big|_{\beta_0}^{\beta_1}(x_k)\Big|_{\alpha_0}^{\alpha_1}\left(\frac{x_i^2}{2}\right)\Big|_{\gamma_0}^{\gamma_1}\hat{x}_i \\
&= \frac{1}{2}(\alpha_1-\alpha_0)(\beta_1-\beta_0)\left(\gamma_1^2-\gamma_0^2\right)\hat{x}_i
\end{aligned} \tag{6.0.4}$$

Substituting this result in the equation 6.0.3 and doing the algebra, we can see:

$$\begin{aligned}
\vec{r}_{COMs} &= \frac{1}{LWH}\int_{z_0}^{z_1}\int_{y_0}^{y_1}\int_{x_0}^{x_1} x\hat{x} + y\hat{y} + z\hat{z}dxdydz \\
&= \frac{1}{LWH}[\frac{1}{2}(z_1-z_0)(y_1-y_0)\left(x_1^2-x_0^2\right)\hat{x} + \frac{1}{2}(z_1-z_0)(x_1-x_0)\left(y_1^2-y_0^2\right)\hat{y} \\
&+ \frac{1}{2}(x_1-x_0)(y_1-y_0)\left(z_1^2-z_0^2\right)\hat{z}]
\end{aligned} \tag{6.0.5}$$

If the integration limits are: $x_0 = -\frac{L}{2}$, $x_1 = \frac{L}{2}$ $y_0 = -\frac{W}{2}$, $x_1 = \frac{W}{2}$ $z_0 = -H$, $z_1 = 0$. Then, the position vector of the COM of the structure is:

$$\begin{aligned}\vec{r}_{COMs} &= \frac{1}{LWH}[\frac{1}{2}(z_1 - z_0)(y_1 - y_0)\left(x_1^2 - x_0^2\right)\hat{x} + \frac{1}{2}(z_1 - z_0)(x_1 - x_0)\left(y_1^2 - y_0^2\right)\hat{y} \\ &+ \frac{1}{2}(x_1 - x_0)(y_1 - y_0)\left(z_1^2 - z_0^2\right)\hat{z}] \\ &= \frac{1}{2LWH}\left[0\hat{x} + 0\hat{y} - LWH^2\hat{z}\right] \\ &= 0\hat{x} + 0\hat{y} - \frac{H}{2}\hat{z}.\end{aligned} \tag{6.0.6}$$

As we know, the height of the structure is h_s, then the position vector is:

$$\vec{r}_{COMs} = 0\hat{x} + 0\hat{y} - \frac{h_s}{2}\hat{z} \tag{6.0.7}$$

.

Now we can reduce the dimensions of the motion because the movement of the structure is constrained to a plane, specifically the x-z plane. Then, the position vector now is:

$$\vec{r}_{COMs} = 0\hat{x} - \frac{h_s}{2}\hat{z} \tag{6.0.8}$$

Using the transform relations to change the cartesian coordinates to polar coordinates:

$$r = \sqrt{x^2 + z^2} = \sqrt{0 + \left(\frac{h_s}{2}\right)^2} = \frac{h_s}{2}, \tag{6.0.9}$$

$$\theta = \frac{3\pi}{2}. \tag{6.0.10}$$

Therefore, the position of the COM in polar coordinates is:

$$\vec{r}_{COMs} = \frac{h_s}{2}\hat{r} \tag{6.0.11}$$

Analogously, we can do the same process to the magnet body, then the position vector of COM of the magnet is:

$$\vec{r}_{COMm} = \frac{h_m}{2}\hat{r} \tag{6.0.12}$$

But, as we know, the magnet is under the structure, then the relative height of the magnet with respect of the origin is $h_s + h_m$. Therefore, the position vector of COM of the magnet is:

$$\vec{r}_{COMm} = \left(h_s + \frac{h_m}{2}\right)\hat{r} \tag{6.0.13}$$

Then, the COM of the system has the following form:

$$\begin{aligned}
\vec{R}_{COM} &= \frac{m_s\vec{r}_{COMs} + m_m\vec{r}_{COMm}}{m_m + m_s} \\
&= \frac{1}{m_s + m_m}\left(\frac{m_s h_s}{2} + m_m\left\{h_s + \frac{h_m}{2}\right\}\right)\hat{r} \\
&= \frac{1}{m_s + m_m}\left[\frac{1}{2}\left(m_s h_s + 2m_m h_s + m_m h_m\right)\hat{r}\right] \\
&= \frac{1}{2}\left(\frac{1}{m_s + m_m}\right)\left[h_s(m_m + m_s) + m_m(h_s + h_m)\right]\hat{r} \\
\boxed{\therefore \vec{R}_{COM} = \frac{1}{2}\left[h_s + \frac{m_m}{m_s + m_m}(h_s + h_m)\right]\hat{r}}&.
\end{aligned} \tag{6.0.14}$$

The previous result is important because it lets us know that the position of the center of mass of the system depends on the masses involved. The center of mass is closer to the larger mass and the distance to the origin is always constant.

Another important thing is to define the space where our pendulum will be acting. We have the frame structure in the form of a rectangle that has a width of d and a height of h, like:

At the top, in the middle of the width of the frame structure, we place a pivot. We decide to place the origin of the coordinate system in the pivot. Then, the coordinates of the pivot are $(0, 0)$.

As we know, we are going to use a pendulum formed by two bodies, then our pendulum is a ***"physical pendulum"***, and the length of that pendulum (ℓ) is the sum of the height of the prism and the magnet ($\ell = h_s + h_m$). Now, we can find some geometric relations between the length of the pendulum and the frame structure, which are:

$$\begin{aligned}
&d \geq 2\ell \implies \ell \leq \frac{d}{2}, \\
&h \geq \ell,
\end{aligned} \tag{6.0.15}$$

These equations define the conditions for a pendulum acting within our frame structure.

We can restrict the movement of the pendulum by placing two magnets in the system. These magnets are static and are located symmetrically about the system's origin. They are at a vertical distance η, with the horizontal distance from the middle of the structure where they are $\frac{\delta}{2}$. Then, the coordinates are $(\frac{\delta}{2}, -\eta)$ for the right magnet and $(-\frac{\delta}{2}, -\eta)$ for the left magnet. With the coordinates of the fixed magnets, we can find the maximum angle (Θ) that our system could have:

$$\delta = 2\ell\sin(\Theta) \implies \sin(\Theta) = \frac{\delta}{2\ell} \implies \Theta = \arcsin\left(\frac{\delta}{2\ell}\right), \tag{6.0.16}$$

$$\eta = -\ell\cos(\Theta) \implies \cos(\Theta) = -\frac{\eta}{\ell} \implies \Theta = \arccos\left(-\frac{\eta}{\ell}\right). \tag{6.0.17}$$

Both expressions give us a relation to find the maximum angle (Θ) of the system, so we can find a relation between the arguments of the trigonometric functions; this relation is the Pythagorean relation of a unitary circle, then:

$$\left(\frac{\delta}{2\ell}\right)^2 + \left(-\frac{\eta}{\ell}\right)^2 = 1 \implies \ell^2 = \frac{\delta^2}{4} + \eta^2. \tag{6.0.18}$$

The previous result gives us a relation between the length of our pendulum and the coordinates of the fixed magnets.

Then, we can find the domains of our operative space and structural parameters. The operative domain in the horizontal direction is:

$$-\frac{d}{2} \leq x \leq \frac{d}{2} \implies Dom(x) = \left[-\frac{d}{2}, \frac{d}{2}\right]. \tag{6.0.19}$$

In the vertical direction, the domain is:

$$0 \leq z \leq h \implies Dom(z) = [-h, 0]. \tag{6.0.20}$$

Then, the domain of the horizontal position of the fixed magnets is:

$$0 \leq \delta \leq \frac{d}{2} \implies Dom(\delta) = \left[0, \frac{d}{2}\right]. \tag{6.0.21}$$

Finally, the domain of angle θ is:

$$-\Theta \leq \theta \leq \Theta \implies Dom(\theta) = [-\Theta, \Theta] \tag{6.0.22}$$

It is important to highlight that the system is going to be analyzed in polar coordinates, because the movement follows a curved trajectory.

6.1 Ideal case

For the ideal case, as we know, the dissipation forces are considered as null. Then, the only forces that play in our system are the gravitational force, the magnetic repulsive forces, and the torque of the structure. Then, the system of forces is described by:

$$\vec{F} = \vec{T} + \vec{g} + \vec{F}_{m1} + \vec{F}_{m2} \tag{6.1.1}$$

We know that this way is too complicated, so, as Landau and Lifshits said, it is simplest to work with the energies that act on the system. For that reason, we are going to use the Lagrangian and Hamiltonian analysis.

6.1.1 Lagrangian analysis

As we have seen previously, analytical mechanics requires knowing the expressions for the energies. For that reason, we are going to find them.

To find the expressions for the energies, we need to remember that we are working with a body, so we can express everything in terms of the center of mass and the moments of inertia of the body. The center of mass of a quadrangular prism is:

$$I_{COM} = \frac{1}{12} m(w^2 + h^2), \tag{6.1.2}$$

where w is the width of our prism. Something important to remember is that our body is a pendulum, so we can use the parallel axis theorem to find the moment of inertia at the pivot point.

$$I_{pivot} = I_{COM} + mR_i^2, \tag{6.1.3}$$

where R_i is the distance from the origin to the center of mass of our body. It is important to remember that we have two bodies, so we need to use the pivot moment of inertia of both bodies.

$$\begin{aligned} I_0 &= I_s + I_m \\ &= I_{COMs} + mR_s^2 + I_{COMm} + mR_m^2 \\ &= \frac{1}{12} m_s(w^2 + h_s^2) + m\left(\frac{h_s}{2}\right)^2 + \frac{1}{12} m_m(w^2 + h_m^2) + m\left(h_s + \frac{h_m}{2}\right)^2. \end{aligned} \tag{6.1.4}$$

It is important to highlight that I_0 is a constant because every term in the moment of inertia is defined by the geometry of our body.

Now, we can find the expression of the kinetic energy of a system. This can be found by the following equation:

$$T = \frac{1}{2} I_0 \dot{\theta}^2. \tag{6.1.5}$$

The potential energy is integrated by the gravitational potential energy and the two magnetic repulsion energies. The gravitational potential energy is the same as that of a simple pendulum.

$$U_g = mgR(1 - \cos(\theta)), \tag{6.1.6}$$

where R is the distance between the pivot and the center of mass of our system ($R = |\vec{R}_{COM}|$), m is the mass of all the system ($m = m_s + m_m$), g is the gravity acceleration and θ is the angle of the position of our system. In the case of the magnetic repulsion energy, we need to consider the angle of the pendulum. Now checking how the distance between magnets changes in terms of the angle is, for the right magnet:

$$
\begin{aligned}
r_R(\theta) &= \sqrt{\left(\frac{\delta}{2} - \ell\sin(\theta)\right)^2 + (\eta + \ell\cos(\theta))^2} \\
&= \sqrt{\frac{\delta^2}{4} - \delta\ell\sin(\theta) + \ell^2\sin^2(\theta) + \eta^2 + 2\eta\ell\cos(\theta) + \ell^2\cos^2(\theta)} \\
&= \sqrt{\frac{\delta^2}{4} + \eta^2 + \ell^2 - \ell(\delta\sin(\theta) - 2\eta\cos(\theta))},
\end{aligned}
\tag{6.1.7}
$$

Now, using the Pythagorean relation (6.0.18)and the relations between position and maximum angle (6.0.16, 6.0.17):

$$
\begin{aligned}
r_R(\theta) &= \sqrt{2\ell^2 - \ell(\delta\sin(\theta) - 2\eta\cos(\theta))} \\
&= \sqrt{2\ell^2 - \ell(2\ell\sin(\Theta)\sin(\theta) + 2\ell\cos(\Theta)\cos(\theta))} \\
&= \sqrt{2\ell^2 - 2\ell^2(\sin(\Theta)\sin(\theta) + \cos(\Theta)\cos(\theta))} \\
\therefore r_R(\theta) &= \ell\sqrt{2(1 - \cos(\Theta - \theta))}
\end{aligned}
\tag{6.1.8}
$$

Now, for the left magnet:

$$
\begin{aligned}
r_L(\theta) &= \sqrt{\left(-\frac{\delta}{2} - \ell\sin(\theta)\right)^2 + (\eta + \ell\cos(\theta))^2} \\
&= \sqrt{\frac{\delta^2}{4} + \delta\ell\sin(\theta) + \ell^2\sin^2(\theta) + \eta^2 + 2\eta\ell\cos(\theta) + \ell^2\cos^2(\theta)} \\
&= \sqrt{2\ell^2 + \ell(\delta\sin(\theta) + 2\eta\cos(\theta))} \\
&= \sqrt{2\ell^2 + \ell(2\ell\sin(\Theta)\sin(\theta) - 2\ell\cos(\Theta)\cos(\theta))} \\
&= \sqrt{2\ell^2 - 2\ell^2(\cos(\Theta)\cos(\theta) - \sin(\Theta)\sin(\theta))} \\
\therefore r_L(\theta) &= \ell\sqrt{2(1 - \cos(\Theta + \theta))}
\end{aligned}
\tag{6.1.9}
$$

With the distance between the fixed magnets and the free movement magnet, we can find the expression of the repulsive magnetic energy in terms of the distance; for this, we applied the expression of the magnetic energy that we used in the previous chapters. Then, the magnetic energies are:

$$
U_{MR} = M\ln(r_R(\theta)) \quad \text{and} \quad U_{ML} = M\ln(r_L(\theta)). \tag{6.1.10}
$$

Then the Lagrangian of our system is:

$$
\begin{aligned}
L &= T - U \\
&= \frac{1}{2}I_0\dot{\theta}^2 + mgR(1 - \cos(\theta)) + M\ln(r_R(\theta)) + M\ln(r_L(\theta))
\end{aligned}
\tag{6.1.11}
$$

Now, applying the Euler-Lagrange equation:

$$
\begin{aligned}
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} &= 0 \\
\frac{d}{dt}\left(I_0\dot{\theta}\right) + mgR\sin(\theta) - \frac{M}{r_R(\theta)}\frac{dr_R(\theta)}{d\theta} - \frac{M}{r_L(\theta)}\frac{dr_L(\theta)}{d\theta} &= 0 \\
I_0\ddot{\theta} + mgR\sin(\theta) - \frac{M}{r_R(\theta)}\frac{dr_R(\theta)}{d\theta} - \frac{M}{r_L(\theta)}\frac{dr_L(\theta)}{d\theta} &= 0 \\
I_0\ddot{\theta} + mgR\sin(\theta) - \frac{M}{r_R(\theta)}\left(\frac{\ell\sin(\Theta+\theta)}{\sqrt{2(1-\cos(\Theta+\theta))}}\right) - \frac{M}{r_L(\theta)}\left(-\frac{\ell\sin(\Theta-\theta)}{\sqrt{2(1-\cos(\Theta-\theta))}}\right) &= 0 \\
I_0\ddot{\theta} + mgR\sin(\theta) - \frac{M\sin(\Theta+\theta)}{2(1-\cos(\Theta+\theta))} + \frac{M\sin(\Theta-\theta)}{2(1-\cos(\Theta-\theta))} &= 0 \\
I_0\ddot{\theta} + mgR\sin(\theta) - \frac{M}{2}\left(\frac{\sin(\Theta+\theta)}{1-\cos(\Theta+\theta)} - \frac{\sin(\Theta-\theta)}{1-\cos(\Theta-\theta)}\right) &= 0.
\end{aligned}
\tag{6.1.12}
$$

Now, using trigonometric identities and doing some algebra to simplify the magnetic term:

$$
\begin{aligned}
\frac{\sin(\Theta+\theta)}{1-\cos(\Theta+\theta)} - \frac{\sin(\Theta-\theta)}{1-\cos(\Theta-\theta)} =& \frac{\sin(\Theta+\theta)(1-\cos(\Theta-\theta)) - \sin(\Theta-\theta)(1-\cos(\Theta+\theta))}{(1-\cos(\Theta+\theta))(1-\cos(\Theta-\theta))} \\
=& \frac{\sin(\Theta+\theta) - \frac{1}{2}(\sin(2\Theta)+\sin(2\theta)) - \sin(\Theta-\theta))}{1-\cos(\Theta-\theta)-\cos(\Theta+\theta)+\cos(\Theta+\theta)\cos(\Theta-\theta)} \\
& - \frac{\frac{1}{4}(\sin(2\Theta)-\sin(2\theta))}{1-\cos(\Theta-\theta)-\cos(\Theta+\theta)+\cos(\Theta+\theta)\cos(\Theta-\theta)} \\
=& \frac{\sin(\Theta+\theta)-\sin(\Theta-\theta)-\sin(2\theta)}{(\cos(\theta)-\cos(\Theta))^2} \\
=& -\frac{2\sin(\theta)(\cos(\theta)-\cos(\Theta))}{(\cos(\theta)-\cos(\Theta))^2} \\
=& -\frac{2\sin(\theta)}{\cos(\theta)-\cos(\Theta)}.
\end{aligned}
\tag{6.1.13}
$$

Then, our differential equation in terms of the maximum angle is:

$$
\boxed{I_0\ddot{\theta} + \left[mgR + \frac{M}{\cos(\theta)-\cos(\Theta)}\right]\sin(\theta) = 0}. \tag{6.1.14}
$$

It is important to highlight that our system does not depend on the length of the pendulum or the coordinates of the fixed magnets. Our system depends on the maximum angle (Θ), the distance between the pivot and the system's center of mass (R), the mass of the structure (m), the magnetic energy (M), and the structure's moment of inertia (I_0).

Then, we can find that our differential equation is:

$$
\ddot{\theta} + \left[\frac{mgR}{I_0} + \frac{M}{I_0}\frac{1}{\cos(\theta)-\cos(\Theta)}\right]\sin(\theta) = 0. \tag{6.1.15}
$$

This last result is very important because it lets us see that if the magnets are eliminated ($M = 0$), the equation of motion is the same as that of a physical pendulum. To simplify our differential equation, we define two constants: the natural frequency of a physical pendulum (ω_{0p}) as $\omega_{0p}^2 = \frac{mgR}{I_o}$ and the frequency of the magnetic repulsion (ϕ_M) as $\phi_M^2 = \frac{M}{I_0}$

$$\ddot{\theta} + \left[\omega_{0p}^2 + \frac{\phi_M^2}{\cos(\theta) - \cos(\Theta)}\right] \sin(\theta) = 0. \tag{6.1.16}$$

The last equation is similar to the equation of motion of a simple pendulum. So, using the analogy, we can conclude that the natural frequency of the system (ω_0) is all the terms that multiply $\sin(\theta)$. Then, our system is as follows:

$$\omega_0^2(\theta) = \omega_{0p}^2 + \frac{\phi_M^2}{\cos(\theta) - \cos(\Theta)} \implies \boxed{\ddot{\theta} + \omega_0^2(\theta)\sin(\theta) = 0}. \tag{6.1.17}$$

It is important to note that the natural frequency of the system is a function of the angle (θ), and it can be a constant for a specific case, that is, the small angle case ($\theta \ll 1$). For the small angle case our equation of motion turns:

$$\omega_0^2 = \omega_{0p}^2 + \frac{\phi_M^2}{\cos(\Theta) + 1} \implies \boxed{\ddot{\theta} + \omega_0^2\theta = 0}. \tag{6.1.18}$$

6.1.2 Hamiltonian analysis

Now, with our Lagrangian, we are going to use the definition of the Hamiltonian to find it for our system. With the Hamiltonian, we can generate a phase portrait to see the behavior of the system. Although we know that the system oscillates like a physical pendulum, because the equation of motion (6.1.15) is the same as that of a physical pendulum when the magnetic energy is null. The principal reason for finding the Hamiltonian of our system is to find the expression of the energy that is in the system.

Remembering, the Hamiltonian is defined through the Legendre transformation of the Lagrangian. For a system described by a generalized coordinate θ and its conjugate momentum p_θ, the Hamiltonian is defined as:

$$H = p_\theta\dot{\theta} - L. \tag{6.1.19}$$

The generalized momentum associated with the coordinate θ is obtained from the Lagrangian as:

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}}. \tag{6.1.20}$$

Using the Lagrangian obtained previously (6.1.11), the Hamiltonian of the system takes the following expression:

$$\boxed{H(\theta, p_\theta) = \frac{p_\theta^2}{2I_0} - mgR(1 - \cos(\theta)) - M\ln(r_R(\theta)) - M\ln(r_L(\theta))} \tag{6.1.21}$$

Since the Hamiltonian does not depend explicitly on time, it represents a conserved quantity of the system. Physically, this corresponds to the total mechanical energy of the pendulum, constituted by the kinetic energy and the potential energies associated with gravity and magnetic interaction.

The equations of motion of the system can now be obtained from Hamilton's equations:

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta}, \tag{6.1.22}$$

$$\dot{p}_\theta = -\frac{\partial H}{\partial \theta}. \tag{6.1.23}$$

Computing the derivatives, we obtain

$$\dot{\theta} = \frac{p_\theta}{I_0}, \tag{6.1.24}$$

and

$$\dot{p}_\theta = -\left[mgR + \frac{M}{\cos(\theta) - \cos(\Theta)}\right]\sin(\theta). \tag{6.1.25}$$

Now, we can generate the phase portrait diagram. It is important to see that the magnetic energy (M) and the maximum angle (Θ) change the phase portrait. To see the impact of M and Θ on the dynamic, we plot a set of diagrams with different values of M and Θ.

For the following diagrams, we want to see how the system reacts to different values of the magnetic energy M, and we use $m = 2\,kg$, $g = 10\,\frac{m}{s^2}$, $R = \frac{5}{8}\,m$, $I_0 = \frac{15}{8}\,kg\,m^2$, and $\Theta = 1.57$.

Now, we are going to see how the system reacts to different values for the maximum angle and a fixed value for the magnetic energy $M = 1$.

As we can see in both sets of diagrams, the system oscillates, but the region of oscillation changes when Θ and M change.

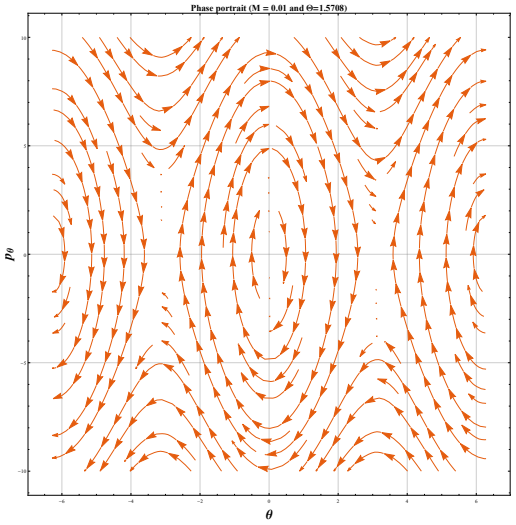
Another thing that we can find with Hamilton's equations is the equilibrium point. To do this, we need $\dot{p}_\theta = 0$, so the equilibrium is:

$$-\left[mgR + \frac{M}{\cos(\theta) - \cos(\Theta)}\right]\sin(\theta) = 0 \implies \begin{cases} \sin(\theta) = 0, & \theta = 0 \\ mgR + \frac{M}{\cos(\theta) - \cos(\Theta)} = 0, & \theta = \pm \arccos\left(\frac{-M - mgR\cos(\Theta)}{mgR}\right) \end{cases} \tag{6.1.26}$$

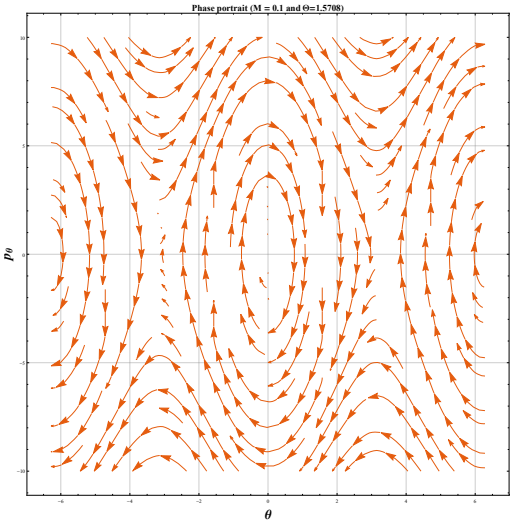
6.1.3 Dimensional analysis

Now, something important to analyze in our differential equation is that the angle does not have units, so the system is partially dimensionless; to make our differential equation dimensionless, we need to work with time.

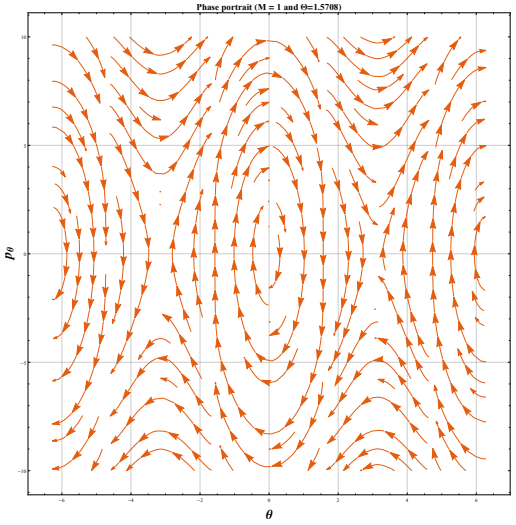
Firstly, we can obtain the dimensions of all the factors that appear in our differential equation (6.1.15). These factors, with their dimensions, are:



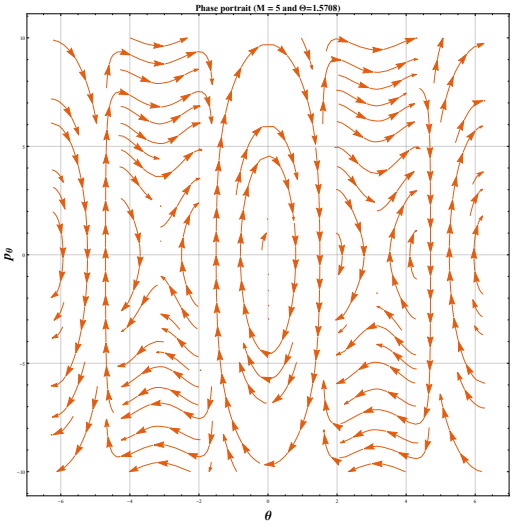
((a)) Phase portrait of the system with $M = 0.01$



((b)) Phase portrait of the system with $M = 0.1$



((c)) Phase portrait of the system with $M = 1$



((d)) Phase portrait of the system with $M = 5$

Figure 6.2: System reaction to different values of M.

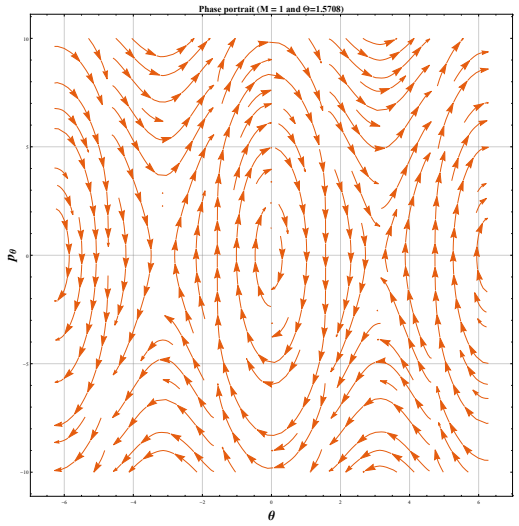
$$[m] = M, [R] = L, [g] = LT^{-2}, [t] = T, [I_0] = ML^2, [M] = ML^2T^{-2}. \tag{6.1.27}$$

It is important that Θ and θ are dimensionless.

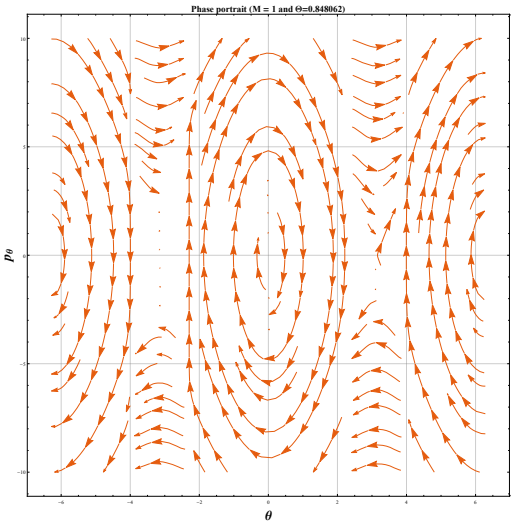
For this system, we are going to use R, g, and I_0 as dimensional variables, because R is the geometric parameter of the system, g is the cinematic parameter, and I_0 is the dynamic parameter. Now, expressing our parameters in terms of the dimensionless groups:

$$\Pi_i = \delta R^\alpha g^\beta I_0^\lambda, \tag{6.1.28}$$

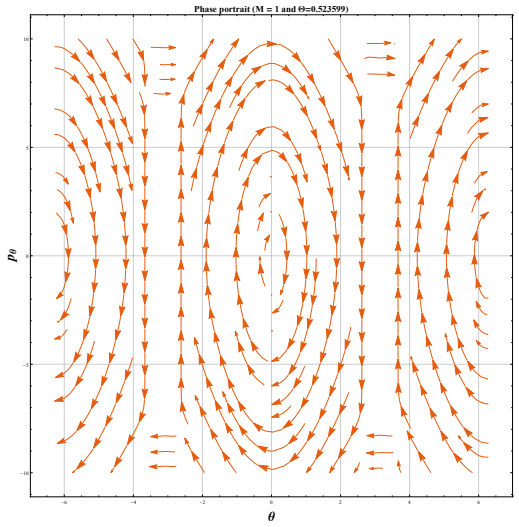
where δ is the variable that we want to be dimensionless, in our system we have the variables m, t, and M. And α, β, and λ are the powers that generate our dimensionless groups.



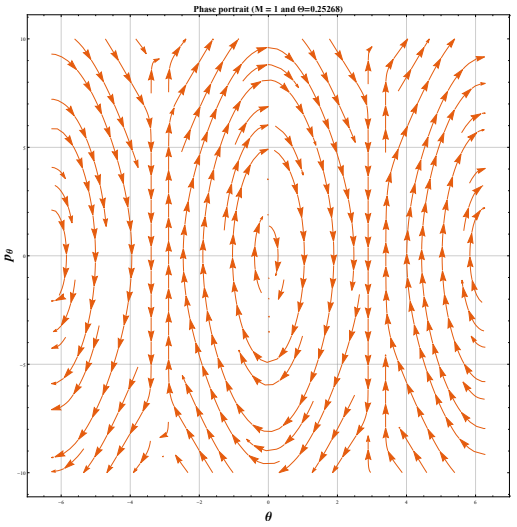
((a)) Phase portrait of the system with $\Theta = 1.57$



((b)) Phase portrait of the system with $\Theta = 0.84$



((c)) Phase portrait of the system with $\Theta = 0.52$



((d)) Phase portrait of the system with $\Theta = 0.25$

Figure 6.3: System reaction to different values of Θ.

To find the powers of our dimensional variables that dimensionless every group, we can construct an equation system. This system is.

$$\begin{pmatrix} L \\ M \\ T \end{pmatrix} + \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & -2 & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \lambda \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \implies \begin{pmatrix} \alpha \\ \beta \\ \lambda \end{pmatrix} = \begin{pmatrix} 2M - L - \frac{T}{2} \\ \frac{T}{2} \\ -M. \end{pmatrix} \tag{6.1.29}$$

It is important to remember that every dimensionless group can be expressed as a fraction. So, the dimensionless groups can be rewritten as:

$$\Pi_i = \frac{\delta}{\delta_s} = \tilde{\delta}, \tag{6.1.30}$$

where δ_s is the scale factor of the variable δ y $\tilde{\delta}$ is the dimensionless form of the variable δ. Now, we can obtain dimensionless groups for the variables m, t, and M.

Dimensionless group of the mass and the Inertia number

The m's group is:

$$\begin{pmatrix}\alpha\\ \beta\\ \lambda\end{pmatrix} = \begin{pmatrix}2\\ 0\\ -1\end{pmatrix} \quad \therefore \quad \boxed{\Pi_1 = \frac{m}{m_s} = \frac{mR^2}{I_0} \implies m_s = \frac{I_0}{R^2}} \quad . \tag{6.1.31}$$

With this we can define a dimensionless number that compares the inertia of the center of mass with the moment of inertia of the structure. This number is:

$$\chi \equiv \Pi_1 = \frac{mR^2}{I_0} \tag{6.1.32}$$

Dimensionless group of the time

The group of t is.

$$\begin{pmatrix}\alpha\\ \beta\\ \lambda\end{pmatrix} = \begin{pmatrix}-\frac{1}{2}\\ \frac{1}{2}\\ 0\end{pmatrix} \quad \therefore \quad \boxed{\Pi_2 = \frac{t}{t_s} = t\sqrt{\frac{g}{R}} \implies t_s = \sqrt{\frac{R}{g}}} \quad . \tag{6.1.33}$$

This result is important; the time scale is in terms of the distance between the pivot and the center of mass of the structure. With this, we can find the frequency scale.

$$\omega_s = \frac{1}{t_s} = \sqrt{\frac{g}{h}} \quad \wedge \quad T_s = \frac{2\pi}{\omega_s} = 2\pi\sqrt{\frac{h}{g}}. \tag{6.1.34}$$

These are the frequency and period of time of a pendulum. Now, we use the time scale to find the derivative of time.

$$t = t_s\tau \implies dt = t_s d\tau \implies dt^2 = t_s^2 d\tau^2 \tag{6.1.35}$$

Dimensionless group of the magnetic energy and the Energie's number

Now, the group of M:

$$\begin{pmatrix}\alpha\\ \beta\\ \lambda\end{pmatrix} = \begin{pmatrix}1\\ -1\\ -1\end{pmatrix} \quad \therefore \quad \boxed{\Pi_3 = \frac{M}{M_s} = \frac{MR}{I_0 g} \implies M_s = \frac{gI_0}{R}} \quad . \tag{6.1.36}$$

This dimensionless group relates the magnetic energy and the mechanical potential energy. Using this dimensionless group, we are going to define Energie's number as:

$$\Lambda \equiv \Pi_3 = \frac{MR}{gI_0}. \tag{6.1.37}$$

The Energie's number is going to be so useful because it relates all the variables that are not affected by time. An example of some quantity that is not affected by time is the equilibrium distance, so we can rewrite this number in terms of the Energie's number as:

6.1.4 Dimensionless system

Once we get the dimensionless groups, we can dimensionless our differential equation (6.1.15).

$$\begin{aligned}
\frac{d^2\theta}{dt^2} + \left[\frac{mgR}{I_0} + \frac{M}{I_0}\frac{1}{\cos(\theta) - \cos(\Theta)}\right]\sin(\theta) = 0 \\
\frac{d^2\theta}{d(t_s\tau)^2} + \left[\frac{mgR}{I_0} + \frac{M}{I_0}\frac{1}{\cos(\theta) - \cos(\Theta)}\right]\sin(\theta) = 0 \\
\frac{1}{t_s^2}\frac{d^2\theta}{d\tau^2} + \left[\frac{mgR}{I_0} + \frac{M}{I_0}\frac{1}{\cos(\theta) - \cos(\Theta)}\right]\sin(\theta) = 0 \\
\frac{d^2\theta}{d\tau^2} + \left[\frac{mgRt_s^2}{I_0} + \frac{Mt_s^2}{I_0}\frac{1}{\cos(\theta) - \cos(\Theta)}\right]\sin(\theta) = 0.
\end{aligned} \tag{6.1.38}$$

Now, using the time scale factor, and the deffinition of our dimensionless numbers:

$$\begin{aligned}
\frac{d^2\theta}{d\tau^2} + \left[\frac{mgRt_s^2}{I_0} + \frac{Mt_s^2}{I_0}\frac{1}{\cos(\theta) - \cos(\Theta)}\right]\sin(\theta) = 0 \\
\frac{d^2\theta}{d\tau^2} + \left[\frac{mgR}{I_0}\left(\frac{R}{g}\right) + \frac{M}{I_0}\left(\frac{R}{g}\right)\frac{1}{\cos(\theta) - \cos(\Theta)}\right]\sin(\theta) = 0 \\
\frac{d^2\theta}{d\tau^2} + \left[\frac{mgR^2}{gI_0} + \frac{MR}{gI_0}\frac{1}{\cos(\theta) - \cos(\Theta)}\right]\sin(\theta) = 0 \\
\therefore \boxed{\frac{d^2\theta}{d\tau^2} + \left[\chi + \frac{\Lambda}{\cos(\theta) - \cos(\Theta)}\right]\sin(\theta) = 0}\,.
\end{aligned} \tag{6.1.39}$$

Something important to note is that the dimensionless numbers χ and Λ are dimensionless frequencies of the system. This can be seen in the following equations:

$$\chi = \frac{mR^2}{I_0} \implies \chi = \frac{mgR}{I_0}\left(\frac{R}{g}\right) \therefore \chi = \omega_{0p}^2 t_s^2\,, \tag{6.1.40}$$

$$\Lambda = \frac{MR}{gI_0} \implies \Lambda = \frac{M}{I_0}\left(\frac{R}{g}\right) \therefore \Lambda = \phi_M^2 t_s^2\,. \tag{6.1.41}$$

Our differential equation is similar to a simple pendulum system. Then, just as that system, this system is analytically unsolvable. The unique case where our differential equation is analytically solvable is the small-angle system, but for a general case, we need to use a numerical method.

6.1.5 Small-angle approximation

In the case of the small-angle system ($\theta \ll 1$), using Taylor's series, we obtain:

$$\sin\theta \approx \theta \quad \text{and} \quad \cos(\theta) \approx 1. \tag{6.1.42}$$

Then our differential equation transfroms into the following system:

$$\ddot{\theta} + \left[\chi + \frac{\Lambda}{1-\cos(\Theta)}\right]\theta = 0. \tag{6.1.43}$$

Now, using the trigonometric identity of sine square, we obtain.

$$\ddot{\theta} + \left[\chi + \frac{\Lambda}{1-\cos(\Theta)}\right]\theta = \ddot{\theta} + \left[\chi + \frac{\Lambda}{2\sin^2\left(\frac{\Theta}{2}\right)}\right]\theta = 0. \tag{6.1.44}$$

As we know, the operative range of θ is defined by Θ, then we can conclude that for our system, the maximum angle is also less than one. This can be shown in the following equation:

$$\ddot{\theta} + \left[\chi + \frac{\Lambda}{2\left(\frac{\Theta}{2}\right)^2}\right]\theta = 0 \implies \ddot{\theta} + \left(\chi + \frac{2\Lambda}{\Theta^2}\right)\theta = 0. \tag{6.1.45}$$

Now we can find the dimensionless natural frequency of our system, which is:

$$\Omega_0^2 = \chi + \frac{2\Lambda}{\Theta^2} \implies \Omega_0 = \sqrt{\chi + \frac{2\Lambda}{\Theta^2}}. \tag{6.1.46}$$

It is important to see that if $\Lambda = 0$, then the system returns to being completely a physical pendulum. Now, we can find the solution for our small-angle approximation differential equation, that is:

$$\theta(\tau) = A\cos(\Omega_0\tau) + B\sin(\Omega_0\tau). \tag{6.1.47}$$

Using the initial conditions, which are: $\theta(0) = \Theta$ and $\dot{\theta}(0) = 0$, we obtain:

$$\theta(\tau) = \Theta\cos(\Omega_0\tau). \tag{6.1.48}$$

6.1.6 Stability

As we can see in the phase portrait diagrams, the system generates closed trajectories. So we can conclude that the system is stable. A more formal demonstration of the idea is that our Hamiltonian system has only one degree of freedom, and for the Poincaré-Bendixson theorem, chaos cannot occur in systems of one degree of freedom.

6.2 Damping system

The next step in the analysis of our system is adding the damping term. It is important because the damping system lets us know how the dissipative forces affect the behavior of the ideal system and adds some stability to the forced system.

If we remember, the damping force is the damping coefficient (β) times the speed of the system (v). Then, we can express it as:

$$F_d = -\beta v. \tag{6.2.1}$$

Now, we rewrite the speed in terms of the angle. Using the relation between the tangential velocity and the angular velocity, it is important to note that tangential velocity is applied over the center of mass of the system, we obtain.

$$F_d = -\beta v \implies F_d = -\beta R\dot{\theta}. \tag{6.2.2}$$

But our system works with torques. Then, dissipative torque of the system (τ_d) is:

$$\tau_d = F_d R \implies \tau_d = -\underbrace{\beta R^2}_{b}\dot{\theta} \therefore \tau_d = -b\dot{\theta}, \tag{6.2.3}$$

where b is a damping coefficient in terms of torques.

6.2.1 Lagrangian analysis

For this analysis, we are going to use the same Lagrangian that we find for the ideal case (6.1.11). It is important to remember that in this system, we are adding the damping term, and it is a non-conservative force. As we have seen previously, when the system has a non-conservative force, we can add it to our Euler-Lagrange equation:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \left(\frac{\partial L}{\partial \theta}\right) = Q_i, \tag{6.2.4}$$

where Q_i is the non-conservative force, that in our system is the damping torque ($\tau_d = -b\dot{\theta}$). Therefore, the equation of motion has the following form:

$$I_0\ddot{\theta} + b\dot{\theta} + \left[mgR + \frac{M}{\cos(\theta) - \cos(\Theta)}\right]\sin(\theta) = 0. \tag{6.2.5}$$

Dividing in terms of the moment of inertia (I_0).

$$\ddot{\theta} + \frac{b}{I_0}\dot{\theta} + \left[\frac{mgR}{I_0} + \frac{M/I_0}{\cos(\theta) - \cos(\Theta)}\right]\sin(\theta) = 0. \tag{6.2.6}$$

Now that we have our differential equation, we can do the Hamiltonian analysis to see the behavior of our system.

6.2.2 Hamiltonian analysis

The Hamiltonian of our system is the same as that of the ideal system (6.1.21). Something important to remember is that the Hamiltonian of the ideal system does not include the damping term, because it is a conservative Hamiltonian; but in Hamilton's equations, the damping term is included. Hamilton's equations are:

$$\begin{pmatrix}\dot{\theta}\\ \dot{p_\theta}\end{pmatrix} = \begin{pmatrix}\frac{\partial H}{\partial p_\theta}\\ -\frac{\partial H}{\partial \theta} - \frac{bp_\theta}{I_0}\end{pmatrix} = \begin{pmatrix}\frac{p_\theta}{I_0}\\ -\left[mgR + \frac{M}{\cos(\theta)-\cos(\Theta)}\right]\sin(\theta) - \frac{bp_\theta}{I_0}\end{pmatrix}. \tag{6.2.7}$$

With our dynamical system, we can plot the phase portrait of our system with the intention of seeing the behavior that the system follows.

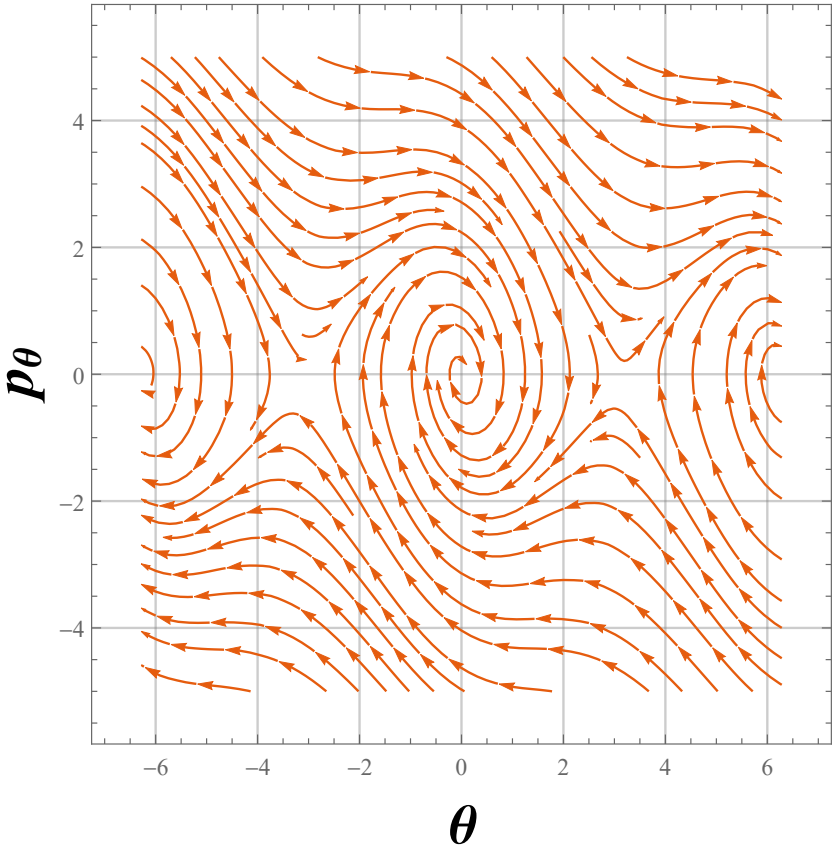


Figure 6.4: Phase portrait of the damiping system.

6.2.3 Dimensional analysis

The next step for us is to dimensionless our differential equation (6.2.6). To do this, we are going to continue using the same dimensionless variable as in the ideal case, but we are going to find the dimensionless group for the damping term

Dimensionless group of the damping coefficient and the Damping number

The b's group is:

$$\begin{pmatrix}\alpha\\ \beta\\ \lambda\end{pmatrix} = \begin{pmatrix}\frac{1}{2}\\ -\frac{1}{2}\\ -1\end{pmatrix} \quad \therefore \quad \boxed{\Pi_4 = \frac{b}{b_s} = \frac{b}{I_0}\sqrt{\frac{R}{g}} \implies b_s = I_0\sqrt{\frac{g}{R}} = \frac{I_0}{t_s}} \quad . \tag{6.2.8}$$

With this, we can define a dimensionless number that is the ratio of damping torque to inertial torque. This number is:

$$2\Phi \equiv \Pi_4 = \frac{b}{I_0}\sqrt{\frac{R}{g}} = \frac{bt_s}{I_0} \tag{6.2.9}$$

6.2.4 Dimensionless system

Now, applying our dimensionless groups, we obtain our dimensionless system.

$$
\begin{aligned}
\frac{d^2\theta}{dt^2} + \frac{b}{I_0}\frac{d\theta}{dt} + \left[\frac{mgR}{I_0} + \frac{M}{I_0}\frac{1}{\cos(\theta) - \cos(\Theta)}\right]\sin(\theta) = 0 \\
\frac{d^2\theta}{d(t_s\tau)^2} + \frac{b}{I_0}\frac{d\theta}{d(t_s\tau)} + \left[\frac{mgR}{I_0} + \frac{M}{I_0}\frac{1}{\cos(\theta) - \cos(\Theta)}\right]\sin(\theta) = 0 \\
\frac{1}{t_s^2}\frac{d^2\theta}{d\tau^2} + \frac{b}{I_0 t_s}\frac{d\theta}{d\tau} + \left[\frac{mgR}{I_0} + \frac{M}{I_0}\frac{1}{\cos(\theta) - \cos(\Theta)}\right]\sin(\theta) = 0 \\
\frac{d^2\theta}{d\tau^2} + \frac{bt_s}{I_0}\frac{d\theta}{d\tau} + \left[\frac{mgRt_s^2}{I_0} + \frac{Mt_s^2}{I_0}\frac{1}{\cos(\theta) - \cos(\Theta)}\right]\sin(\theta) = 0 \\
\boxed{\frac{d^2\theta}{d\tau^2} + 2\Phi\frac{d\theta}{d\tau} + \left[\chi + \frac{\Lambda}{\cos(\theta) - \cos(\Theta)}\right]\sin(\theta) = 0}.
\end{aligned}
\tag{6.2.10}
$$

This last dimensionless differential equation is, as in the ideal case (6.1.39), analytically unsolvable. And as in the ideal case, the unique case that we can solve analytically is the small-angle approximation.

6.2.5 Small-angle approximation

Remembering the case of the small-angle system ($\theta \ll 1$), using Taylor's series, we obtain:

$$
\sin\theta \approx \theta \quad \text{and} \quad \cos(\theta) \approx 1. \tag{6.2.11}
$$

Then our differential equation is now:

$$
\ddot{\theta} + 2\Phi\dot{\theta} + \left[\chi + \frac{\Lambda}{1 - \cos(\Theta)}\right]\theta = 0. \tag{6.2.12}
$$

As we know, the operative range of θ is defined by Θ, then we can conclude that for our system, the maximum angle is also less than one. This can be shown in the following equation:

$$
\ddot{\theta} + 2\Phi\dot{\theta} + \left[\chi + \frac{\Lambda}{2\sin^2\left(\frac{\Theta}{2}\right)}\right]\theta = 0 \implies \ddot{\theta} + 2\Phi\dot{\theta} + \left(\chi + \frac{2\Lambda}{\Theta^2}\right)\theta = 0. \tag{6.2.13}
$$

Using the definition of the natural frequency of the system:

$$
\ddot{\theta} + 2\Phi\dot{\theta} + \Omega_0^2\theta = 0. \tag{6.2.14}
$$

Now, finding the characteristic equation of our differential equation.

$$
s^2 + 2\Phi s + \Omega_0^2 = 0. \tag{6.2.15}
$$

The solutions of our characteristic equations are:

$$
s = -\Phi \pm \sqrt{\Phi^2 - \Omega_0^2} = -\Phi \pm i\sqrt{\Omega_0^2 - \Phi^2} = -\Phi \pm i\eth, \tag{6.2.16}
$$

where $\eth$ is the discriminant of our quadratic solution, and it is defined as $\eth = \sqrt{\Omega_0^2 - \Phi^2}$. With the solutions of our characteristic equation, the solution of our differential equation is:

$$\theta(\tau) = Ae^{s_1\tau} + Be^{s_2\tau} \implies \theta(\tau) = Ae^{(-\Phi+i\eth)\tau} + Be^{(-\Phi-i\eth)\tau}. \tag{6.2.17}$$

Doing some algebra and using Euler's complex identity:

$$\begin{aligned}
\theta(\tau) &= Ae^{(-\Phi+i\eth)\tau} + Be^{(-\Phi-i\eth)\tau} \\
&= e^{-\Phi\tau}\left(Ae^{i\eth\tau} + Be^{-i\eth\tau}\right) \\
&= e^{-\Phi\tau}\left(A\cos(\eth\tau) + iA\sin(\eth\tau) + B\cos(\eth\tau) - iB\sin(\eth\tau)\right) \\
&= e^{-\Phi\tau}\left[(A+B)\cos(\eth\tau) + i(A-B)\sin(\eth\tau)\right] \\
\therefore \boxed{\theta(\tau) = e^{-\Phi\tau}\left(C\cos(\eth\tau) + D\sin(\eth\tau)\right)}&.
\end{aligned} \tag{6.2.18}$$

Using the initial conditions, which are: $\theta(0) = \Theta$ and $\dot{\theta}(0) = 0$. Using the first initial condition, we obtain:

$$\begin{aligned}
\theta(0) = \Theta &= e^{-\Phi(0)}\left(C\cos(\eth(0)) + D\sin(\eth(0))\right) \\
&= 1(C\cos(0) + D\sin(0)) \\
\therefore \boxed{\Theta = C}&.
\end{aligned} \tag{6.2.19}$$

Obtaining the first derivative of θ.

$$\begin{aligned}
\dot{\theta}(\tau) &= -\Phi e^{-\Phi\tau}\left(\Theta\cos(\eth\tau) + D\sin(\eth\tau)\right) + e^{-\Phi\tau}\left(-\eth\Theta\sin(\eth\tau) + \eth D\cos(\eth\tau)\right) \\
&= e^{-\Phi\tau}\left[(\eth D - \Phi\Theta)\cos(\eth\tau) - (\Phi D + \eth\Theta)\sin(\eth\tau)\right].
\end{aligned} \tag{6.2.20}$$

Now, applying the second initial condition:

$$\begin{aligned}
\dot{\theta}(0) = 0 &= e^{-\Phi(0)}\left[(\eth D - \Phi\Theta)\cos(\eth(0)) - (\Phi D + \eth\Theta)\sin(\eth(0))\right] \\
&= 1\left[(\eth D - \Phi\Theta)\cos(0) - (\Phi D + \eth\Theta)\sin(0)\right] \\
0 = \eth D - \Phi\Theta &\implies \eth D = \Phi\Theta \\
\therefore \boxed{D = \frac{\Phi\Theta}{\eth}}&.
\end{aligned} \tag{6.2.21}$$

So, the solution of our differential equation is:

$$\theta(\tau) = \Theta e^{-\Phi\tau}\left[\cos(\eth\tau) + \frac{\Phi}{\eth}\sin(\eth\tau)\right] \tag{6.2.22}$$

6.3 Forced system

We now introduce an external force into our system. This force is applied over the structure and generates a torque in the system. For this reason, in our differential equation, the external force is expressed as a torque (T_0).

For our system, we are going to consider a periodic external force. Then, it generates a periodic external torque $T_0\cos(\omega t)$, where ω is the frequency of the external torque.

6.3.1 Lagrangian analysis

As with the damped system, the forced system has the same Lagrangian (6.1.11). To find the equation of motion, we apply the Euler-Lagrange equation, adding as the non-conservative force, the damping force, and the external torque. The generalize external force is:

$$Q_i = T_0 \cos(\omega t) - b\dot{\theta}. \tag{6.3.1}$$

Then, applying the Euler-Lagrange equation, our differential equation is:

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} &= Q_i \\ I_0\ddot{\theta} + \left[mgR + \frac{M}{\cos(\theta) - \cos(\Theta)}\right]\sin(\theta) &= T_0\cos(\omega t) - b\dot{\theta} \\ I_0\ddot{\theta} + b\dot{\theta} + \left[mgR + \frac{M}{\cos(\theta) - \cos(\Theta)}\right]\sin(\theta) &= T_0\cos(\omega t). \end{aligned} \tag{6.3.2}$$

For our differential equation, we can see that this system is non-autonomous, because there is a term in our differential equation that has a clear time dependence. For that reason, we cannot generate a phase portrait that describes the system at every time because the system changes in every instant of time. To know if our system generates a closed trajectory, we need to analyze the stability of the system, but before doing that, we are going to dimensionless our differential equation.

6.3.2 Dimensional analysis

To dimensionless our differential equation, we are going to continue using the same dimensionless variables and dimensionless groups that we used in the ideal and damped cases. For that reason, we only need to find the dimensionless groups of the external torque and the frequency.

Dimensionless group of the external torque and the torque number

The T_0's group is:

$$\begin{pmatrix}\alpha\\ \beta\\ \lambda\end{pmatrix} = \begin{pmatrix}1\\ -1\\ -1\end{pmatrix} \quad \therefore \quad \boxed{\Pi_5 = \frac{T_0}{T_s} = \frac{T_0 R}{gI_0} \implies T_s = \frac{gI_0}{R} = \frac{I_0}{t_s^2}} \quad . \tag{6.3.3}$$

With this, we can define a dimensionless number as the ratio of external torque to inertial torque. This number is:

$$\sigma \equiv \Pi_5 = \frac{T_0 R}{I_0 g} = \frac{T_0 t_s^2}{I_0} \tag{6.3.4}$$

Dimensionless group of frequency and the frequency number

The ω's group is:

$$\begin{pmatrix}\alpha\\ \beta\\ \lambda\end{pmatrix} = \begin{pmatrix}\frac{1}{2}\\ -\frac{1}{2}\\ 0\end{pmatrix} \quad \therefore \quad \boxed{\Pi_6 = \frac{\omega}{\omega_s} = \omega\sqrt{\frac{R}{g}} \implies \omega_s = \sqrt{\frac{g}{R}} = \frac{1}{t_s}} \quad . \tag{6.3.5}$$

With this, we can define a dimensionless number as the ratio of the external frequency and the frequency of a pendulum. This number is:

$$\Omega \equiv \Pi_6 = \omega\sqrt{\frac{R}{g}} = \omega t_s. \tag{6.3.6}$$

With

6.3.3 Dimensionless system

Now, applying our dimensionless groups, we obtain our dimensionless system.

$$\begin{aligned}
I_0\frac{d^2\theta}{dt^2} + b\frac{d\theta}{dt} + \left[mgR + \frac{M}{\cos(\theta)-\cos(\Theta)}\right]\sin(\theta) &= T_0\cos(\omega t)\\
\frac{d^2\theta}{dt^2} + \frac{b}{I_0}\frac{d\theta}{dt} + \left[\frac{mgR}{I_0} + \frac{M}{I_0}\frac{1}{\cos(\theta)-\cos(\Theta)}\right]\sin(\theta) &= \frac{T_0}{I_0}\cos(\omega t)\\
\frac{d^2\theta}{d(t_s\tau)^2} + \frac{b}{I_0}\frac{d\theta}{d(t_s\tau)} + \left[\frac{mgR}{I_0} + \frac{M}{I_0}\frac{1}{\cos(\theta)-\cos(\Theta)}\right]\sin(\theta) &= \frac{T_0}{I_0}\cos(\omega(t_s\tau))\\
\frac{1}{t_s^2}\frac{d^2\theta}{d\tau^2} + \frac{b}{I_0 t_s}\frac{d\theta}{d\tau} + \left[\frac{mgR}{I_0} + \frac{M}{I_0}\frac{1}{\cos(\theta)-\cos(\Theta)}\right]\sin(\theta) &= \frac{T_0}{I_0}\cos(\omega t_s\tau)\\
\frac{d^2\theta}{d\tau^2} + \frac{bt_s}{I_0}\frac{d\theta}{d\tau} + \left[\frac{mgRt_s^2}{I_0} + \frac{Mt_s^2}{I_0}\frac{1}{\cos(\theta)-\cos(\Theta)}\right]\sin(\theta) &= \frac{T_0t_s^2}{I_0}\cos(\omega t_s\tau)\\
\boxed{\frac{d^2\theta}{d\tau^2} + 2\Phi\frac{d\theta}{d\tau} + \left[\chi + \frac{\Lambda}{\cos(\theta)-\cos(\Theta)}\right]\sin(\theta) = \sigma\cos(\Omega\tau)}\,.&
\end{aligned} \tag{6.3.7}$$

6.3.4 Stability

Starting from the idea that the general solution is analytically unsolvable, we need to know if our system is stable or chaotic. To know this, we are going to analyze the stability of the system.

Linearizing our differential equation, we obtain the following set of first-order differential equations.

$$v(\theta_1,\theta_2,\theta_3) = \begin{pmatrix}\dot{\theta_1}\\ \dot{\theta_2}\\ \dot{\theta_3}\end{pmatrix} = \begin{pmatrix}\theta_2\\ \sigma\cos(\theta_3) - 2\Phi\theta_2 - \left[\chi + \frac{\Lambda}{\cos(\theta_1)-\cos(\Theta)}\right]\sin(\theta_1)\\ \Omega\end{pmatrix}, \tag{6.3.8}$$

where $\theta_3 = \Omega\tau$ is the angle induced by the forcing. With our dynamical system vector, we can build our Jacobian matrix. The Jacobian matrix is:

$$\mathbf{J}(\theta_1,\theta_2,\theta_3)=\begin{pmatrix}\frac{\partial v_1}{\partial\theta_1} & \frac{\partial v_1}{\partial\theta_2} & \frac{\partial v_1}{\partial\theta_3}\\ \frac{\partial v_2}{\partial\theta_1} & \frac{\partial v_2}{\partial\theta_2} & \frac{\partial v_2}{\partial\theta_3}\\ \frac{\partial v_3}{\partial\theta_1} & \frac{\partial v_3}{\partial\theta_2} & \frac{\partial v_3}{\partial\theta_3}\end{pmatrix}$$

$$\implies \mathbf{J}(\theta_1,\theta_2,\theta_3)=\begin{pmatrix}0 & 1 & 0\\ -\left(\left[\chi+\frac{\Lambda}{\cos(\theta_1)-\cos(\Theta)}\right]\cos(\theta_1)+\frac{\Lambda\sin^2(\theta_1)}{(\cos(\theta_1)-\cos(\Theta))^2}\right) & -2\Phi & -\sigma\sin(\theta_3)\\ 0 & 0 & 0\end{pmatrix}=\mathbf{J}(\tau). \tag{6.3.9}$$

Then, the trace of this Jacobian is:

$$\mathrm{tr}(\mathbf{J}(\tau))=-2\Phi. \tag{6.3.10}$$

The following step is to find the Monodromy matrix (, and it follows the next:

$$\dot{\mathbf{M}}(\tau)=\mathbf{J}(\tau)\mathbf{M}(\tau),\quad \mathbf{M}(0)=\mathbf{I}. \tag{6.3.11}$$

Now, we can use Liouville's formula:

$$\det\left(\mathbf{M}(\tau)\right)=\det\left(\mathbf{M}(t_0)\right)e^{\int_{t_0}^{T}\mathrm{tr}(\mathbf{J}(\tau))d\tau}. \tag{6.3.12}$$

Integrating from 0 to T, we obtain:

$$\det\left(\mathbf{M}(T)\right)=e^{-2\Phi T}. \tag{6.3.13}$$

With this result, we can find the Floquet multipliers, which are the eigenvalues of the Monodromy matrix ($\mathbf{M}(T)$). Using the definition of Floquet multipliers, we obtain.

$$\rho_1\cdot\rho_2\cdot\rho_3=\det\left(\mathbf{M}(T)\right)=e^{-2\Phi T}, \tag{6.3.14}$$

where ρ_1, ρ_2, and ρ_3 are the Floquet multipliers. It is important to remember that in an autonomous system in 3D that has a periodic orbit, one of the multipliers is always 1, then $\rho_3=1$. The Floque multipliers let us know if the trajectory that the system follows is stable or unstable. To know this, we have the following table:

Condition	**How many?**	**System behavior**
$\lvert\rho_i\rvert<1$	All except the one equal to 1	Stable limit cycle (perturbations decay)
$\lvert\rho_i\rvert=1$	At least one other besides the neutral 1	Marginal stability (bifurcation point)
$\lvert\rho_i\rvert>1$	At least one (excluding the neutral 1)	Unstable limit cycle (perturbations grow)

Table 6.1: Stability criteria based on Floquet multipliers ρ_i for an autonomous system with a periodic orbit. One multiplier is always 1 (neutral direction along the orbit). The table applies to the remaining multipliers.

For a damped system ($\Phi>0$), if a periodic orbit exists, the product of its Floquet multipliers is $e^{-2\Phi T}<1$. This is a necessary condition for asymptotic stability (all non-neutral multipliers inside the unit circle). To determine whether such a periodic orbit actually exists, we construct a Poincaré map.

To make our Poincaré map, we use the Runge-Kutta method on our dynamical system, and we record the values of $(\theta, \dot{\theta})$ at times that are multiples of the period T (i.e., a stroboscopic map). Doing this, we obtain the following Poincaré map.

Lyapunov exponents

While Floquet multipliers characterize the stability of a periodic orbit, many systems, in particular the forced nonlinear oscillators, can exhibit chaotic behavior where no periodic orbit exists. In such cases, a more general tool is needed: the ***Lyapunov exponents***. The behavior of the system in terms of the Lyapunov exponents follows the next table:

Condition	**How many?**	**System behavior**
$\lambda < 0$	All (except possibly one zero for a periodic orbit)	Stable dynamics
$\lambda = 0$	At least one (others < 0)	Marginal stability
$\lambda > 0$	At least one	Chaotic dynamics

Table 6.2: Interpretation of Lyapunov exponents λ_i (ordered $\lambda_1 \geq \lambda_2 \geq \cdots$) for an autonomous system. For a stable periodic orbit, one exponent is zero (direction along the orbit) and the remaining are negative. A positive exponent indicates chaos.

When a system is stable, there exists a relation between the Floquet multipliers and the Lyapunov exponents. That relation is:

$$\lambda_i = \frac{1}{T} \ln(\rho_i) \tag{6.3.15}$$

Now, using the Lyapunov exponents equation:

$$\lambda_1+\lambda_2+\lambda_3 = \frac{1}{T}\ln(\rho_1)+\frac{1}{T}\ln(\rho_2)+\frac{1}{T}\ln(\rho_3) = \frac{1}{T}\ln(\rho_1\cdot\rho_2\cdot\rho_3) = \frac{1}{T}\ln(\rho_1\cdot\rho_2) \tag{6.3.16}$$

Using the value of $\rho_3 = 1 \implies \lambda_3 = 0$, and our Floquet product (6.3.14)

$$\lambda_1 + \lambda_2 = \frac{1}{T}\ln\left(e^{-2\Phi T}\right) \implies \lambda_1 + \lambda_2 = -2\Phi \tag{6.3.17}$$

Using the condition of stability:

$$\lambda_1 + \lambda_2 = -2\Phi < 0 \implies \Phi > 0, \tag{6.3.18}$$

this result implies that if the friction number (Φ) increases, the stability increases too. But the amplitude of the oscillation decreases. Another important thing that we can find is the values of Lyapunov exponents that give us stable system

$$\lambda_1 + \lambda_2 = -2\Phi \implies \lambda_1 = -(2\Phi + \lambda_2), \tag{6.3.19}$$

we know that a stable system needs $\lambda_i < 0$, then:

$$\lambda_1 < 0 \implies -(2\Phi + \lambda_2) < 0 \implies 2\Phi + \lambda_2 > 0 \implies \lambda_2 > -2\Phi, \tag{6.3.20}$$

but both Lyapunov exponents need to be less than one; therefore:

$$-2\Phi < \lambda_2 < 0. \tag{6.3.21}$$

Now, we can find the limit for the first Lyapunov exponent domain for a stable system:

$$\begin{aligned} \lambda_1 &= -(2\Phi + 0) = -2\Phi \\ \lambda_1 &= -(2\Phi - 2\Phi) = 0 \\ \therefore &-2\Phi < \lambda_1 < 0. \end{aligned} \tag{6.3.22}$$

We can see that both Lyapunov exponents have the same domain if the system is stable, but it is important to highlight that they can have different values; the only thing that they need to follow is that the sum of both is -2Φ as is presented in the stability condition equation 6.3.18.

To see how the system responds to the external moment, we present a Lyapunov exponent graph. We are going to study the stability of the system with low friction. For this analysis, we are going to use $\Phi = 1$. Then, the Lyapunov diagram is.

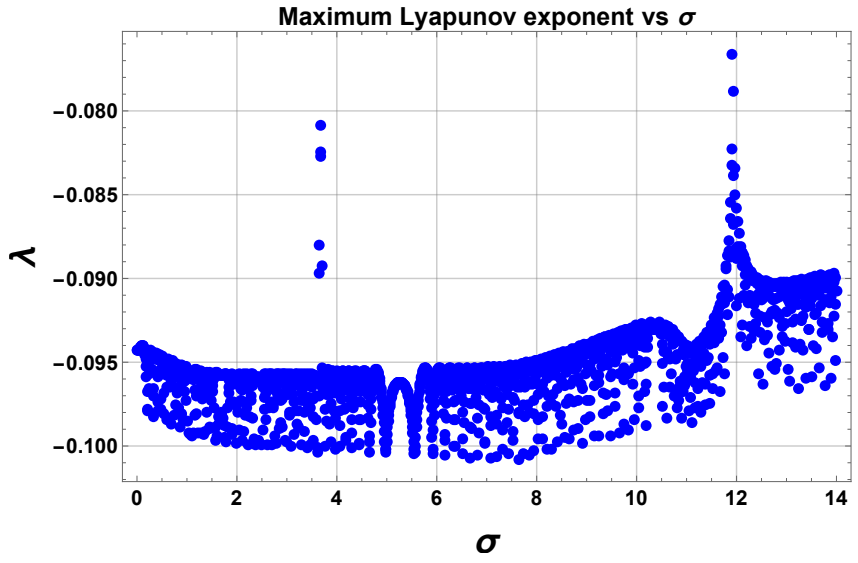


Figure 6.5: Lyapunov diagram of the pendulum system in terms of the external moment σ with a friction number of $\Phi = 0.1$.

For the previous figure, we can see that our system is very stable. The previous result indicates that chaos in the system is completely related to the force (σ) and frequency (Ω) parameters.

Now, we are going to present some bifurcation diagrams. For the first diagram, we use a $\Phi = 0.1$, and we obtain.

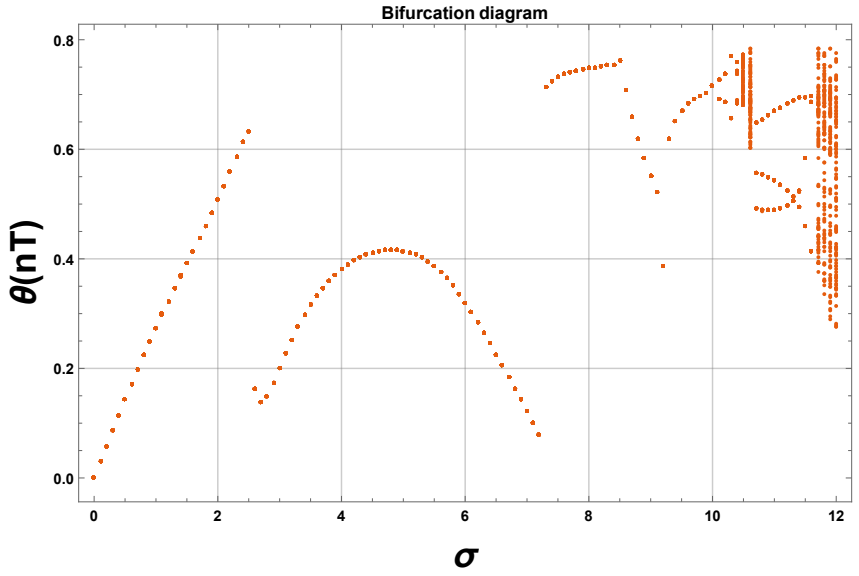


Figure 6.6: Bifurcation diagram for different values of the force with $\Lambda = 1$, $\Phi = 0.1$, $\Theta = \frac{\pi}{4}$ and $\Omega = 1$

Now, we are going to increase the damping number to see how the system reacts to an increase in the damping. We are going to go from 0.1 to 0.25.

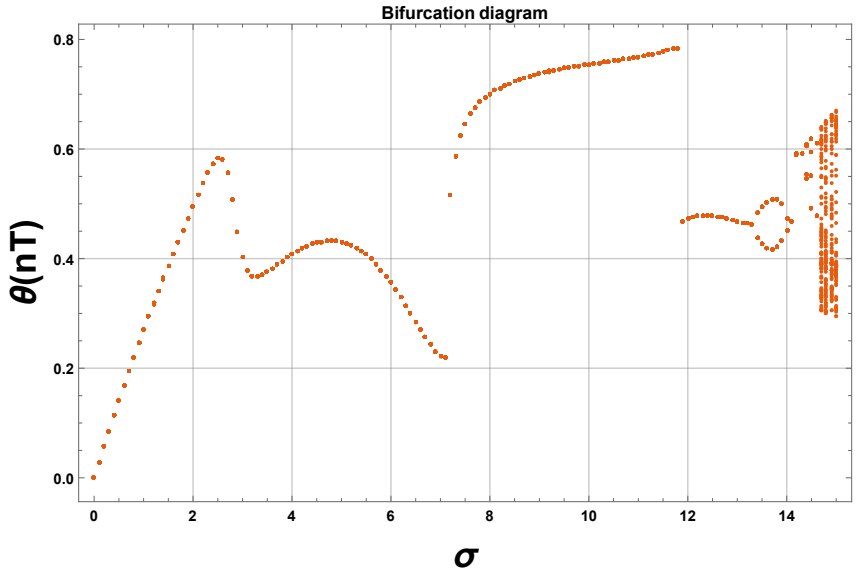


Figure 6.7: Bifurcation diagram for different values of the force with $\Lambda = 1$, $\Phi = 0.1$, $\Theta = \frac{\pi}{4}$ and $\Omega = 1$

As we can see in the previous figure, when the damping increases, the necessary force to generate chaos also increases. Now, we are going to check when the damping is $\Phi = 0.5$

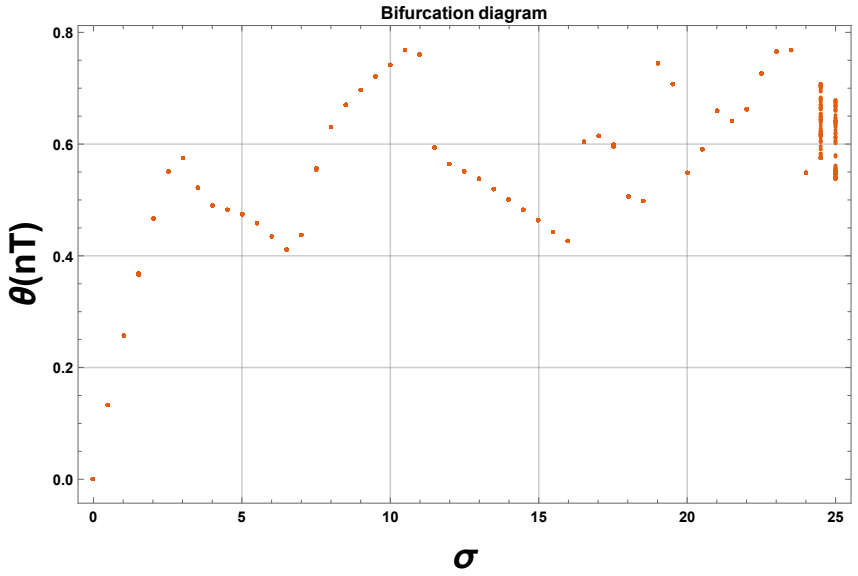


Figure 6.8: Bifurcation diagram for different values of the force with $\Lambda = 1$, $\Phi = 0.1$, $\Theta = \frac{\pi}{4}$ and $\Omega = 1$

From all the previous bifurcation diagrams, we can see that stability needs the damping term. And the increases in the damping term increase the stability of the system.

6.3.5 Numerical

Now, we know that our dynamical system is unsolvable analytically, except for the small-angle approximation. So, to solve a general system, we need to use a numerical method to generate a numerical solution. We use the Runge-Kutta method using the NDSolve function.

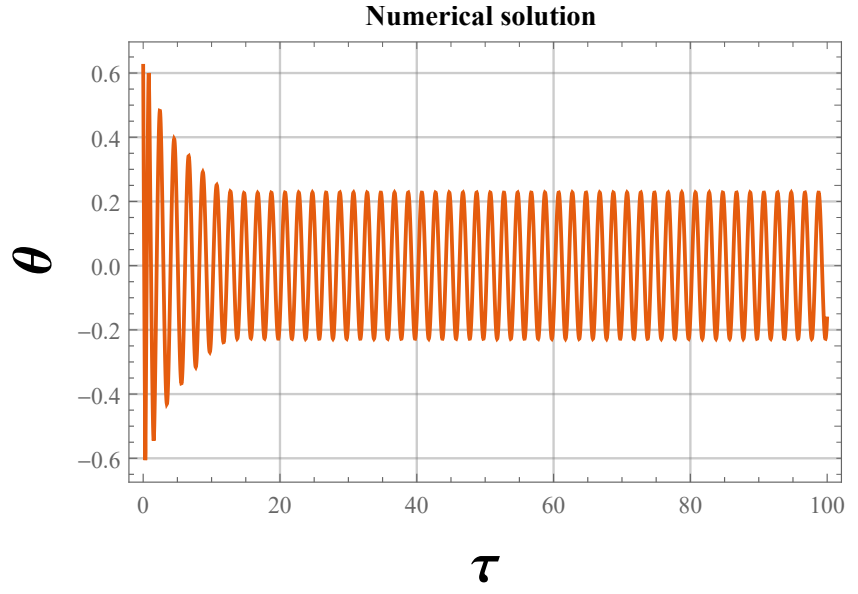


Figure 6.9: Numerical solution of the system with $\Lambda = 1$, $\Phi = 0.5$, $\Theta = \frac{\pi}{5}$, σ=1 and $\Omega = \pi$

With the solution, we are able to find how the angle changes in terms of the dimensionless numbers, for the Energie's number, we obtain.

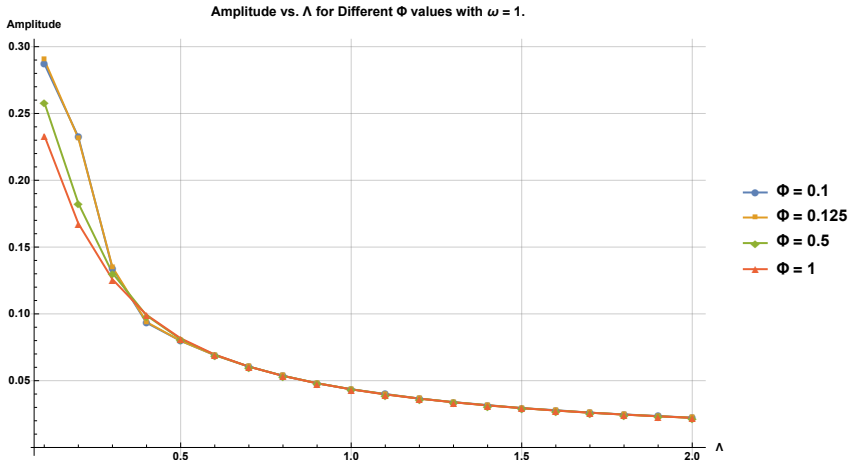


Figure 6.10: Bifurcation diagram for different values of the force with $\Lambda = 1$, $\Phi = 0.1$, $\Theta = \frac{\pi}{4}$ and $\Omega = 1$

6.3.6 Small-angle approximation

As we did in the previous sections, we are going to analyze our system under the small-angle approximation. Using Taylor's series, we obtain:

$$\sin\theta \approx \theta \quad \text{and} \quad \cos(\theta) \approx 1. \tag{6.3.23}$$

Then our differential equation transforms into the following system:

$$\ddot{\theta} + 2\Phi\dot{\theta} + \left[\chi + \frac{\Lambda}{1-\cos(\Theta)}\right]\theta = \sigma\cos(\Omega\tau). \tag{6.3.24}$$

Using the sine square identity and knowing that the operative range of θ is defined by Θ, then we can conclude that for our system, the maximum angle is also less than one. This can be shown in the following equation:

$$\ddot{\theta} + 2\Phi\dot{\theta} + \left[\chi + \frac{\Lambda}{2\sin^2\left(\frac{\Theta}{2}\right)}\right]\theta = \sigma\cos(\Omega\tau) \implies \ddot{\theta} + 2\Phi\dot{\theta} + \left(\chi + \frac{2\Lambda}{\Theta^2}\right)\theta = \sigma\cos(\Omega\tau). \tag{6.3.25}$$

With the previous result, we know the dimensionless natural frequency of our system, which is:

$$\Omega_0^2 = \chi + \frac{2\Lambda}{\Theta^2} \implies \Omega_0 = \sqrt{\chi + \frac{2\Lambda}{\Theta^2}}. \tag{6.3.26}$$

Using the natural frequency, our system for the small-angle approximation is:

$$\ddot{\theta} + 2\Phi\dot{\theta} + \Omega_0^2\theta = \sigma\cos(\Omega\tau). \tag{6.3.27}$$

With our differential system in its simplest form, we can solve it. To solve the system, we need to solve the homogeneous term and the particular term.

To solve the homogeneous term, we need the characteristic equation of the system.

$$s^2 + 2\Phi s + \Omega_0^2 = 0. \tag{6.3.28}$$

The solutions of our characteristic equations are:

$$s = -\Phi \pm \sqrt{\Phi^2 - \Omega_0^2} = -\Phi \pm i\sqrt{\Omega_0^2 - \Phi^2} = -\Phi \pm i\eth, \tag{6.3.29}$$

where $\eth$ is the discriminant of our quadratic solution, and it is defined as $\eth = \sqrt{\Omega_0^2 - \Phi^2}$. With the solutions of our characteristic equation, the solution of our differential equation is:

$$\theta_h(\tau) = Ae^{s_1\tau} + Be^{s_2\tau} \implies \theta_h(\tau) = Ae^{(-\Phi+i\eth)\tau} + Be^{(-\Phi-i\eth)\tau}. \tag{6.3.30}$$

Doing some algebra and using Euler's complex identity:

$$\begin{aligned}
\theta_h(\tau) &= Ae^{(-\Phi+i\eth)\tau} + Be^{(-\Phi-i\eth)\tau} \\
&= e^{-\Phi\tau}\left(Ae^{i\eth\tau} + Be^{-i\eth\tau}\right) \\
&= e^{-\Phi\tau}\left(A\cos(\eth\tau) + iA\sin(\eth\tau) + B\cos(\eth\tau) - iB\sin(\eth\tau)\right) \\
&= e^{-\Phi\tau}\left[(A+B)\cos(\eth\tau) + i(A-B)\sin(\eth\tau)\right] \\
\therefore \boxed{\theta_h(\tau)} &= \boxed{e^{-\Phi\tau}\left(C\cos(\eth\tau) + D\sin(\eth\tau)\right)}.
\end{aligned} \tag{6.3.31}$$

Now, solving the particular term of the system, we obtain:

$$\theta_p(\tau) = Y\cos(\Omega\tau) + X\sin(\Omega\tau), \tag{6.3.32}$$

with

$$Y = \frac{\sigma(\Omega_0^2 - \Omega^2)}{(\Omega_0^2 - \Omega^2)^2 + (2\Phi\Omega)^2}, \qquad X = \frac{\sigma(2\Phi\Omega)}{(\Omega_0^2 - \Omega^2)^2 + (2\Phi\Omega)^2}. \tag{6.3.33}$$

Equivalently,

$$\theta_p(\tau) = A\cos(\Omega\tau - \delta), \quad A = \frac{\sigma}{\sqrt{(\Omega_0^2 - \Omega^2)^2 + (2\Phi\Omega)^2}}, \quad \tan\delta = \frac{2\Phi\Omega}{\Omega_0^2 - \Omega^2},\ \delta \in [0, \pi]. \tag{6.3.34}$$

The complete solution of the system is:

$$\theta(\tau) = \theta_h(\tau) + \theta_p(\tau) \tag{6.3.35}$$

Using the initial conditions:

$$\theta(0) = \Theta = C_1 + Y \quad \Rightarrow \quad C_1 = \Theta - Y, \tag{6.3.36}$$

$$\dot{\theta}(0) = 0 = -\Phi C_1 + \eth C_2 + X\Omega \quad \Rightarrow \quad C_2 = \frac{\Phi C_1 - X\Omega}{\eth}. \tag{6.3.37}$$

Then our system is:

$$\boxed{\theta(\tau) = e^{-\Phi\tau}\left[(\Theta - M)\cos(\eth\tau) + \frac{\Phi(\Theta - M) - N\Omega}{\eth}\sin(\eth\tau)\right] + M\cos(\Omega\tau) + N\sin(\Omega\tau)}. \tag{6.3.38}$$

Now that we know the behavior of the system in small-angle approximation, we can find the behavior of the system when the external force is the force of a fluid.

6.3.7 Fluid force

As we mentioned in the fluid chapter, there are a lot of dimensionless numbers that help us to describe the behavior of the fluid, but for the VIV nature, the most important number is the Strouhal number. Because it describes the vortex shedding behind a body.

As we mentioned, the Gravity-inertial number previously compares the inertial forces to the gravitational forces in a fluid, more specifically, because we have a pendulum.

Now, we are going to rewrite our differential equation using the fluid expressions. A lift force of a fluid, as we have studied previously, has the following form.

$$F_f = \frac{1}{2}\rho U^2 S C_L, \tag{6.3.39}$$

where C_L is the lift coefficient, S is the surface area of the pendulum that interacts with the fluid, and ρ is the density of our fluid. Now, using this force to find the torque of a fluid force, we have.

$$T_f = R \times F_f = \frac{1}{2}\rho U^2 S R C_L, \tag{6.3.40}$$

where R is the distance between the center of mass and the pivot. Using the dimensionless group of the torque and the dimensionless length scale number D, we obtain:

$$\begin{aligned} \sigma = \frac{T_f t_s^2}{I_0} \implies \sigma = \frac{\rho U^2 S C_L R t_s^2}{2I_0} \\ \sigma = \frac{\rho U^2 S R^2 C_L}{2I_0 g} \implies \sigma = \frac{C_L}{2}\left(\frac{U^2}{gD}\right)\left(\frac{\rho S R^2 D}{I_0}\right) \\ \sigma = \frac{C_L \Gamma^2}{2}\left(\frac{\rho S D \chi}{m}\right) \therefore \sigma = \frac{\chi C_L \Gamma^2}{2m^*}. \end{aligned} \tag{6.3.41}$$

We define m^* as the mass ratio between the mass of the structure and the mass of the fluid, that is, $m^* = \left(\frac{m}{\rho D S}\right)$. Something that is relevant to highlight is that the lift coefficient (C_L) could be a function of time, which would depend on the geometry, but in this specific example, we are going to work with a constant lift coefficient.

$$\frac{d^2\theta}{d\tau^2} + 2\Phi\frac{d\theta}{d\tau} + \left[\chi + \frac{\Lambda}{\cos(\theta) - \cos(\Theta)}\right]\sin(\theta) = \frac{\chi C_L \Gamma^2}{2m^*}\cos(\Omega\tau), \tag{6.3.42}$$

as in the general case of a forced system, the amplitude of our oscillation will depend on the Inertia number (χ) and the frequency of the vortices, which in this case is the Strouhal number (St).

Another thing that we need to consider when we are working with a fluid force is the geometry of our structure, cause its shape will define the lift coefficient and the Strouhal number (St). The Stouhal number definition is: $St = \frac{f_s D}{U}$. Then we can find a relation between the Strouhal number and the frequency number. This relation is:

$$\begin{aligned} \omega = 2\pi f_s \implies \frac{\Omega}{t_s} &= 2\pi \left(\frac{StU}{D}\right) \\ \implies \Omega = \frac{2\pi StU t_s}{D} \implies \Omega &= \frac{2\pi StU}{D}\sqrt{\frac{R}{g}} \\ \implies \Omega = \frac{2\pi StU}{\sqrt{gD}}\sqrt{\frac{R}{D}} \implies &\boxed{\Omega = 2\pi St\Gamma\sqrt{\frac{R}{D}}}. \end{aligned} \tag{6.3.43}$$

Then, the frequency of the system is related to the Strouhal and the Gravity-inertial number. If the characteristic length is equal to the distance between the center of mass and the pivot ($D = R$), the frequency number can be simplified as:

$$\Omega = 2\pi St\Gamma. \tag{6.3.44}$$

Then, we can rewrite our dimensionless differential equation as:

$$\frac{d^2\theta}{d\tau^2} + 2\Phi\frac{d\theta}{d\tau} + \left[\chi + \frac{\Lambda}{\cos(\theta) - \cos(\Theta)}\right]\sin(\theta) = \frac{\chi C_L \Gamma^2}{2m^*}\cos(2\pi St\Gamma\tau), \tag{6.3.45}$$

It is important to see that our system now depends on the values of the Strouhal (St), Gravity-inertia (Γ), Energie's (Λ), Inertia (χ), and the friction number (Φ). In this system, the most important numbers are the Strouhal and Gravity-inertial numbers because they define the frequency of the oscillation. The principal problem is that we have a frequency that is a product of two variables. But we can rewrite the system with a new dimensionless time definition to express the frequency of the external force in terms of only one parameter, the Strouhal number.

New dimensionless equation

If we pay attention to the Strouhal number-frequency relation (equation (6.3.43)), we find that there is another possible dimensionless form that could help us to let the dimensionless frequency be only in terms of the Strouhal numbers. This new dimensionless uses fluid and structure properties. The new relation is:

$$\Omega = \frac{2\pi StU t_s}{D} \implies \Omega = 2\pi St \implies \frac{Ut_s}{D} = 1 \quad \therefore \quad \boxed{t_s = \frac{D}{U} = \frac{R}{U}}, \tag{6.3.46}$$

this new time scale lets us rewrite our differential equation. To rewrite it, we are going to use the dimensionless groups that we obtained previously. Starting the new dimensionless analysis in the equation of motion (4.3.5).

Applying the dimensionless groups to our differential equation, we can find the equation of motion

$$\begin{aligned}
I_0\ddot{\theta} + b\dot{\theta} + \left[mgR + \frac{M}{\cos(\theta) - \cos(\Theta)}\right]\sin(\theta) &= T_0\cos(\omega t) \\
\implies \frac{1}{t_s^2}\frac{d^2\theta}{d\tau^2} + \frac{b}{I_0 t_s}\frac{d\theta}{d\tau} + \left[\frac{mgR}{I_0} + \frac{M}{I_0}\frac{1}{\cos(\theta) - \cos(\Theta)}\right]\sin(\theta) &= \frac{T_0}{I_0}\cos(\Omega\tau) \\
\implies \frac{d^2\theta}{d\tau^2} + \frac{bt_s}{I_0}\frac{d\theta}{d\tau} + \left[\frac{mgRt_s^2}{I_0} + \frac{/}{Mt_s^2}I_0\frac{1}{\cos(\theta) - \cos(\Theta)}\right]\sin(\theta) &= \frac{T_0 t_s^2}{I_0}\cos(\Omega\tau) \\
\implies \frac{d^2\theta}{d\tau^2} + \frac{bR}{I_0 U}\frac{d\theta}{d\tau} + \left[\frac{mgRR^2}{I_0U^2} + \frac{MR^2}{I_0U^2}\frac{1}{\cos(\theta) - \cos(\Theta)}\right]\sin(\theta) &= \frac{T_0R^2}{I_0U^2}\cos(\Omega\tau) \\
\implies \frac{d^2\theta}{d\tau^2} + 2\Phi\frac{\sqrt{gR}}{U}\frac{d\theta}{d\tau} + \left[\frac{\chi gR}{U^2} + \frac{\Lambda gR}{U^2}\frac{1}{\cos(\theta) - \cos(\Theta)}\right]\sin(\theta) &= \frac{\sigma gR}{U^2}\cos(\Omega\tau)
\end{aligned} \tag{6.3.47}$$

Using the definition of the Gravity-inertia number ($\Gamma = \frac{U}{\sqrt{gR}}$), we simplify our differential equation as:

$$\begin{aligned}
\frac{d^2\theta}{d\tau^2} + 2\Phi\frac{\sqrt{gR}}{U}\frac{d\theta}{d\tau} + \left[\frac{\chi gR}{U^2} + \frac{\Lambda gR}{U^2}\frac{1}{\cos(\theta) - \cos(\Theta)}\right]\sin(\theta) &= \frac{\sigma gR}{U^2}\cos(\Omega\tau) \\
\implies \frac{d^2\theta}{d\tau^2} + \frac{2\Phi}{\Gamma}\frac{d\theta}{d\tau} + \left[\frac{\chi}{\Gamma^2} + \frac{\Lambda}{\Gamma^2}\frac{1}{\cos(\theta) - \cos(\Theta)}\right]\sin(\theta) &= \frac{\sigma}{\Gamma^2}\cos(\Omega\tau) \\
\implies \frac{d^2\theta}{d\tau^2} + 2\Phi\Gamma\frac{d\theta}{d\tau} + \left[\chi + \frac{\Lambda}{\cos(\theta) - \cos(\Theta)}\right]\sin(\theta) &= \sigma\cos(\Omega\tau)
\end{aligned} \tag{6.3.48}$$

Now, using the fluid form of the dimensionless torque, our differential equation is:

$$\frac{d^2\theta}{d\tau^2} + 2\Phi\Gamma\frac{d\theta}{d\tau} + \left[\chi + \frac{\Lambda}{\cos(\theta) - \cos(\Theta)}\right]\sin(\theta) = \frac{\chi C_L\Gamma^2}{2m^*}\cos(\Omega\tau). \tag{6.3.49}$$

6.3.8 Harnessing energy model

Once we know the way the motion of our system works, we can find an expression to use the motion to generate electricity. To generate it, we are going to use the induction Faraday's law, that is.

$$\varepsilon = -n\frac{d\Phi_B}{dt}, \tag{6.3.50}$$

where ε is the electromotive force induced for the motion of the magnetic flux, n is the number of turns of a coil, and Φ_B is the magnetic flux. We know that the magnetic flux is defined as.

$$\Phi_B = BS\cos(\alpha), \tag{6.3.51}$$

where B is the intensity of the magnetic field, S is the surface area of the coil, and α is the angle between the magnetic field and the normal vector of the surface area of the coil. Now, using the definition of the magnetic flux in Faraday's law.

$$\varepsilon = -n\frac{d}{dt}\left(BS\cos\left(\alpha\right)\right) \implies \varepsilon = nBS\sin\left(\alpha\right)\dot{\alpha}. \tag{6.3.52}$$

Now, since we have the expression for the electromotive force (ε), we need to link it to the motion of the pendulum. To do that, we need to find a relation between α, which is the angle between the magnetic field and the surface normal vector of the coil, and θ, which is the angle of the pendulum.

If we assume that the coil is placed in some way so that its center is aligned with the origin, then it has a fixed angle ν that tells us it position. Then, the angle α is the difference between the pendulum position angle (θ) and the coil position angle (ν).

Then, we can obtain the relation between α and θ and their derivatives.

$$\alpha = \theta - \nu \implies \dot{\alpha} = \dot{\theta}. \tag{6.3.53}$$

Therefore, the induced voltage in the coils is.

$$\varepsilon = nBS\sin\left(\theta - \nu\right)\dot{\theta}. \tag{6.3.54}$$

This last expression is important because we can see the voltage in each coil. However, we know that induction is only generated when the magnet passes over the coil, so the magnet induces a current only in the arc length that the coil uses. To express that we use the Heaviside step function. If each coil has the same width, and that width is w, and its radius is the height of the structure (h). Then, we find that the angle (ϑ) of the length of the arc of each coil is.

$$\vartheta = \frac{w}{h}. \tag{6.3.55}$$

Then, the operative angle of the induction in each coil is:

$$[\nu_i - \frac{\vartheta}{2}, \nu_i + \frac{\vartheta}{2}]. \tag{6.3.56}$$

But, each coil has its unique position angle ν_i. So, it is necessary to find a recursive expression that helps us to express the position of each coil. To do that, we can allocate the space based on the number of coils we want to use. If we distribute the coils geometrically, we obtain that the position angle of each coil is

$$\nu_i = \frac{\Theta(2i - N - 1)}{N + 1}, \quad i = 1, 2, \ldots, N. \tag{6.3.57}$$

where N is the number of coils that we are going to use in our system, and i is the i-th coil that we study. Then, the voltage of each coil can be expressed as.

$$\varepsilon_i = nBS\sin\left(\theta - \frac{\Theta(2i - N - 1)}{N + 1}\right)\dot{\theta}\left[H\left(\theta - \frac{\Theta(2i - N - 1)}{N + 1} + \frac{w}{2h}\right) - H\left(\theta - \frac{\Theta(2i - N - 1)}{N + 1} - \frac{w}{2h}\right)\right]. \tag{6.3.58}$$

or in a simple form.

$$\varepsilon_i = nBS\sin\left(\theta - \nu_i\right)\left(H\left(\theta - \nu_i + \frac{\vartheta}{2}\right) - H\left(\theta - \nu_i - \frac{\vartheta}{2}\right)\right)\dot{\theta}. \tag{6.3.59}$$

Now, to find the voltage generated by the system we sum the voltage generated by each coil. The total voltage is.

$$\varepsilon = \sum_{i=1}^{N} \varepsilon_i. \tag{6.3.60}$$

With the previous information, we can generate a model to see the total voltage generated by our system. We are going to generate a numerical model. The system that we are going to model is a pendulum that has $\Lambda = 1$, $\chi = 1$, $\Phi = 0.5$, $\sigma = 2$, $\Theta = \frac{\pi}{4}$; the induction systema has the following properties, each coil has ten turns ($n = 10$), each coil has an area of one ($S = 1m^2$), the magnet on the pendulum has a magnetic field of one tesla ($B = 1T$), the width of each coils is one meter ($w = 1m$), and the height of the structure is the ten meters ($h = 10m$), and we have five coils ($N = 5$). This generate the following figure.

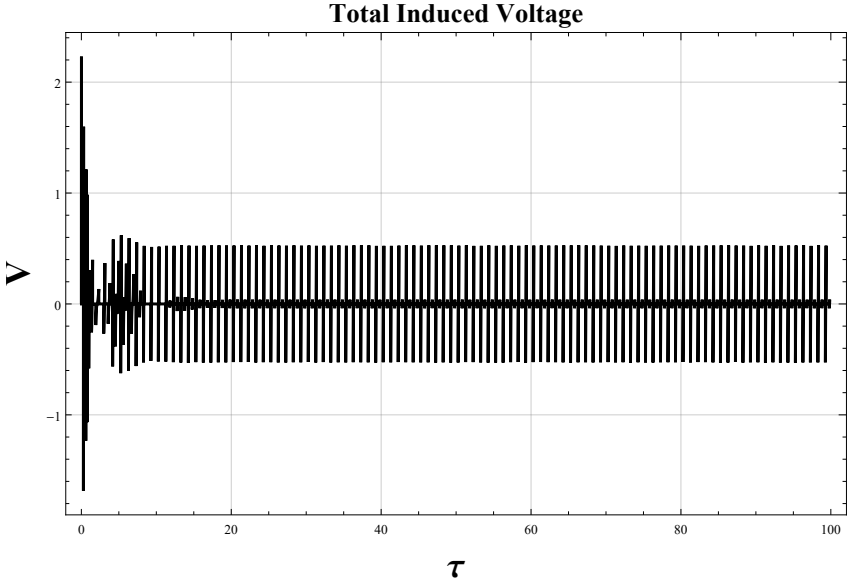


Figure 6.11: Numerical solution of the total voltage in terms of time with $\Lambda = 1$, $\chi = 1$, $\Phi = 0.5$, $\sigma = 2$, $\Theta = \frac{\pi}{4}$, $n = 10$, $S = 1m^2$, $B = 1T$, $w = 1m$, $h = 10m$, and $N = 5$

As we have seen previously, the solution of our dynamical system is extremely dependent on the damping number Φ and the torque number σ. So, we are going to see how the voltage reacts to different values of the damping number Φ. For the following figure that lets us see how the voltage responds to the damping number, we are going to use the same system as the previous figure, with a change in the torque number. For this simulation, we are going to use $\sigma = 1$. The voltage of the system with different values of damping is.

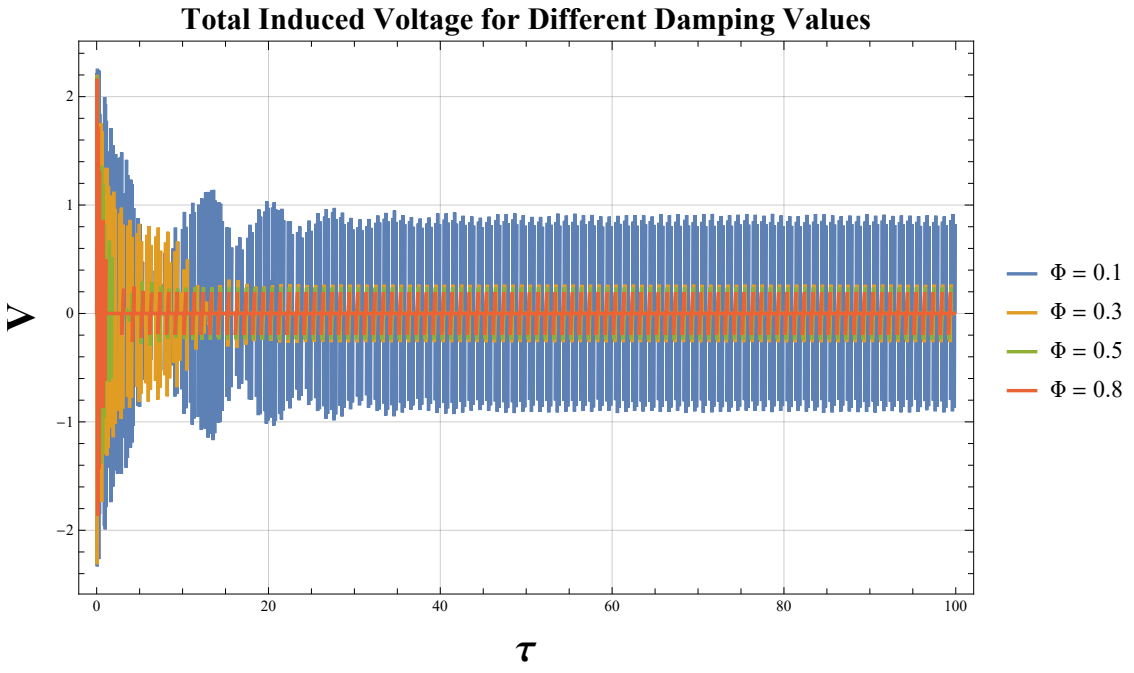


Figure 6.12: Voltage generated with different values of damping

6.4 Conclusions

This dynamical system is a very stable system. The stability is generated by the friction of the environment and the repulsive magnetic force that the magnets generate. But the system could be in chaos if the torque generated by the external force is large enough.

The oscillation frequency of the system depends on the moment of inertia of the structure (I_0), the repulsive magnetic force between the magnets, and the maximum angle of oscillation (Θ). This can be shown easily when the system is dimensionless; we can find three dimensionless numbers that govern the natural frequency of the oscillation; these numbers are the Inertia number (χ), maximum angle of the oscillation (Θ), and the Energie's number (Λ). If the system is in the regime of small oscillations, then the natural frequency is.

$$\Omega_0 = \sqrt{\chi + \frac{2\Lambda}{\Theta^2}}. \tag{6.4.1}$$

This implies that the frequency increases when χ and Λ increase, and when the maximum angle decreases, the frequency also increases. This is logical because, if the intensity of the magnets increases, the pendulum will return faster, and when the maximum angle is short, the distance between the magnets is short. Then, the repulsive force increases faster, causing more oscillations, and that implies a higher frequency.

Something important to highlight is that if the system is undamped and if an external force has enough strength, the system oscillates and decays to a stable state. This stable state keeps the oscillation constant over time. Talking about the external force, this force can be the force of a fluid in motion. Using the lift force of a fluid, we find that the dimensionless torque depends on four dimensionless parameters, the inertia number (χ), the lift coefficient

(C_L), the mass ratio number (m^*), and the Inertial-Gravitational interaction number (Γ). So, these numbers define the magnitude of the torque that the system obtains for the fluid.

Another important parameter to obtain the stable state is the frequency of the external force (Ω). When the frequency of the external force is similar to the natural frequency of the system, the system begins to resonate with the external force. In terms of fluids, the frequency of the external force is defined by the vortex shedding frequency generated for the fluid-structure interaction; then, the vortices that are generated for our system are the source of the oscillation. The frequency of the vortices is defined by the Strouhal number (St). In this system, we can see that the frequency of the external force is always proportional to the Strouhal number, but the definition of the dimensionless number can simplify or complicate the expression of the external frequency. If we only attend to the geometric parameters of the system, the time scale and frequency of the vortices are.

$$t_s = \sqrt{\frac{R}{g}} \quad , \quad \Omega = 2\pi St\Gamma. \tag{6.4.2}$$

And, if we pay attention to the geometric system parameters and the fluid parameters, the time scale and the frequency of the vortices are.

$$t_s = \frac{D}{U} \quad , \quad \Omega = 2\pi St. \tag{6.4.3}$$

These expressions are important because they can define the frequency of the external force. The first expression (6.4.2) lets us control the frequency in terms of the build parameters, and in the second (6.4.3), the frequency is independent of the structure parameters.

Finally, the harnessing energy model lets us see that the system could transform the motion of the pendulum into electricity. Moreover, if we pay attention, the system, when it is in the stable state, generates a voltage that oscillates between a maximum and a minimum voltage, which has the same value if we think in terms of magnitude. So, we can think that this system is analogous to an alternating voltage system.

Conclusions

In this project, we worked with three different systems that used the magnetic force as a restoring force to generate an oscillation. To keep the movement of the systems, we introduced an external force, which was the force of a fluid in motion. So, finally, we built three forced oscillator systems, with the intention of extracting energy from the movement of a structure pushed by a fluid force.

The systems that we studied oscillate, and the external force that is provided for the motion of a fluid generates Vortex-Induced Vibrations (VIV), which create gradients of pressure, causing our systems to start to oscillate in synchrony with the VIV's frequency. So, we can control the frequency of the oscillations with the shape of the structure and the motion of the fluid.

Each system has its own dimensionless numbers that govern and explain the motion of the system. But some dimensionless numbers appear in every system, and those are the dimensionless numbers that are related to the movement of the fluid. That is logical because all of them have the fluid force as the external force that impels the system. The fluid dimensionless numbers that always appear are:

- The Strouhal number (St): The frequency of the VIV's in our system.
- The Inertial-Gravitational interaction number (Γ): The ratio between the movement of the fluid and the movement of the structure driving the gravity.
- The mass ratio (m^*): The ratio of the mass of the fluid that is interacting with the structure, which has its own mass.
- The lift coefficient (C_L): The coefficient that is given by the shape of the body and helps to lift when the fluid passes around it.

Another dimensionless number that always appeared was the friction number (Φ), and it is because of nature that every system always has some friction with the environment. This number is so important because it generates the stability of the system.

From the structural systems, when the movement is locked in a specific region, we found an important dimensionless number that lets us know how the magnetic energy and the gravitational potential energy interact, this number is the Energie's number (Λ). And it is defined as.

$$\Lambda = \frac{M}{mgh}. \tag{6.4.4}$$

Where M is the magnetic energy that magnets have, m is the mass of our magnet that oscillates, g is the gravitational acceleration, and h is the height of the system. This number is important because in closed systems, the system with three magnets (chapter 5) and the pendulum system (chapter 6), the natural frequency has this number inside. So, Energie's number controls the movement of the system.

For the open system, the magneto-gravitational system (chapter 4), we can not find the Energie's number because the forcing increases the height; for that reason, we can not find the Energie's number from the beginning of the movement, and that implies that it does not govern the behavior of the system.

In terms of harnessing energy, we found that we can generate voltage in an open circuit using Faraday's law. In every system, we can induce voltage. The most effective system to generate voltage is the pendulum system, this is because in the length of the arc that it follows, we can put more coils than in the other systems, which can only have one or two coils.

In conclusion, the systems we studied can transform the mechanical energy of a structure into electric energy using Vortex Induced-Vibrations generated by the motion of a fluid. We obtained the theoretical expressions that let us estimate the voltage that we can induce in coils by the force of the vortex-induced vibrations. If we can control the frequency of the VIV with precision, we can build our system with specific dimensions to harness the energy of the fluids effectively. It is important to study the shape of the structures to find the best shape for the movement in each fluid, with the intention of improving the voltage induction.

Appendix

In this appendix, we present the various Wolfram Mathematica codes we wrote throughout the project to simulate our systems and generate our plots.

It is important to note that the codes we present here represent one version, as most of the code can be used across all systems with only minor changes. These changes are the properties or equations of each system.

Phase portrait diagram

This code lets us visualize the phase portrait diagram that a system has. This code needs to know the position and momentum expressions previously and the values of the parameters and conditions that our system has. It is important to note that the code we present at this moment is for the ideal case of a system; for other systems, we need to add the damping and forcing terms.

```
    (*Parameters*)g = 9.81;
m = 1;
M = 10;

(*System of ODEs*)
vx[x_, v_] := v;
vv[x_, v_] := -g + M/(m x);

StreamPlot[{vx[x, v], vv[x, v]}, {x, 0.5, 5}, {v, -5, 5},
 PlotTheme -> "Scientific",(*Visual consistency we agreed on*)
 Frame -> True, Axes -> False, GridLines -> Automatic,
 GridLinesStyle -> Directive[GrayLevel[0.7, 0.35]],
 StreamPoints -> Fine, StreamStyle -> Thick,
 ColorFunction -> ColorData["DeepSeaColors"],
 ColorFunctionScaling -> True,(* You requested this:FRAME LABELS*)
 FrameLabel -> {Style["x", Black, Bold, Large],
   Style["p", Black, Bold, Large]},
 PlotLabel -> Style["Phase Portrait", Black, Bold, Large],
 ImageSize -> 450]
```

Contrast of numerical and asymptotic solutions

This code contrasts the numerical solution of a system with the asymptotic solution that we obtain. This code requires you to know previously the asymptotic solution that you want to contrast.

```
\[CapitalPhi] = 0.01;
\[Delta] = Sqrt[1 - \[CapitalPhi]^2];

glffo = {{, }, {, }, {, }, {, }};

    For[i = 1, i < Length[fel] + 1, i++,
 sf = NDSolve[{x[t] x''[t] + 2 \[CapitalPhi] x[t] x'[t] + x[t] - 1 ==
     0, x[0] == 1 + fel[[i]], x'[0] == 0}, x, {t, 0, 30}];
 ffopf = Plot[{Evaluate[x[t] /. sf],
    1 + E^(-\[CapitalPhi] t)
       fel[[i]] (Cos[\[Delta] t] + \[CapitalPhi]/\[Delta] \
Sin[\[Delta] t])}, {t, 0, 30}, PlotRange -> All,
   PlotLabel ->
    Style["First order solution with \[Epsilon]="@fel[[i]], Black,
     Bold], FrameLabel -> {Style["\[Tau]", Black, Bold, Large],
     Style["\!\(\*OverscriptBox[\(x\), \(~\)]\)", Black, Bold,
      Large]}, PlotTheme -> "Scientific", GridLines -> Automatic,
   PlotStyle -> {Thick, Dashed}];
 ffospf =
  ParametricPlot[{{Evaluate[{x[t], x'[t]} /. sf]}, {1 +
      E^(-\[CapitalPhi] t)
        fel[[i]] (Cos[\[Delta] t] + \[CapitalPhi]/\[Delta] Sin[\
\[Delta] t]),
     fel[[i]] (-((
        E^(-t \[CapitalPhi]) (\[Delta]^2 + \[CapitalPhi]^2) Sin[
          t \[Delta]])/\[Delta])) }}, {t, 0, 20},
   PlotLabel ->
    Style["First order with \[Epsilon]="@fel[[i]], Black, Bold],
   FrameLabel -> {Style["\!\(\*OverscriptBox[\(x\), \(~\)]\)", Black,
      Bold, Large],
     Style["\!\(\*OverscriptBox[OverscriptBox[\(x\), \(~\)], \
\(.\)]\)", Black, Bold, Large]}, PlotTheme -> "Scientific",
   GridLines -> Automatic, PlotStyle -> {Thick, Dashed}];
 glffo[[i, 1]] = ffopf;
 glffo[[i, 2]] = ffospf;
 ]
GraphicsGrid[glffo]
```

Dimensioless groups

This code helps us to find the different dimensionless groups and the scale factors of each dimension of a system. This code needs to introduce the dimensions of each variable that we have in the system and the dimensions of the variables that we use to dimensionless

the system. The code returns us a table that gives us the dimensionless groups of each variable.

```
    vdsys = {{x, 1, 0, 0}, {t, 0, 0, 1}, {A, 1, 0, 0}, {mv, 0, 1, 0}};
kvdsys = {{rv, 1, 0, 0}, {gv, 1, 0, -2}, {Mv, 2, 1, -2}};
TableForm[
 Table[{Solve[{vdsys[[i, 2]] + kvdsys[[1, 2]] \[Alpha] +
       kvdsys[[2, 2]] \[Beta] + kvdsys[[3, 2]] \[Lambda] == 0,
     vdsys[[i, 3]] + kvdsys[[1, 3]] \[Alpha] +
       kvdsys[[2, 3]] \[Beta] + kvdsys[[3, 3]] \[Lambda] == 0,
     vdsys[[i, 4]] + kvdsys[[1, 4]] \[Alpha] +
       kvdsys[[2, 4]] \[Beta] + kvdsys[[3, 4]] \[Lambda] ==
      0}, {\[Alpha], \[Beta], \[Lambda]}],
   vdsys[[i, 1]] kvdsys[[1, 1]]^\[Alpha] kvdsys[[2,
      1]]^\[Beta] kvdsys[[3, 1]]^\[Lambda] /.
    Solve[{vdsys[[i, 2]] + kvdsys[[1, 2]] \[Alpha] +
        kvdsys[[2, 2]] \[Beta] + kvdsys[[3, 2]] \[Lambda] == 0,
      vdsys[[i, 3]] + kvdsys[[1, 3]] \[Alpha] +
        kvdsys[[2, 3]] \[Beta] + kvdsys[[3, 3]] \[Lambda] == 0,
      vdsys[[i, 4]] + kvdsys[[1, 4]] \[Alpha] +
        kvdsys[[2, 4]] \[Beta] + kvdsys[[3, 4]] \[Lambda] ==
       0}, {\[Alpha], \[Beta], \[Lambda]}],
   kvdsys[[1, 1]]^-\[Alpha] kvdsys[[2, 1]]^-\[Beta] kvdsys[[3,
      1]]^-\[Lambda] /.
    Solve[{vdsys[[i, 2]] + kvdsys[[1, 2]] \[Alpha] +
        kvdsys[[2, 2]] \[Beta] + kvdsys[[3, 2]] \[Lambda] == 0,
      vdsys[[i, 3]] + kvdsys[[1, 3]] \[Alpha] +
        kvdsys[[2, 3]] \[Beta] + kvdsys[[3, 3]] \[Lambda] == 0,
      vdsys[[i, 4]] + kvdsys[[1, 4]] \[Alpha] +
        kvdsys[[2, 4]] \[Beta] + kvdsys[[3, 4]] \[Lambda] ==
       0}, {\[Alpha], \[Beta], \[Lambda]}]}, {i, 1, Length[vdsys]}],
 TableHeadings -> {Table[vdsys[[i, 1]], {i, 1, 4}], {"",
    "\[CapitalPi]-group", "scale factor"}}]
 TableForm[{{{{\[Alpha] -> 1, \[Beta] -> 1, \[Lambda] -> -1}}, {((
      g m)/M) x}, {(g^(-1)/m)
      M}}, {{{\[Alpha] -> Rational[
      1, 2], \[Beta] -> 1, \[Lambda] -> Rational[-1, 2]}}, {((
      g m^Rational[1, 2]) M^Rational[-1, 2]) t}, {(
     g^(-1) m^Rational[-1, 2])
      M^Rational[
      1, 2]}}, {{{\[Alpha] -> 1, \[Beta] -> 1, \[Lambda] -> -1}}, {(
     A g) (m/M)}, {(g^(-1)/m) M}}},
 TableHeadings -> {{x, t, A}, {
   "", "\[CapitalPi]-group", "scale factor"}}]
```

Numerical solution

This code generates the numerical solution of a dynamical system, and after that, we plot the solution to see the evolution of the system.

In particular, the code is designed for the pendulum system. To use for the other systems, we need to introduce the correct differential equation and its variables.

```
    \[Chi] = 1;
\[CapitalLambda] = 1;
\[CapitalPhi] = 0.5;
\[Sigma] = 1;
\[CapitalOmega] = \[Pi];
\[CapitalTheta] = \[Pi]/5;

f = NDSolve[{\[Theta]''[\[Tau]] +
     2 \[CapitalPhi] \[Theta]'[\[Tau]] + (\[Chi] + \
\[CapitalLambda]/(-Cos[\[CapitalTheta]] + Cos[\[Theta][\[Tau]]] +
         10^-6)) Sin[\[Theta][\[Tau]]] == \[Sigma] \
Cos[\[CapitalOmega] \[Tau]], \[Theta][
     0] == \[CapitalTheta], \[Theta]'[0] == 0}, \[Theta], {\[Tau], 0,
   500}]

Plot[Evaluate[\[Theta][\[Tau]] /. f], {\[Tau], 0, 100},
 PlotRange -> All,
 PlotLabel -> Style["Numerical solution ", Black, Bold],
 FrameLabel -> {Style["\[Tau]", Black, Bold, Large],
   Style["\[Theta]", Black, Bold, Large]}, PlotTheme -> "Scientific",
 GridLines -> Automatic]
```

Bifurcation

This code is used to generate the bifurcation diagrams. Here, the code solves the system numerically for a specific value of the external force, obtaining the stable states of the system for each value of the external force, and after that, we plot those points.

```
    (*-----USER PARAMETERS-----*)
\[CapitalPhi] = 0.1;(*damping coefficient*)
\[CapitalOmega] = 1;(*forcing frequency*)
T = 2 \[Pi]/\[CapitalOmega];(*forcing period*)
\[CapitalLambda] = 1; (*Energies number*)
\[Chi] = 1;(*Frequency pendulum number*)
\[CapitalTheta] = \[Pi]/4;(*Max angule value*)
(*Integration times (in periods)*)transientPeriods = 200;(*discard \
this many periods at the start*)samplePeriods = 100;(*sample this \
many periods after the transient*)tran =
 transientPeriods T;(*time after which we start sampling*)total = \
(transientPeriods +
    samplePeriods) T;(*total integration time*)(*Parameter range for \
F*)\[Sigma]min = 0;
\[Sigma]max = 12;
\[Sigma]step = 0.1;
\[Sigma]values = Range[\[Sigma]min, \[Sigma]max, \[Sigma]step];

(*-----FUNCTION:compute Poincaré points for a single F-----*)
```

```
getPoincare[\[Sigma]_] :=
  Module[{sol, points},
   sol = NDSolve[{\[Theta]''[\[Tau]] +
        2 \[CapitalPhi] \[Theta]'[\[Tau]] + (\[Chi] + \
\[CapitalLambda]/(-Cos[\[CapitalTheta]] + Cos[\[Theta][\[Tau]]] +
           10^-6)) Sin[\[Theta][\[Tau]]] == \[Sigma] Cos[\
\[CapitalOmega] \[Tau]], \[Theta][0] == \[CapitalTheta], \[Theta]'[
        0] == 0}, \[Theta], {\[Tau], 0, total}, MaxSteps -> Infinity,
     Method ->
      "StiffnessSwitching"    (*helps with potential stiffness*)];
   (*Extract x at times t=tran,tran+T,...,total*)
   points =
    Table[{\[Sigma], \[Theta][\[Tau]] /. sol[[1]]}, {\[Tau], tran,
      total, T}];
   points];

(*-----GENERATE DATA FOR ALL F VALUES-----*)
data = Flatten[
   Table[getPoincare[\[Sigma]], {\[Sigma], \[Sigma]values}], 1];

(*-----PLOT THE BIFURCATION DIAGRAM-----*)
ListPlot[data, PlotRange -> All,
 AxesLabel -> {"Forcing amplitude \[Sigma]", "\[Theta](nT)"},
 PlotStyle -> {PointSize[0.005]},
 LabelStyle -> Directive[Bold, Black],
 PlotLabel -> Style["Bifurcation diagram", Black, Bold],
 PlotRange -> All, ImageSize -> Large, GridLines -> Automatic,
 PlotTheme -> "Scientific",
 FrameLabel -> {Style["\[Sigma]", Black, Bold, Large],
   Style["\[Theta](nT)", Black, Bold, Large]}]
```

Lyapunov exponents

The following code is for generating the diagram of Lyapunov exponents vs the external torque. This code gives us the Lyapunov exponents values for each value of σ.

```
(* ================================================*)(*Maximum \
Lyapunov exponent vs sigma*)(* \
================================================*)ClearAll["Global`*"]

(*----------Fixed parameters----------*)

\[CapitalPhi] = 0.1;(*damping coefficient*)\[CapitalOmega] = \
1.0;(*forcing frequency*)\[CapitalLambda] = 1.0;(*nonlinear \
strength*)\[Chi] = 1.0;(*linear restoring parameter*)\[CapitalTheta] \
= 0.3;(*singular angle parameter*)tmax = 400;(*total integration \
time*)dt = 0.5;(*renormalization interval*)(*----------Initial \
conditions----------*)theta0 = 0.1;
v0 = 0.0;
```

```
\[Delta]0 =
 10^-8;(*initial perturbation norm*)(*----------Range of sigma \
values----------*)sigmaValues = Range[0, 14, 0.01];

(* ================================================*)
(*Main loop over sigma*)
(* ================================================*)

results = Table[(*current state and perturbation*)state = {theta0, v0};
   pert = {\[Delta]0, 0};
   t = 0;
   sum = 0;
   While[t < tmax,(*Solve nonlinear system+variational equations*)
    sol = NDSolve[{(*----------Original system----------*)\[Theta]'[\
\[Tau]] == v[\[Tau]],
        v'[\[Tau]] ==
         sigma Cos[\[CapitalOmega] \[Tau]] -
          2 \[CapitalPhi] v[\[Tau]] - (\[Chi] + \
\[CapitalLambda]/(Cos[\[Theta][\[Tau]]] -
                Cos[\[CapitalTheta]])) \
Sin[\[Theta][\[Tau]]],(*----------Variational equations----------*)\
\[Delta]\[Theta]'[\[Tau]] == \[Delta]v[\[Tau]], \[Delta]v'[\[Tau]] == \
(-\[Chi] Cos[\[Theta][\[Tau]]] - \
\[CapitalLambda]/(Cos[\[Theta][\[Tau]]] - Cos[\[CapitalTheta]])*
             Cos[\[Theta][\[Tau]]] - \[CapitalLambda] Sin[\[Theta][\
\[Tau]]]^2/(Cos[\[Theta][\[Tau]]] -
                  Cos[\[CapitalTheta]])^2) \[Delta]\[Theta][\[Tau]] -
          2 \[CapitalPhi] \[Delta]v[\[Tau]],(*----------Initial \
conditions----------*)\[Theta][0] == state[[1]],
        v[0] == state[[2]], \[Delta]\[Theta][0] ==
         pert[[1]], \[Delta]v[0] == pert[[2]]}, {\[Theta],
        v, \[Delta]\[Theta], \[Delta]v}, {\[Tau], 0, dt},
       MaxSteps -> Infinity, Method -> {"StiffnessSwitching"}][[1]];
    (*----------Update state----------*)
    state = {\[Theta][dt] /. sol, v[dt] /. sol};
    (*----------Update perturbation----------*)
    pert = {\[Delta]\[Theta][dt] /. sol, \[Delta]v[dt] /. sol};
    (*----------Renormalization----------*)norm = Norm[pert];
    sum = sum + Log[norm/\[Delta]0];
    pert = \[Delta]0*pert/norm;
    t = t + dt;];
   (*----------Lyapunov exponent----------*)lyap = sum/tmax;
   {sigma, lyap}, {sigma, sigmaValues}];

(* ================================================*)
(*Plotting*)
(* ================================================*)
```

```
ListPlot[results, PlotRange -> All,
 PlotStyle -> {PointSize[0.015], Blue},
 AxesLabel -> {"\[Sigma]", "\[Lambda]"},
 LabelStyle -> Directive[Bold, Black, 14],
 PlotLabel ->
  Style["Maximum Lyapunov exponent vs \[Sigma]", Black, Bold],
 ImageSize -> Large, GridLines -> Automatic,
 PlotTheme -> "Scientific", Frame -> True,
 FrameLabel -> {Style["\[Sigma]", Black, Bold, Large],
   Style["\[Lambda]", Black, Bold, Large]}, AxesStyle -> Thick]

(*Optional export*)
(*Export["lyapunov_vs_sigma.pdf",%]*)
```

Bibliography

[1] William A. Adkins and Mark G. Davidson. *Ordinary Differential Equations*. United States: Springer, 2012. ISBN: 978-1-4614-3617-1. DOI: 10.1007/978-1-4614-3618-8.

[2] R. Ahamed, I. Howard, and K McKee. "Dynamic analysis of magnetic spring-based nonlinear oscillator system". In: *Nonlinear Dynamics* 111 (2023), pp. 15705–15736. DOI: https://doi.org/10.1007/s11071-023-08669-3.

[3] Peter R. Andronov et al. In: *Aerospace Science and Technology* 93 (2019). DOI: https://doi.org/10.1016/j.ast.2019.105348.

[4] Francois Axisa and Jose Antunes. *Modelling of mechanical systems: Fluid Structure Interaction*. Great Britain: Elsevier Inc., 2007. ISBN: 978-0-750-66847-7.

[5] Sandip Banerjee. *Mathematical modeling: Models, Analysis and Applications*. India: CRC Press, 2014. ISBN: 978-1-4822-2916-5.

[6] Martin Bazant and Joey Gu. *MIT 10.50.CH03x. Analysis of Transport Phenomena*. Massachusetts, United States, 2023. URL: https://courses.mitxonline.mit.edu/learn/course/course-v1:MITxT+10.50.CH03x+1T2023/home.

[7] Maksymilian Bednarek, Donat Lewandowski, and Jan Awrejcewicz. "Magnetic and Electromagnetic Springs Forces: Determination and Usage in Damping Vibrations". In: *Advances in Nonlinear Dynamics*. Ed. by Walter Lacarbonara et al. Cham: Springer International Publishing, 2022, pp. 115–124. ISBN: 978-3-030-81166-2.

[8] Jiri Blazek. *Computational Fluid Dynamics: Principles and Applications*. 3rd. United Kingdom: Elsevier, 2015. ISBN: 978-0-08-099995-1.

[9] M. Brocchini and F. Trivellato. *Vorticity and Turbulence Effects in Fluid Structure Interaction: An Application to Hydraulic Structure Design*. Great Britain: WIT Press, 2006. ISBN: 1-84564-052-7.

[10] Butcher, J. C. *NUMERICAL METHODS FOR ORDINARY DIFFERENTIAL EQUATIONS*. 3rd. Wiley, 2016.

[11] M.G. Calkin. *Lagrangian and Hamiltonian Mechanics*. Singapore: World Scientific Publishing Co., 1996. ISBN: 981-02-2672-1.

[12] Frank S. Crawford Jr. *Ondas: Berkeley physics course-volumen 3 (Juan T. D'alessio, Trad.)* México: Editorial Reverté (Obra original publicada en 1968), 1965. ISBN: 84-291-4023-9.

[13] Aytekin Duranay, Omer Kemal Kinaci, and Michael Bernitsas. "Effect of aspect ratio on hydrokinetic energy harnessing using cylinders in VIV". In: *Journal of Ocean Engineering and Marine Energy* 8 (2022), pp. 217–232. DOI: https://doi.org/10.1007/s40722-022-00226-1.

[14] C. Foias et al. *Navier–Stokes Equations and Turbulence*. United Kingdom: Cambridge University Press, 2004. ISBN: 0-511-03936-0.

[15] S.K. Godunov. *Ecuaciones de la Física Matemática (Juán José Tolosa, Trad.)* México: MIR, 1978.

[16] Herbert Goldstein, Charles P. Poole, and John Safko. *Classical Mechanics*. 3rd. Pearson, 2013. ISBN: 9781292026558.

[17] Peng Han et al. In: *Ocean Engineering* 282 (2023). DOI: https://doi.org/10.1016/j.oceaneng.2023.114869.

[18] Sadri Hassani. *Mathematical Physics: A modern introduction to its Foundations*. 2nd. United States: Springer, 2013. ISBN: 978-3-319-01194-3. DOI: 10.1007/978-3-319-01195-0.

[19] John K. Hunter. *Asymptotic Analysis and Singular Perturbation Theory*. United States: University of California at Davis, 2004. URL: https://www.math.ucdavis.edu/~hunter/notes/asy.pdf.

[20] Hyunjoong Kim and CChee-Han Tan. *Math 6730: Asymptotic and Perturbation Methods*. Universit of Utah, United States, 2018.

[21] L. D. Landau and E. M. Lifshits. *Mechanics*. Russia: Mir, 1960.

[22] Enzo Levi. *Teorías y Métodos de las Matemáticas Aplicadas*. México: Facultad de Ingeniería UNAM, 1965.

[23] J. David Logan. *Applied Partial Differential Equations*. 3th. United States: Springer, 2015. ISBN: 978-3-319-12492-6. DOI: 10.1007/978-3-319-12493-3.

[24] Teruo Matsushita. *Electricity and Magnetism: New Formulation by Introduction of Superconductivity*. Japan: Springer, 2014. ISBN: 978-4-431-54525-5. DOI: 10.1007/978-4-431-54526-2.

[25] Efstathios E (Sthatis) Michaelides. *Alternative Energy Sources*. Germany: Springer, 2012. ISBN: 978-3-642-20950-5. DOI: 10.1007/978-3-642-20951-2.

[26] Yahya Modarres-Sadegh. *Introduction to Fluid-Structure Interactions*. Switzerland: Springer, 2021. ISBN: 978-3-030-85882-7.

[27] Peter J. Olver. *Introduction to Partial Differential Equations*. United States: Springer, 2014. ISBN: 978-3-319-02098-3. DOI: 10.1007/978-3-319-02099-0.

[28] Michael P. Païdoussis, Stuart J. Price, and Emmanuel de Langre. *Fluid-Structure Interactions: CROSS-FLOW-INDUCED INSTABILITIES*. United States of America: Cambrige University Press, 2011. ISBN: 978-0-521-11942-9.

[29] Press, W. H. and Teukolsky, S. A. and Vetterling, W. T. and Flannery, B. P. *NUMERICAL RECIPES: THE ART OF SCIENTIFIC COMPUTING*. 3rd. Cambridge University Press, 2007.

[30] Hazef A. Radi and John O. Rasmussen. *Principles of Physics: For Scientists and Engineers*. Germany: Springer, 2013. ISBN: 978-3-642-23026-4. DOI: 10.1007/978-3-642-23026-4.

[31] José Rogan C. and Víctor Muñoz G. *Apuntes de un curso de FÍSICA MATEMATICA*. 3ra. Chile: Departamento de Física.

[32] Anne Rooney. *La historia de la física: De la filosofía natural al enigma de la materia oscura (Luigi Freda Eslava, Trad.)* México: Grupo Editorial Tomo (Obra original publicada en 2011), 2013. ISBN: 978-607-415-473-3.

[33] Soldovieri C., T. *INTRODUCCION A LA MECANICA DE LAGRAGE Y HAMILTON*. 1era. Venezuela: Preprint, 2019.

[34] Ain A. Sonin. *The Physical Basis of DIMENSIONAL ANALYSIS*. 2nd. Cambridge, Massachusetts, United States: Department of Mechanical Engineering MIT, 2001.

[35] Iain Stewart. *MIT OpenCourseWare 8.09 | Fall 2014 | Undergraduate Classical Mechanics III*. Massachusetts, United States, 2014. URL: https://ocw.mit.edu/courses/8-09-classical-mechanics-iii-fall-2014/.

[36] Jing Tang Xing. *Fluid-Solid Interaction Dynamics: Theory, Variational Principles, Numerical Methods, and Applications*. United Kingdom: Elsevier Inc., 2019. ISBN: 978-0-12-819352-5.

[37] Stephen T. Thornton and Jerry B. Marion. *Classical Dynamics of Particles and Systems*. Cengage Learning, 2003. ISBN: 9780534408961.

[38] Jiyuan Tu, Guan-Heng Yeoh, and Chaoqun Liu. *Computational Fluid Dynamics: A practial approach*. 3rd. Elsevier, 2018. ISBN: 978-0-08-101127-0.

[39] Arnt Inge Vistnes. *Physics of oscillations and waves: With use of Mathlab and Python*. Oslo, Norway: Springer, 2018. ISBN: 978-3-319-72313-6. DOI: https://doi.org/10.1007/978-3-319-72314-3.

[40] Xiaodong (Sheldon) Wang. *Fundamentals of Fluid-Structure Interactions: Analytical and a Computational approaches*. Hungary: Springer, 2008. ISBN: 978-0-444-52807-0.

[41] Frank M. White. *Fluid Mechanics*. 8th. United States of America: McGraw-Hill Education, 2016. ISBN: 978-9-38-596549-4.

[42] C.H.K Williamson and R. Govardhan. "Vortex-Induced Vibrations". In: *Annual Reviews Fluid Mechanics* 36 (2004), pp. 413–455. DOI: 10.1146/annurev.fluid.36.050802.122128.

[43] Thomas Witelski and Mark Bowen. *Methods of Mathematical Modelling: Continuous Systems and Differential Equations*. USA and Japan: Springer, 2015. ISBN: 978-3-319-23041-2. DOI: 10.1007/978-3-319-23042-9.

[44] Thomas Witelski and Mark Bowen. *Physics of oscillations and waves: With use of Mathlab and Python*. Oslo, Norway: Springer, 2018. ISBN: 978-3-319-72313-6. DOI: https://doi.org/10.1007/978-3-319-72314-3.

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TESIS

Que como parte de los requisitos para obtener el Grado de

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Presenta:

Axayacatl Hernández Fuentes

Dirigido por:

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T E S I S

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Director de tesis

Dr. Raúl Alejandro Ávalos Zúñiga

Querétaro, Qro.
2026

Este trabajo de tesis está dedicado a mi familia.
A mi mamá y mi hermano por ser mi soporte y apoyo incondicional.
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Resumen

Esta tesis presenta el desarrollo teórico de un convertidor de energía hidroeléctrica basado en resortes magnéticos y vibraciones inducidas por vórtices (VIV). Se proponen, modelan y analizan tres sistemas oscilatorios no lineales: un oscilador magnetogravitacional, un sistema oscilatorio de tres imanes y un péndulo rígido con repulsión magnética. Se estudió el comportamiento de los sistemas dinámicos mediante mecánica lagrangiana y hamiltoniana, análisis dimensional, métodos de perturbación asintótica y simulaciones numéricas (Runge-Kutta). Se estudiaron los sistemas en condiciones ideales, amortiguadas y forzadas. La interacción fluido-estructura se incorpora mediante VIV, donde la fuerza de sustentación y el número de Strouhal definen la excitación externa. La estabilidad y el caos se evalúan mediante exponentes de Lyapunov, multiplicadores de Floquet y diagramas de bifurcación. Finalmente, se aplica la ley de inducción de Faraday para estimar la tensión eléctrica generada en las bobinas, demostrando la viabilidad de convertir la oscilación mecánica en energía eléctrica. Los resultados resaltan el papel de los parámetros adimensionales (por ejemplo, el número de Energie, el número de Strouhal, la relación gravedad-inercia) y muestran que la configuración basada en péndulo ofrece el mayor potencial para una recolección de energía eficiente.

Abstract

This thesis presents the theoretical development of a hydroelectric energy converter based on magnetic springs and vortex-induced vibrations (VIV). Three nonlinear oscillatory systems are proposed, modeled, and analyzed: a magneto-gravitational oscillator, a three-magnets oscillator system, and a rigid pendulum with magnetic repulsion. We studied the behavior of the dynamic systems using Lagrangian and Hamiltonian mechanics, dimensional analysis, asymptotic perturbation methods, and numerical simulations (Runge-Kutta). We studied our systems under ideal, damped, and forced conditions. The fluid-structure interaction is incorporated through VIV, where the lift force and Strouhal number define the external excitation. Stability and chaos are assessed via Lyapunov exponents, Floquet multipliers, and bifurcation diagrams. Finally, Faraday's law of induction is applied to estimate the electrical voltage generated in coils, demonstrating the feasibility of converting mechanical oscillation into electrical energy. The results highlight the role of dimensionless parameters (e.g., Energie's number, Strouhal number, gravity-inertia ratio) and show that the pendulum-based configuration offers the highest potential for efficient energy harvesting.

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Introduction

Since Ancient times, the first civilizations of humanity sought energy sources that would facilitate labor and manufacturing processes, such as mills pulled by oxen, horses, or slaves, and typical water mills or even Archimedes' screw; but it wasn't until the Industrial Revolution that, taking advantage of thermodynamics knowledge and flourishing science, humans had the capacity to provide machines with enough energy to reduce both production time and labor.

The method used in the Industrial Revolution was refined over time, building increasingly efficient machines, but using the same type of energy source: fossil fuels, which later would prove to be harmful not only to human health but to the entire biosphere, generating a phenomenon known as climate change, as fuel residues disperse in the environment causing the Earth to absorb more radiation, warming it and changing the climate of different ecosystems around the world.

Due to the above, the United Nations (UN) in 1979 during the 1st World Climate Conference identified climate change as an urgent global problem and urged nations to seek new energy sources, which would not leave such a marked impact on nature. Since then, humanity has been looking for renewable energy sources, some examples of these energies are solar, wind, nuclear, etc.

Renewable energies are at their peak. The best-known renewable energies are wind and solar. But as mentioned by Michaelides, two-thirds of the planet is covered with water, water that is in constant movement and has kinetic energy, which is not usually harnessed, which is why an alternative to other renewable energy sources with great potential is found in water flows and fluid movement ([25]).

This work will study different mechanical systems that implement repulsive magnetic forces to generate an oscillatory system, with the intetion to develop a theoretical analysis related to the conversion of mechanical energy to electrical energy from fluid-structure interaction, since a fluid in motion possesses kinetic energy, which can be transformed into another type of energy, especially electrical energy. To convert kinetic energy into electrical energy, a ***hydroelectric converter*** is required, which is a system that allows us to convert the kinetic energy from the moving fluid into electrical energy.

We are going to work with three different systems or forms to build our hydroelectric converter. The first system consists of two magnets: one fixed and one free to move vertically, where the restoring force comes from the combination of gravity and magnetic repulsion. The second system uses three magnets: two fixed at the ends of a vertical channel and one free to move between them, generating a symmetric oscillation. The third

system is a rigid pendulum with a magnet at its tip, whose oscillation is limited by two fixed magnets placed symmetrically.

The operation of the hydroelectric converter is as follows: when the moving fluid interacts with the structure, it causes it to experience an oscillating pressure force, pushing the structure towards one of the channel walls, where as it approaches, the repulsive forces increase, repelling the structure from the wall in the direction of the other wall, the process is repeated generating a forced oscillation between the structure and the fluid. It is important to highlight that the restitution force present in the system is a magnetic force, which is non-linear.

It is important to know the theoretical behavior of such a system because it allows us to estimate how much energy can be captured and transformed.

We can explain our motivation like:

- Fluid–structure interaction appears naturally in many environments: ocean currents, wind flows, and vortex streets generated around obstacles. These phenomena contain mechanical energy that is often wasted.
- Magnetic–gravitational oscillators offer a simple, low-cost way to capture part of this natural energy without the need for electronics, complex materials, or external power sources.
- By using only magnets, gravity, and fluid motion, this system proposes an alternative pathway toward clean and passive energy harvesting.
- Understanding the dynamics of these oscillators helps us design devices capable of converting environmental flows into useful oscillations, contributing to sustainable technologies and small-scale renewable systems.
- This project aims to analyze, model, and simulate these mechanisms to demonstrate their potential as energy–extraction tools for a cleaner future.

The first, second, and third chapters are theoretical chapters. Here, we present all the theory that we are going to need to work on our systems. Each chapter is focused on a specific topic with the idea of concentrating the same information in the same block.

In the first chapter, we discuss the mathematical techniques we need to apply to the differential equations of our systems to solve and analyze them. We explore the concept and power of differential equations, their classification, and how we can analyze them before to solve it, to know their behavior and stability. Another important topic that we inspect is the Perturbative analysis, and how it is useful to solve differential equations that are commonly unsolvable. We introduce and explain the idea behind dimensional analysis and how transforming systems to a dimensionless system could help in the analysis. Another group of techniques that we address in this chapter is the mechanical analytical techniques (Lagrangian and Hamiltonian analysis). Finally, we talk about numerical solutions for differential equations that are analytically unsolvable, in particular, the Runge-Kutta method.

The second chapter focuses on the oscillation theory. Here, we talk about the different types of oscillations (harmonic, damped, and forced oscillations), the different natures of the restoring forces on an oscillatory system, and how to solve them. In every system, we can obtain the expression of the energy, so we find the expression of the energy for each system. For our systems, which have an external force adding energy, it is important to

understand and apply the idea of resonance, so at the end of this chapter, we study this crucial phenomenon.

As we mentioned previously, we are going to extract energy from the motion of fluids. So, it is important to know how fluids act in nature, and the third chapter of our project is centered on that idea. So we explore the fluid dynamics and the fluid-structure interaction. The fluid-structure interaction generates a kind of vortices that are generated at some period of time, these vortices are the Vortex Induced Vibrations (VIV), and they are so useful because they are the form that we can estimate the energy that a structure or mechanical system can extract from a fluid in motion.

The chapters fourth, fifth, and sixth are focused on solving the different systems that we analyze in our project. Each chapter is focused on a specific system and its solutions.

The fourth chapter is dedicated to the simplest system, the magneto-gravitational system. In this system, we establish our methodology to work in all systems, and we apply the techniques that we explain in the theoretical chapters. This system is characterized by the presence of gravity and the repulsive force between two magnets. And we can find some important dimensionless forms of our system that can help us to describe the movement and the voltage induction in an open circuit.

The fifth chapter extends the analysis to a system with three magnets, where the free magnet oscillates between two fixed magnets. This configuration offers a symmetric potential well and allows us to study the effect of confinement on the oscillation. We analyze the ideal, damped, and forced cases, and we also study the stability and chaos conditions using Lyapunov exponents and bifurcation diagrams. Finally, we model the energy harvesting capability of this system.

The sixth chapter is dedicated to the analysis of a physical pendulum that has a magnet at the bottom of it, and its oscillation is limited by two fixed magnets. Applying the mathematical analysis techniques, we find that this system is the most stable among the three, but also the most complex due to the coupling between gravitational torque and magnetic forces. We derive the dimensionless equations, study the small-angle approximation, and develop a multi-coil model to estimate the total induced voltage, demonstrating that this configuration is the most effective for energy harvesting.

The final chapter, the seventh, is dedicated to the conclusions. In this chapter, we are going to resume and recap the conclusions of the fourth, fifth, and sixth chapters, and we establish how dimensionless numbers and our theoretical tools were effective for this work. We also discuss the main findings, the limitations of our models, and suggest future lines of research to optimize these systems for practical applications in renewable energy generation.

0.1 Objectives

The general objective of our project is to develop a theoretical framework for a hydroelectric converter that transforms the kinetic energy of a moving fluid into electrical energy using vortex-induced vibrations acting on oscillatory systems that use the magnetic force as a restorative force, using analytical mechanics, dimensional analysis, asymptotic approximations, and numerical methods.

0.1.1 Specific objectives

The specific objectives of this project are:

- Derive the equations of motion of our dynamical systems using Lagrangian and Hamiltonian mechanics.
- Non-dimensionalize the systems through dimensional analysis to identify key dimensionless parameters (Energie's number, friction number, Strouhal number, gravity-inertia ratio) and solve the resulting equations using asymptotic approximations and numerical methods (Runge-Kutta).
- Analyze the dynamical stability of the forced systems by computing Lyapunov exponents, Floquet multipliers, and bifurcation diagrams to determine conditions for periodic, stable, or chaotic behavior under vortex-induced vibrations.
- Evaluate the energy harvesting potential by applying Faraday's law to estimate induced voltage in coils, comparing the three proposed configurations to identify the most effective design for converting fluid motion into electrical energy.

CHAPTER 1

Mathematical Analysis Techniques

> Philosophy is written in this grand book, the universe, which stands continually open to our gaze, but it cannot be understood unless one first learns to comprehend the language and read the characters in which it is written. It is written in the language of mathematics, and its characters are triangles, circles, and other geometric figures.
>
> Galileo Galilei, *"The Assayer" ("Il Saggiatore")*, 1623

In this chapter, the various mathematical analyses required to study dynamic systems are presented. Primarily, it delves into the study of differential equations and their methods of resolution, especially the solution by asymptotic series. Additionally, it discusses the study of physical dimensions in engineering problems and how to address them to generate a dimensionless system, meaning applicable to all systems, as well as the use of analytical mechanics (Lagrangian and Hamiltonian mechanics) to understand and characterize the behavior of a system, particularly oscillating systems with forces different from those of Hooke, fluid systems, and those involving fluid-structure interaction.

1.1 Analysis of Differential Equations

Differential equations, as Adkins and Davidson [1] states, are a tool used in multiple problems in science and engineering. They are employed when a description of a measurable quantity is needed in terms of another independent measurable quantity; generally, the independent quantity is time. This can be described using Leibniz's notation, where the differential represents the rate of change of the dependent variable concerning the independent one, which corresponds to a value or function, mathematically expressed as:

$$\frac{df(x)}{dx} = g(x), \tag{1.1.1}$$

where $f(x)$ represents the measurable quantity whose behavior we want to understand (property), x is the independent quantity against which the behavior of $f(x)$ is contrasted (parameter), and $g(x)$ is the behavior of $f(x)$ as x changes. The differential equation is

called *ordinary* if our quantity to measure depends only on one independent quantity, and when it depends on more, it is called *partial* ([27]). An example of a partial differential equation is the following:

$$\frac{\partial f(x,y)}{\partial x} + \frac{\partial f(x,y)}{\partial y} = g(x,y), \tag{1.1.2}$$

where x and y are the parameters, $f(x,y)$ is the property whose description we want to know as x and y vary, and $g(x,y)$ is the known behavior of $f(x,y)$ when the parameters change.

Something important to note is that the properties ($f(x)$) can be expressed in vector form to encompass all the properties of the system in a simple expression. Therefore, the rate of change ($g(x)$) can also be a vector.

$$\frac{d\vec{f}(x)}{dx} = \vec{g}(x), \tag{1.1.3}$$

where $\vec{f}(x)$ is the vector of n parameters that depend on x, and $\vec{g}(x)$ is the vector of the rates of change of those parameters.

For the nature of differential equations, when a differential equation cannot be solved analytically, we can use certain techniques to analyze the system's behavior and, with that, build an approximate solution.

Phase line

When we are working with differential equations, the easiest way to know the behavior of a differential equation is by plotting it, which helps us to find the critical points that can be or not equilibrium points.

The phase line technique is the simplest analysis and, in essence, consists of finding the critical points and in which direction a point in the function will move.

As we know, the critical points of a function are found using the first derivative criteria. In a math form, we have:

$$\frac{df(x)}{dx} = g(x) \quad \text{if} \quad g(x_0) = 0, \tag{1.1.4}$$

where $f(x)$ is the function that we want to understand, x is the independent variable, $g(x)$ is the first derivative of the $f(x)$ function and x_0 is a critical point of the $f(x)$ function. The number of critical points is proportional to the grade of the $g(x)$ function. This can be expressed as.

$$\forall\ g(x) = \sum_{i=0}^{n} a_i x^i : g(x) = 0 \implies \exists\ n \text{ roots or critical points.} \tag{1.1.5}$$

Now that we know the fundamental idea, we can introduce the function of this technique. The phase line consists of putting in a line, as the number line, the critical points of a function, with the intention where the equilibrium points can be. Once we know the equilibrium points of our function, we need to know if they are stable or unstable.

The stability of a point can easily be understood. If a neighbor point of a critical point is going away when the time increases, then the critical point is unstable and is called ***asymptotically unstable***. On the other hand, when the neighbor point is attracted to the critical point, we said that is ***asymptotically stable***.

To understand this technique better, we are going to do an example.

Example

The function that we are going to test is $f(x) = x - 2x^2 - 3x^3 + x^4 + x^{21} + 0.2$, and the grafic of this function is

Phase plane

This analysis is the dimension extension of the phase line analysis because in the phase plane, we can see the inflection points and, more importantly, the behavior of the system. So, we can determine with this analysis whether a system follows a cyclic movement.

For this technique, we require a couple of sets of equations. The equations must have been independent but with some relation between them. In the case of physics, the relation is the energy, the force, and other fundamental physics definitions.

Nullclines

Nullclines are curves in the phase plane that help us to divide the space into different regions that have different behaviors. This technique, as Witelski and Bowen ([44]) highlight, does not provide the solution of the system but lets us know how the trajectories act when they pass through the nullclines curves.

To work with nullclines, we need to start considering that the derivates of our function, at least at a point, are zero, and we are trying to find those points. This can be expressed as

$$\frac{df(x,y)}{dx} = g(x,y) = 0 \ \wedge \ \frac{df(x,y)}{dy} = h(x,y) = 0, \tag{1.1.6}$$

where $f(x,y)$ is our function that we want to analize, x and y are the parameters that affect our function, $g(x,y)$ and $h(x,y)$ are the derivative functios of $f(x,y)$ by each independet variable. That $g(x,y)$ and $h(x,y)$ are equal to zero is so important because in the points where these functions are zero, the trajectories change.

Then, we are going to analyze what nullclines mean. We will begin with the x analysis. $\frac{df(x,y)}{dx} = 0$ is called an x-nullcline and means that the function is completely vertical (because the function does not change in the x direction). For analogy, the analysis of y is the y-nullcline and is when the function has a purely horizontal behavior, or in the math form $\frac{df(x,y)}{dy} = 0$.

The direction of the trajectories can be known with the functions $g(x,y)$ and $h(x,y)$, these functions let us know what the trajectories are outside the nullclines points. The nullcline points are sometimes called "equilibrium points".

Non analitical solution

As Rogan C. and Muñoz G. mentions, in physics, understanding the nature of a force leads us to a motion equation, which in turn derives into a differential equation, which is

generally a partial differential equation. Furthermore, we know from what Thornton and Marion has stated that the motion equation derived from forces or the Lagrangian is a second-order differential equation.

One of the challenges with differential equations, both ordinary and partial, is that often the nature of the problem described mathematically is too complex to be solved analytically ([23]), which is why they are usually solved numerically or through some type of approximation.

Limit cycles

As we mentioned previously, differential equations have the property of modeling the world, and it is important to know what their behavior will be. Some dynamical systems present oscillations that do not depend on the initial conditions, but these oscillations are related to the nature of the phenomena; in other words, some systems begin to oscillate due to an intrinsic property. These behaviors can be shown in the phase plane as a closed trajectory, and they are called ***Limit Cycles.***

Something that is important to highlight is that limit cycles are isolated closed trajectories. Unlike the centers found in linear systems (which come in continuous families), a limit cycle stands alone—nearby trajectories are either attracted to it or repelled from it, but they are not themselves closed orbits. This isolation is a defining characteristic of limit cycles.

Depending on the behavior of nearby trajectories, limit cycles are classified as:

- **Stable (attracting):** All nearby trajectories approach the limit cycle as $t \to +\infty$. These model self-sustained oscillations in real-world systems, such as the beating of a heart or the vibrations of a bridge in strong winds.
- **Unstable (repelling):** Nearby trajectories move away from the limit cycle.
- **Semi-stable:** Attracting from one side and repelling from the other.

The presence of a limit cycle is a hallmark of nonlinearity—linear systems cannot produce isolated periodic orbits. Their study is therefore essential for understanding any phenomenon that exhibits persistent, self-generated oscillations.

Detecting Limit Cycles

Proving the existence and uniqueness of limit cycles is one of the most challenging problems in dynamical systems (it is related to Hilbert's 16th problem). However, several powerful tools exist:

1. **Poincaré–Bendixson Theorem:** This is the most important existence theorem for limit cycles in the plane. It states that if a trajectory is confined to a closed, bounded region of the phase plane that contains no fixed points, then the trajectory must itself approach a closed orbit (a limit cycle). To apply it, one must construct a **trapping region**—an annular region where the vector field points inward on the outer boundary and outward on the inner boundary (or vice versa)—and verify that the region contains no equilibria.
2. **Bendixson's Criterion:** This is a negative criterion. If, in a simply connected domain D, the divergence $\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}$ does not change sign and is not identically zero,

then the system has **no** closed orbits lying entirely in D. This is useful for ruling out limit cycles in specific regions.

3. **Dulac's Criterion:** A generalization of Bendixson's criterion. If there exists a function $B(x, y)$ (a Dulac function) such that $\frac{\partial(Bf)}{\partial x} + \frac{\partial(Bg)}{\partial y}$ has a constant sign in a simply connected region, then no closed orbit can exist entirely in that region.

The analysis of limit cycles is crucial for understanding systems that exhibit sustained oscillations, which can be of any nature, biological, chemical, or otherwise.

Poincaré map

The phase plane, as we have seen, shows us a continuous picture of the dynamics of a system, but it can be complex to analyze particularly if the system presents periodic orbits. The **Poincaré map** is a mathematical technique that reduces the study of a continuous-time system to the study of a discrete-time map.

The construction of a Poincaré map is as follows:

1. **Define a Surface:** In the phase space, we place a surface Σ of codimension one (a line in the 2D plane, a plane in 3D space) that is transverse to the flow of the dynamical system. This surface is called a **Poincaré section**.
2. **Intersect and Record:** For a given periodic orbit, we start at a point x_k on Σ and follow the trajectory until it next intersects Σ. This new point of intersection is called x_{k+1}.
3. **Define the Map:** The Poincaré map $P : \Sigma \to \Sigma$ is defined by the rule $x_{k+1} = P(x_k)$.

This process is illustrated schematically in Figure 1.1 (not shown here). Instead of analyzing the continuous flow, we analyze the discrete sequence of points $\{x_0, x_1, x_2, \dots\}$.

Advantages of the Poincaré Map

- **Dimensionality Reduction:** The study of a continuous n-dimensional flow is reduced to the study of an $(n-1)$-dimensional discrete map. For a limit cycle in the plane (2D), the Poincaré section is a line (1D), making analysis trivial.
- **Stability of Periodic Orbits:** A limit cycle in the continuous system corresponds to a **fixed point** of the Poincaré map. The stability of the limit cycle is directly related to the derivative of the map at that fixed point.

  $$\text{If } |P'(x^*)| < 1 \implies \text{fixed point } x^* \text{ is attracting} \implies \text{limit cycle is stable.}$$
  $$\text{If } |P'(x^*)| > 1 \implies \text{fixed point is repelling} \implies \text{limit cycle is unstable.}$$

- **Detecting Chaos:** For more complex systems (three dimensions or more), the Poincaré map can reveal chaotic behavior. A simple closed curve on the Poincaré section suggests a quasiperiodic orbit (a torus), while a dense set of points with fractal structure is a hallmark of a chaotic attractor.

Lyapunov exponents

The concepts of stability discussed so far (stable/unstable fixed points and limit cycles) are qualitative. **Lyapunov exponents** provide a *quantitative* measure of the sensitivity

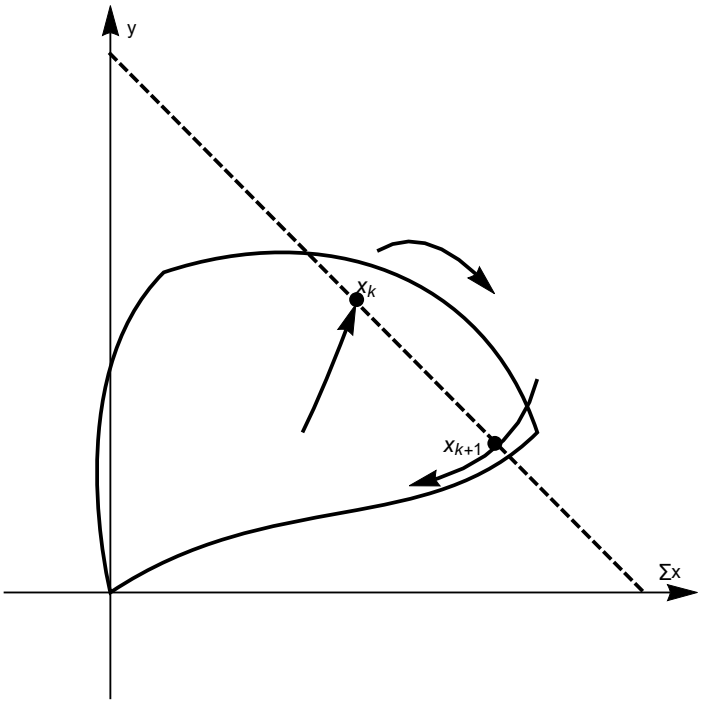


Figure 1.1: Schematic illustration of a Poincaré map. A limit cycle (closed curve) intersects a transverse section Σ at points x_k and x_{k+1}. The flow maps x_k to x_{k+1} after one period.

of a dynamical system to initial conditions. They characterize the rate at which nearby trajectories in phase space diverge or converge.

Imagine two trajectories in phase space starting very close together, separated by an infinitesimal distance $\delta\mathbf{x}(0)$. As the system evolves, this separation will change. For a chaotic system, it will grow exponentially. The (maximal) Lyapunov exponent λ quantifies this growth:

$$\|\delta\mathbf{x}(t)\| \approx e^{\lambda t}\|\delta\mathbf{x}(0)\|. \tag{1.1.7}$$

In an n-dimensional system, there is a spectrum of n Lyapunov exponents, $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$, each describing the average rate of expansion or contraction along a particular direction in phase space.

Interpreting the Lyapunov Spectrum

The signs of the Lyapunov exponents are of paramount importance for classifying the nature of the attractor:

- **Fixed Point (Equilibrium):** All Lyapunov exponents are negative ($\lambda_i < 0$). Perturbations decay in all directions.
- **Limit Cycle (Periodic Orbit):** One Lyapunov exponent is zero ($\lambda_1 = 0$), corresponding to the direction along the flow. All others are negative ($\lambda_{i>1} < 0$).

- **Quasiperiodic Orbit (Torus):** For an m-frequency quasiperiodic orbit, the first m exponents are zero. The rest are negative.
- **Chaotic Attractor:** At least one Lyapunov exponent is positive ($\lambda_1 > 0$). This indicates exponential divergence of nearby trajectories, which is the hallmark of chaos. The sum of all exponents is negative, meaning the dynamics are dissipative and converge to a bounded attractor.

A positive Lyapunov exponent means that even a tiny uncertainty in the initial state will grow rapidly, making long-term prediction impossible—a phenomenon known as the **butterfly effect**.

Numerical Calculation

For most nonlinear systems, Lyapunov exponents cannot be calculated analytically. Then, they are computed numerically by integrating not only the equations of motion for a trajectory but also the **linearized equations of motion** for a set of perturbation vectors.

In conclusion, Lyapunov exponents are the most fundamental diagnostic for chaos. They transform the qualitative idea of "sensitive dependence" into a measurable, calculable quantity that is essential for the rigorous study of nonlinear dynamics.

1.2 Asymptotic or Perturbative Analysis

As presented in the previous section, there is a wide range of differential equations that do not have an analytical solution. For this reason, unsolvable equations are addressed using numerical methods or approximation methods. The most common approximation method for solving differential equations is to generate a solution in the form of a series.

This section introduces asymptotic and perturbative approximations. Furthermore, it presents how these techniques are used for solving equations, particularly differential equations, whether analytically solvable or unsolvable.

1.2.1 The Importance of Asymptotic Approximation

In physics, as in mathematics, there are a limited number of equations (both conventional and differential) that have a solution. To solve problems that lack an exact solution, various techniques are available, such as numerical solutions or approximations. It is in these latter methods that an approximation emerges, namely asymptotic approximation, which involves approximating a set of functions to the real solution.

The main idea of asymptotic approximation is to transition from an expression with the equality sign ($=$) to the asymptotic or similar sign ($\sim$). This is because the equality sign imposes significant restrictions on the possible values obtained, while the asymptotic symbol allows for more flexibility in deriving solutions, permitting the solutions to improve their approximation as the asymptotic function becomes more similar to the original function. This is expressed as follows:

$$f(x) = g(x) \rightarrow f(x) \sim g(x), \tag{1.2.1}$$

where f and g are two functions of x, transitioning from equality between the functions to an approximation of similarity, with the aim of simplifying the treatment between both

functions.

It is important to note that alongside the use of the asymptotic symbol, when using asymptotic approximation, the symbol ϵ is also employed, which represents the perturbative term of the function (that which contains relevant information about the system), with the intention of dividing complex functions into simpler ones for easier manipulation and approximation. Thus, asymptotic approximation is closely related to the perturbative method.

1.2.2 Perturbative Method

The perturbative method consists of decomposing a complex problem into a series of simpler problems; for this purpose, the symbol ϵ is used to represent the perturbative term in the function to be approximated.

This analysis consists of balancing the contribution of any term of a polynomial that is approximated to the solution. The balancing is done for a control parameter ϵ, which tell us how big the contribution of the term in the polynomial to the solution [19].

An example of a perturbation series is:

$$x \approx x_0 + x_1\epsilon + x_2\epsilon^2 + \cdots = \sum_{n=0}^{N} x_n\epsilon^n, \tag{1.2.2}$$

where ϵ is the perturbation term and it helps us fit the polynomials to the solution. The approximation improves with more terms in the perturbation's polynomial [20].

We can estimate the derivation of our perturbation series; we only use the derivation of a sum of functions. The first and second derivatives of the approximation series can be shown as...

$$\frac{d}{dt}\left(\sum_{n=0}^{N} x_n\epsilon^n\right) = \sum_{n=0}^{N} \epsilon^n \frac{dx_n}{dt} \quad \text{and} \quad \frac{d^2}{dt^2}\left(\sum_{n=0}^{N} x_n\epsilon^n\right) = \sum_{n=0}^{N} \epsilon^n \frac{d^2x_n}{dt^2} \tag{1.2.3}$$

To use the perturbative method with differential equations, we only need to apply the differentiation to the approximation series and introduce that expression into the differential equation. After that, we need to balance the contribution of every grade of ϵ in terms of the function that we need to approximate, the initial and boundary conditions.

1.2.3 The Radius of Convergence

When dealing with series approximations, it is always necessary to find the radius of convergence of the series, and the perturbative method is no exception. For this reason, it is important to know how to find the radius of convergence.

It is important to note that the radius of convergence of the perturbative method will grow or shrink depending on the term of the function that contains the perturbation.

1.3 Dimensional Analysis

Units are so important in our lives because with them we can interact with the Universe and we are able to generate precise descriptions of natural phenomena.

Units appear in a natural form in our mind when we are doing a measurement or analyzing phenomena, because we need some constant things to be able to compare our world and the Universe with those constants ([30]). Units are created for humans and respond to the necessities of every civilization, which is the reason why there have been a lot of different units a long of human history for the same phenomenon, like the mile and the meter for the distance or the lunar and solar calendars to measure a year.

Over the years, humanity has learned to abstract its ideas and to write them in a mathematical language. So, we can express a measure as the number of times a unit is used, or in mathematical form.

$$\mathbf{A} = n\mathbf{a}, \tag{1.3.1}$$

where $\mathbf{A}$ is the measure that we do, n is the number of times that we use the unit, and $\mathbf{a}$ is the unit. But the problem arises when we remember that every civilization has its particular unit system, so how can we contrast the results of measurement between some civilizations? The answer to this question has three different solutions, which are:

1. Create a world unit system (as the International System of Units), but as we know, not all countries use this unit system.
2. Knowing the conversion factors of every unit in any system of units, as we use to convert the temperature or distance between the International System and the English Units.
3. Taking advantage of math and dimensionless measure, because the numbers are universal and do not change from country to country, describing the nature in terms of some numbers, as the Reynolds' number.

Dimensional analysis is based on the third option, so in this technique, we are going to describe all the dynamics of a physical system in terms of a set of numbers without dimensions. The idea of this analysis is to generate a method to reduce the complexity of physical systems. The main idea of this analysis is the concept of ***similarity***.

Similarity consists in finding the equivalence between two different systems ([34]), but in math, this concept consists of applying a transformation on the variables to reduce the number of independent variables in the system.

To work with this analysis, we need to follow a set of steps that Sonin said are four steps. These steps are:

1. The independent variables: This step is the most important and consists of identifying the complete set of independent variables of the system. In other words, we need to know how many dimensions act in the evolution of our physical system.
2. Dimensional considerations: We need to know the dimensions of every variable in our system. An example is the dimensions of the gravitational acceleration, in this example its dimensions are: $[g] = LT^{-2}$.
3. Dimensionless variables: In this step, we are going to define the dependent variables in terms of the independent variables with the intention of obtaining a set of dimensionless variables.
4. Buckingham's Π-theorem: We apply the Buckingham's Π-theorem. This theorem tells us: That when we dimensionless the physical quantities, the parameters that

really affect the dynamics of the system are reduced and describe all the systems with no importance of the unit system that we want to use.

Example: The energy of the atomic bomb

The most iconic example of the application of the ***Dimensional analysis*** was when a British scientist (G. I. Taylor) based on a set of photos about the Trinity test, he could estimate the energy of the atomic bomb because the set of photos provided the diameter of the explosion and time lapse.

So, recreating Taylor's work using the steps of the dimensional analysis.

Step 1: Independent variables

The elements that act in this problem are: the density of air (ρ), the radius of the explosion, and the time lapse, and we are searching for the energy that the bomb had. So we have four variables, but energy is a dependent variable; on the other hand, the time, the radius, and the density are independent. To express this in terms of math, we have:

- $j = 4$, where j is the number of variables present in the system.
- $n = 3$, where n is the number of independent dimensions (L, T, M). If we had another dimension, as temperature or substance, we would add it to the independent dimensions.
- $k = j - n = 1$, where k is the number of dimensionless groups that this system has.

Step 2: Dimensional considerations

Now, we are going to find the dimension configuration of every variable that we have. These are those configurations:

- Time: $[t] = T$
- Radius: $[r] = L$
- Density: $[\rho] = \frac{M}{L^3} = ML^{-3}$
- Energy: $[E] = \frac{ML^2}{T^2} = ML^2T^{-2}$

Step 3: Dimensionless variables

Once we have the dimensions of every variable, we select some variables, the same number of independent dimensions present in the system, to create our dimension set (we can choose with liberty wherever variables, but in this example we are looking for the energy; then we do not select as base). So, we have:

$$\Pi_i = f(\rho, t, r, E), \tag{1.3.2}$$

where Π is the dimesionless group. The previous expression tells us that the dimensionless group is a function form by the variables. To generate the dimensionless variables, we use the base variables, and we will assume that every variable can be expressed as the product of the powers of our base variables. This can be shown in the followin equation.

$$v = x^{\alpha} y^{\beta} z^{\delta}, \tag{1.3.3}$$

where v represent the variable, x, y and z are the base variables and α, β and δ are the power of each base variable. To find the values of the powers for every variable, we need to generate an equivalence dimension unit between our variable and our base variables. In other words.

$$[v] = M^{a} L^{b} T^{c} = [x]^{\alpha} [y]^{\beta} [z]^{\delta}, \tag{1.3.4}$$

this equation generates a set of equations that help us to know the configuration for every variable. In our example, we have:

$$[v] = [r]^{\alpha} [\rho]^{\beta} [t]^{\delta} \implies [v] = (L)^{\alpha} \left(ML^{-3}\right)^{\beta} (T)^{\delta} \implies [v] = L^{\alpha - 3\beta} M^{\beta} T^{\delta} \tag{1.3.5}$$

Then, every variable in terms of the base variables and its dimensionless form is:

- Time: The powers to generate the time variable are...

$$\alpha = 0, \beta = 0, \delta = 1 \quad \therefore \quad t = t \implies \tilde{t} = 1$$

- Radius: The powers to generate the radius variable are...

$$\alpha = 1, \beta = 0, \delta = 0 \quad \therefore \quad r = r \implies \tilde{r} = 1$$

- Density: The powers to generate the density variable are...

$$\alpha = 0, \beta = 1, \delta = 0 \quad \therefore \quad \rho = \rho \implies \tilde{\rho} = 1$$

- Energy: The powers to generate the energy variable are...

$$\alpha = 5, \beta = 1, \delta = -2 \quad \therefore \quad E = \frac{\rho r^5}{t^2} \implies \tilde{E} = \frac{E t^2}{\rho r^5}$$

Step 4: Buckingham's Π-theorem

Now that we know the dimensionless form of the energy, we can find the energy expression. To do this, we need to remember that the Π group is dimensionless, then it is equal to some constant. So, we can find:

$$\Pi = f(r, t, \rho, E) = \tilde{E} = c \implies \frac{E t^2}{\rho r^5} = c \quad \therefore \quad E = \frac{c \rho r^5}{t^2}. \tag{1.3.6}$$

This is the result that Taylor obtained, and to find the value of the energy, we only need to substitute the values that appear in the photo set. Taylor estimated the energy of the atomic bomb in 10.4 megatons of TNT, and this result was so close to the exact energy that the US army reported.

Conclusion

Dimensional analysis has the property of giving us functions that describe a phenomenon that we do not know more about than the variables that act on it.

The most important thing about dimensional analysis is that we can express all the dynamics of a phenomenon in a general way, without the necessity of indicating the unit system that we use.

1.4 Lagrangian Analysis

The main issue with Newton's laws, as mentioned by Thornton and Marion [37], is that they are written in vector form, causing many problems to acquire a certain degree of complexity and making their manipulation more laborious. For this reason, it is necessary to use an analysis in which the dynamic system retains all its properties but is not expressed in vector form. This is where Joseph-Louis Lagrange made a significant contribution with his work ***Mécanique analytique***; where he defined a function in terms of coordinates (q) and speeds ($\dot{q}$). This can be expressed as.

$$L(q_1, q_2, \ldots, q_n, \dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n), \tag{1.4.1}$$

where q_1, q_2, ..., and q_n represent the n coordinates that characterize the system; $\dot{q}_1$, $\dot{q}_2$, ..., and $\dot{q}_n$ are the speeds of the n coordinates, and L is the function that describes the system and is called the ***Lagrangian*** of the system.

According to Landau and Lifshits [21], the Lagrangian characterizes the dynamics of a physical system if and only if the Lagrangian minimizes the action, or in other words, we apply the ***principle of least action***. Mathematically, it implies that the action (S), which is expressed in the following integral, takes the minimum possible value.

$$S = \int_{t_1}^{t_2} L(q, \dot{q}, t)dt. \tag{1.4.2}$$

The Lagrangian function is defined as the sum of the kinetic energies minus the potential energies of all the particles in the system.

$$L = T - V \implies L = \sum_{i=1}^{n} \frac{1}{2} m_i \dot{q}_i^2 - V(q_1, q_2, \ldots, q_i) \tag{1.4.3}$$

If we use D'Alembert's principle and the calculus of variations, we can find a set of second-order differential equations Soldovieri C., T. ([33]). This differential equation is known as the ***Euler-Lagrange*** equation and is expressed as follows:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = Q_i, \tag{1.4.4}$$

where q_i represents the i-th coordinate, $\dot{q}_i$ the i-th speed, and Q_i the external force or geometric restriction in the i-th direction. This differential equation is the equation of motion for the n-th particle in the i-th direction. It is important to highlight that the complete system has $n \times i$ equations of motion that we need to solve to know all the

system's behavior. For example, for a particle that moves in the three directions (x, y, z), we have to solve three (1×3) equations of motion.

The relevance of the Lagrangian analysis consists in, if we substitute the definition of the Lagrangian in the Euler-Lagrange equation and consider a simple system (without external forces and geometric restrictions), we can see, as it is shown in the following equation, that Euler-Lagrange equations are equivalent to Newton's second law ([21],[33],[37]).

$$\begin{aligned}
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0 &\implies \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \frac{\partial L}{\partial q_i} \\
\frac{d}{dt}\left(\frac{\partial (T-V)}{\partial \dot{q}_i}\right) = \frac{\partial (T-V)}{\partial q_i} &\implies \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) = \frac{\partial V}{\partial q_i} \\
\frac{d}{dt}\left(\frac{\partial}{\partial \dot{q}_i}\left(\frac{1}{2} m_i \dot{q}_i^2\right)\right) = \frac{\partial V}{\partial q_i} &\implies m_i \frac{d}{dt}\left(\dot{q}_i\right) = \frac{\partial V}{\partial q_i} \\
m_i \ddot{q}_i = \frac{\partial V}{\partial q_i} &\implies m_i a_i = -F_i,
\end{aligned} \tag{1.4.5}$$

1.5 Hamiltonian Analysis

Another important analysis in physics is the ***Hamiltonian analysis***. This analysis consists of analyzing the movement in terms of position (coordinates) and momentum, these are called ***canonical variables***. That is the principal difference between the Lagrangian and Hamiltonian analyses. To change the variables in the Lagrangian to the canonical variables, we have the following relation.

$$p_i = \frac{\partial L(q_i, \dot{q}_i, t)}{\partial \dot{q}_i} \implies p_i = \sum_{i=1}^{n} m_i \dot{q}_i - \frac{\partial V(q_1, q_2, \ldots, q_n)}{\partial \dot{q}_i} \quad \therefore \quad p_i = \sum_{i=1}^{n} m_i \dot{q}_i \tag{1.5.1}$$

The Hamiltonian (H) is defined as the Legendre transform of the Lagrangian ([33]), so the Hamiltonian has the following expression.

$$H(q_i, p_i, t) = \sum_{i=1}^{n} \dot{q}_i p_i - L(q_i, \dot{q}_i, t), \tag{1.5.2}$$

this is the general form of the Hamiltonian, without relevance to whether the system is conservative or not. For a conservative system, the Hamiltonian can be expressed as.

$$
\begin{aligned}
H =& \sum_{i=1}^{n} \dot{q}_i p_i - L(q_i, \dot{q}_i, t) \\
=& \sum_{i=1}^{n} \dot{q}_i p_i - \sum_{i=1}^{n} \frac{1}{2} m_i \dot{q}_i^2 + V(q_1, q_2, \ldots, q_i) \\
=& \sum_{i=1}^{n} \dot{q}_i p_i - \frac{1}{2} \sum_{i=1}^{n} \underbrace{m_i \dot{q}_i}_{p_i} \dot{q}_i + V(q_1, q_2, \ldots, q_i) \\
=& \sum_{i=1}^{n} \dot{q}_i p_i \left(1 - \frac{1}{2}\right) + V(q_1, q_2, \ldots, q_i) \\
=& \sum_{i=1}^{n} \frac{1}{2} m_i \dot{q}_i^2 + V(q_1, q_2, \ldots, q_i) \\
\therefore \quad & \boxed{H = T + V};.
\end{aligned}
\tag{1.5.3}
$$

If we apply the principle of least action:

$$
\begin{aligned}
S =& \int_{t_1}^{t_2} L(q, \dot{q}, t) dt = \int_{t_1}^{t_2} \dot{q}p - H(q, p, t) dt \\
\delta S =& 0 = \delta \int_{t_1}^{t_2} \dot{q}p - H(q, p, t) dt \\
0 =& \int_{t_1}^{t_2} \left(\dot{q}\delta p + p\delta\dot{q} - \frac{\partial H}{\partial p}\delta p - \frac{\partial H}{\partial q}\delta q \right) dt \\
0 =& \int_{t_1}^{t_2} \left(\left\{ \dot{q} - \frac{\partial H}{\partial p} \right\} \delta p - \frac{\partial H}{\partial q}\delta q + \underbrace{p\delta\dot{q}}_{-\dot{p}\delta q} \right) dt \\
0 =& \int_{t_1}^{t_2} \left(\left\{ \dot{q} - \frac{\partial H}{\partial p} \right\} \delta p - \left\{ \frac{\partial H}{\partial q} + \dot{p} \right\} \delta q \right) dt,
\end{aligned}
\tag{1.5.4}
$$

to be real, the two parts of the integral need to be zero. Then we obtain two first-order differential equations, that are called ***Hamilton's equations***. These equations are:

$$
\dot{q} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial q}. \tag{1.5.5}
$$

These equations are so relevant, because they are obtained by the principle of least action, like Euler-Lagrange equation, ergo they have the property to describe the dynamics of the system, with the difference that Hamilton's equations are two first-order differential equations. If we expand the number of particles to n and coordinates to i, then we need to solve $2 \times n \times i$ differential equations of first order. This implies that if we use the Hamiltonian analysis, we need to solve the double of equations as in the Lagrangian analysis, but their equations are simpler than in the Lagrangian analysis.

As we have a set of equations, we can put them together in a vector.

$$\begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} \frac{\partial H}{\partial p} \\ -\frac{\partial H}{\partial q} \end{pmatrix}, \tag{1.5.6}$$

this vector represents how a particle changes its position and its momentum over time. Then, the vector has the property to describe all the dynamics presented in the system. For this reason, the Hamiltonian analysis can be plotted and lets us know the behaviour of the system, this is because it uses the phase plane.

1.6 Numerical Solutions of the Equations of Motion

In the previous sections, the system was analyzed using analytical approaches based on the Lagrangian and Hamiltonian formulations. However, due to the inherent nonlinearities present in the potential terms, the resulting equations of motion do not generally admit analytical solutions. To study the temporal evolution of the system, it becomes necessary to employ numerical methods that approximate the solution through discrete time integration.

Numerical integration methods convert the continuous problem into an iterative process, where the state of the system is updated at small time intervals. These algorithms are essential in nonlinear dynamics, as they allow the exploration of regions of phase space that are analytically inaccessible.

Among the most common numerical schemes are:

- **Euler's method**: the simplest explicit scheme, conceptually straightforward but limited in both accuracy and stability.
- **Runge–Kutta methods**: a family of higher-order explicit algorithms that achieve better accuracy while maintaining reasonable computational cost.
- **Implicit methods**: typically used for stiff equations, where explicit methods become unstable.

In what follows, we briefly describe the fourth-order Runge–Kutta method, which is used in this work to integrate the equations of motion corresponding to the nonlinear magnetic pendulum.

1.6.1 The Fourth-Order Runge–Kutta Method

Consider a general first-order ordinary differential equation:

$$\frac{dy}{dt} = f(t, y), \qquad y(t_0) = y_0. \tag{1.6.1}$$

The Runge–Kutta (RK) methods estimate the value of $y(t)$ at a later time by evaluating f at several intermediate points within each time step. The fourth-order Runge–Kutta

method (RK4), one of the most widely used, updates the solution as:

$$\begin{aligned} k_1 &= f(t_n, y_n), \\ k_2 &= f\left(t_n + \frac{h}{2}, y_n + \frac{h}{2}k_1\right), \\ k_3 &= f\left(t_n + \frac{h}{2}, y_n + \frac{h}{2}k_2\right), \\ k_4 &= f\left(t_n + h, y_n + hk_3\right), \\ y_{n+1} &= y_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4), \end{aligned} \tag{1.6.2}$$

where h is the time step.

This scheme provides a local truncation error of order $\mathcal{O}(h^5)$ and a global error of order $\mathcal{O}(h^4)$, which means that halving the step size approximately reduces the overall error by a factor of 16. The method's robustness and precision make it ideal for integrating nonlinear oscillatory systems.

1.6.2 Illustrative Example: Harmonic Oscillator

To illustrate the method, consider the simple harmonic oscillator described by

$$\ddot{x} + \omega^2 x = 0. \tag{1.6.3}$$

This equation can be written as a system of first-order equations:

$$\begin{aligned} \dot{x} &= v, \\ \dot{v} &= -\omega^2 x. \end{aligned} \tag{1.6.4}$$

The RK4 algorithm can be directly applied to this system, yielding a numerical trajectory $x(t)$ that reproduces the analytical oscillatory motion

$$x(t) = A\cos(\omega t + \phi).$$

This example serves as a validation of the numerical implementation before applying the same procedure to the nonlinear magnetic pendulum analyzed in this chapter.

Implementation in Mathematica

```
 = 1;
eqs = {x'[t] == v[t], v'[t] == -^2 x[t]};
ics = {x[0] == 1, v[0] == 0};
sol = NDSolve[{eqs, ics}, {x, v}, {t, 0, 20}];
Plot[Evaluate[x[t] /. sol], {t, 0, 20},
 PlotLabel -> "Harmonic Oscillator (RK4)",
 AxesLabel -> {"t", "x(t)"}]
```

Implementation in Python

```
import numpy as np
import matplotlib.pyplot as plt

def f(t, y, =1):
    x, v = y
    return np.array([v, -**2 * x])

def rk4_step(f, t, y, h):
    k1 = f(t, y)
    k2 = f(t + h/2, y + h*k1/2)
    k3 = f(t + h/2, y + h*k2/2)
    k4 = f(t + h, y + h*k3)
    return y + h*(k1 + 2*k2 + 2*k3 + k4)/6

t0, tf, h = 0, 20, 0.01
t = np.arange(t0, tf, h)
y = np.zeros((len(t), 2))
y[0] = [1, 0]

for i in range(len(t)-1):
    y[i+1] = rk4_step(f, t[i], y[i], h)

plt.plot(t, y[:,0])
plt.xlabel('t'); plt.ylabel('x(t)')
plt.title('Harmonic Oscillator (Runge-Kutta 4)')
plt.show()
```

Both implementations yield results consistent with the analytical solution, demonstrating the accuracy and stability of the Runge–Kutta approach. Once validated, the same algorithm is employed to integrate the nonlinear equations of motion of the magnetic pendulum presented earlier in this chapter.

CHAPTER 2

Oscillation theory

> The same laws that govern the oscillations of a pendulum govern the oscillations of everything, from the motion of a planet to the vibrations of an atom.
> Richard Feynman, *The Feynman Lectures on Physics. Vol I*, 1964

The study of oscillations is important because they describe many phenomena in nature, such as the movement of a pendulum clock, sound, and more complex interactions like the Earth's movement around the Sun or the behavior of light.

An oscillation is defined as the cyclical motion that a particle follows around a point, called the equilibrium position (x_{eq} or x_0), which is generated by an initial perturbation. This can be appreciated by the following figure.

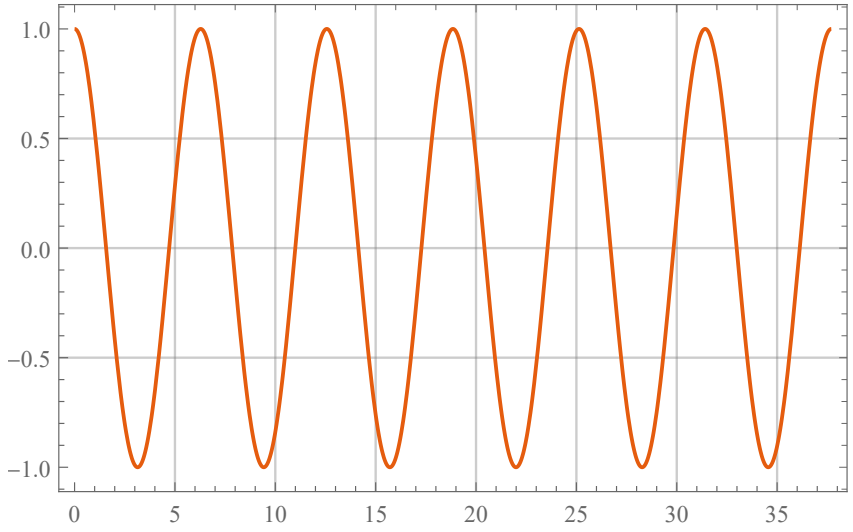


Figure 2.1: An oscillator system.

The oscillations can be categorized into one of three categories:

- ***Harmonic:*** This is the ideal oscillation, where we only consider the restoring force.

- ***Damped:*** This describes a system where we consider both the restoring force and the friction force.
- ***Forced:*** In this category, we consider all the previously mentioned forces, along with the existence of an external force that continuously perturbs the system.

2.1 Harmonic oscillation

A harmonic oscillation is an ideal system, which does not suffer from the effects of external forces or dissipative forces like friction. This means that the system can be described solely by knowing the driving force ($\vec{F}$) and the restoring force present in the system ($\vec{F}_R$).

Using Newton's third law, it can be seen that the sum of both forces must produce mechanical equilibrium, which implies that the restoring force must be opposite to the driving force, leading to the following equation:

$$\vec{F} = -\vec{F}_R \implies \vec{F} + \vec{F}_R = 0, \tag{2.1.1}$$

It is important to note that our equation is a vector equation, which implies that this relationship holds for each of the components of the vector system.

Converting our driving force into a differential equation gives:

$$m\frac{d^2\vec{r}}{dt^2} + \vec{F}_R(\vec{r}) = 0 \implies \sum_{i=1}^{3}\left(m\frac{d^2 r_i}{dt^2} + F_{R\,i}(r_i)\right)\hat{e}_i = 0 \implies \sum_{i=1}^{3}\left(\frac{d^2 r_i}{dt^2} + \frac{F_{R\,i}(r_i)}{m}\right)\hat{e}_i = 0, \tag{2.1.2}$$

where r_i and $F_{R\,i}$ are the i-th component of the position vector and the restoring force vector, respectively, and $\hat{e}_i$ is the direction of the i-th component of the vectors. Therefore, the behavior of the system depends entirely on the form of the components of the restoring force. It is important to highlight that there is no restriction on the restoring force requiring it to consist of a single element (a single force) or to be of the same nature, which implies that our restoring force can be a combination of forces, including mechanical, electrical, magnetic, etc. Thus, the restoring force can be expressed as:

$$F_{R\,i}(r_i) = \sum F_{mec\,i}(r_i) + \sum F_{elec\,i}(r_i) + \sum F_{mag\,i}(r_i) + \cdots \tag{2.1.3}$$

Therefore, it can be concluded that the differential equation, both its nature and whether it is solvable, depends on the expression of the restoring force.

To work with different kinds of restoring forces, we are going to analyze them according to their nature.

2.1.1 Mechanical restoring force

For the mechanical restoring force, we have three different forms of construction. The first involves applying Hooke's law, where the restoring force is applied by a device, like springs, that redirects the restoring force each time the oscillator passes the equilibrium point.

$$F = -k\Delta x, \tag{2.1.4}$$

where k is the restitution's constant and Δx is the displacement of the matter out of the equilibrium point.

The second form associates the force with a set of periodic forces, where the periodic forces have the equilibrium point where the periodic functions are null or where the sum of forces is zero. Then, the restoring force has the following form:

$$F = F_0 \sin(\theta) + F_1 \cos(\theta). \tag{2.1.5}$$

A clear example of this form of restoring force is the gravitational force applied to a pendulum. In that system, the force is expressed as:

$$F = T + F_g = T(\sin(\theta)\hat{e}_x - \cos(\theta)\hat{e}_y) - mg\hat{e}_y \tag{2.1.6}$$

The third form involves constructing a set of forces that change the direction of our oscillator at some distance, such as a set of springs against a wall or the interaction between two tennis players. These forces, which occur intermittently, can be expressed using the Heaviside function ($u(x)$).

$$F = F_1 u(x) + F_2 u(x - x_1). \tag{2.1.7}$$

2.1.2 Electrical restoring force

Before constructing an oscillatory electric system, we are going to recap the electric force interaction. The electric force between two charges is described by Coulomb's law:

$$\vec{F}_e(r) = \frac{k_e q_1 q_2}{r^2}\hat{r}, \tag{2.1.8}$$

where k_e is the proportionality constant, r is the distance between the two charges, q_1 and q_2 are the charges that interact. The electric force can be attractive or repulsive, and this depends on the charges that are interacting. If the charges have different polarities (+- or -+), they generate an attractive force; in the other size, when the charges have the same polarities (++ or –), they generate a repulsive force.

Now that we remember how the electric force acts, we can build an electric system with the property to cause a charge to oscillate in a region of the space. For this, we need three charges: two fixed charges (Q_1 and Q_2) and one mobile (q). To cause the mobile charge has an oscillatory movement, we need it to be between the two fixed charges; and the distance between the fixed charges is d. When q moves to another charge the repulsive force between the charge and q increases, pushing q to r_{eq}, where r_{eq} is the equilibrium point between the repulsive forces $F_{Q_1 q}$ and $F_{Q_2 q}$. This can be visualized by the following scheme.

Then, the electrical restoring force can be expressed as:

$$\vec{F}_R = \vec{F}_{Q_1 q} + \vec{F}_{Q_2 q} = \frac{k_e Q_1 q}{r_1^2}\hat{r} - \frac{k_e Q_2 q}{r_2^2}\hat{r} \implies \vec{F}_R = k_e q\left[\frac{Q_1}{r_1^2} - \frac{Q_2}{r_2^2}\right]\hat{r}. \tag{2.1.9}$$

If $Q_1 = Q_2 = Q$ and the coordinate origin is in one of the fixed charges (in Q_1), we can express the forces in terms of r, where r is the distance of q to the origin. With the previous considerations, our restoring force can be reexpressed as:

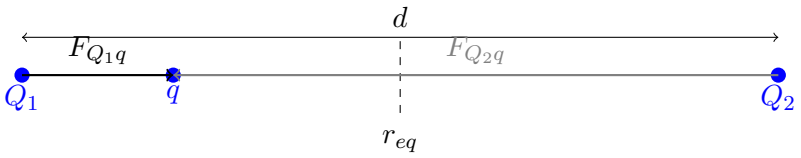


Figure 2.2: Representation of an electric oscillator system, where r_{eq} is the equilibrium point between the repulsive forces F_{Q_1q} and F_{Q_2q}. In our scheme, the intensity of the forces can be visualized in terms of the color intensity, where black is more intense than gray.

$$\begin{aligned} \vec{F}_R &= k_e qQ \left[\frac{1}{r^2} - \frac{1}{(d-r)^2}\right]\hat{r} = k_e qQ \left[\frac{(d-r)^2 - r^2}{r^2(d-r)^2}\right]\hat{r} \\ \implies \vec{F}_R &= k_e qQ \left[\frac{d^2 - 2dr + r^2 - r^2}{r^2(d-r)^2}\right]\hat{r} = k_e qQ \left[\frac{d^2 - 2dr}{r^2(d-r)^2}\right]\hat{r} \\ \therefore \vec{F}_R &= k_e qQd \left[\frac{d-2r}{r^2(d-r)^2}\right]\hat{r}. \end{aligned} \tag{2.1.10}$$

An analog form to attack this phenomenon is introducing the idea of electric fields. The electric field is defined as $\vec{E}(r) = \frac{k_e Q}{r^2}\hat{r}$, so the force in terms of the electric fiel is $\vec{F} = q\vec{E}(r)$. Applying the electric fields in our equation:

$$\vec{F}_R = q\vec{E}_1(r_1) + q\vec{E}_2(r_2) \implies \vec{F}_R = q\left[\vec{E}_1(r) + \vec{E}_2(d-r)\right] \implies \vec{F}_R = q\left[E_1(r) - E_2(d-r)\right]\hat{r}, \tag{2.1.11}$$

where E_1 and E_2 are the electric fields of the fixed charges.

Another way to make an oscillator system with electrical restoring force is when two charges with the same polarity cause a repulsive electric force and a force that acts over the mobile charge.

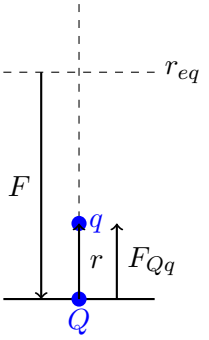


Figure 2.3: Representation of the second electric oscillator system.

Then, the mathematical expression of the system is:

$$\vec{F}_R = \vec{F}_{Qq} + \vec{F} \implies \vec{F}_R = \left(\frac{k_e qQ}{r^2} - F\right)\hat{r}. \tag{2.1.12}$$

We can observe that the repulsive force domains when the distance (r) between the two charges is small, but when the distance is increasing, the attractive force starts to obtain the principal role in the movement. If we use the electric field, our restoring force is.

$$\vec{F}_R = q\vec{E} + \vec{F} \implies \vec{F}_R = (qE(r) - F)\,\hat{r}, \tag{2.1.13}$$

from this last expression, we can see that the direction and intensity of the restoring force ($\vec{F}_R$) depends on r.

2.1.3 Magnetic restoring force

The magnetic case is similar to the electric case, except that we are going to work with magnetic interactions.

In contrast with the electric force, the magnetic force does not have a specific form. The behavior of the magnetic force depends on how the magnetic field is originated. If our magnetic field is generated by an electric current or a charge that is moving, then our magnetic force follows the magnetic part of the Lorentz force.

$$\vec{F} = q\vec{v} \times \vec{B}, \tag{2.1.14}$$

where $\vec{v}$ is the velocity of the charge, $\vec{B}$ is the magnetic field. Something important to highlight is that the magnetic force is a force that is applied in a perpendicular direction to the movement and the magnetic field.

But if we have two permanent magnets, we can build a magnetic analogous to Coulomb's law. For this reason, this equation is sometimes called the magnetic charges force. So, the magnetic force for two permanent magnets is:

$$\vec{F}_M = \frac{\mu_0 m_1 m_2}{4\pi r^2}\hat{r}, \tag{2.1.15}$$

where μ_0 is the magnetic permeability of the free space, $m1$ and $m2$ are the magnetic charges and r is the distance between the two magnets. Something important to see is that if we want to consider the environment where the experiment is done, then we need to change the magnetic permeability of the free space (μ_0) to the magnetic permeability of the environment.

Another way to describe the force between magnets is by using the magnetic energy present in their interaction. Therefore, the equation that models the magnetic force is:

$$\vec{F}_M = \frac{M}{r}\hat{r}, \tag{2.1.16}$$

where M is the magnetic energy in the interaction. This method is useful when we do not know the value of the magnetic charges.

In the same way as the electrical restoring force, the magnetical restoring force has the same two ways to be built. The first is using the interaction between three magnets, two of which are fixed magnets, and the other has the property of moving in the region between the fixed magnets.

The main difference between the magnetic system and the electric system is that magnets have at least two poles, so we need to be cautious with their poles to generate a double

repulsive force in the region. This specific magnets configuration can be shown in the following figure.

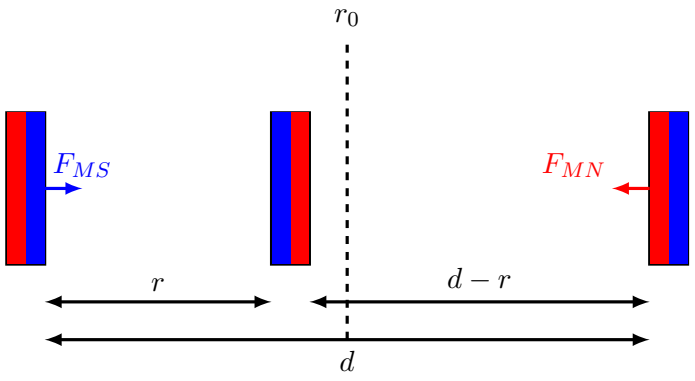


Figure 2.4: Graphic representation of a magnetic system with three permanent magnets that use the repulsive force between magnets to create an oscillatory system.

Expressing the configuration system present in the previous figure (figure 2.4) in an equation.

$$\vec{F}_R = \vec{F}_{MS} + \vec{F}_{MN} \implies \vec{F}_R = \left[\frac{M_S}{r} - \frac{M_N}{d-r}\right]\hat{r}, \tag{2.1.17}$$

where M_S is the magnetic energy present in the interaction by the south poles of a fixed magnet and the mobile magnet and M_N is the energy in the interaction by the north poles. If $M_S = M_N = M$ we can simplify the systema as.

$$\vec{F}_R = M\left[\frac{1}{r} - \frac{1}{d-r}\right]\hat{r} \implies \vec{F}_R = M\left[\frac{d-r-r}{r(d-r)}\right]\hat{r} \implies \vec{F}_R = M\left[\frac{d-2r}{r(d-r)}\right]\hat{r}. \tag{2.1.18}$$

The second magnetic system, like its electric analog system, uses the repulsive force between two magnets and an external force. This can be shown in the figure 2.5.

If we use the magnetic analog of the equation (2.1.12), we can describe our system as.

$$\vec{F}_R = \vec{F}_M + \vec{F} \implies \vec{F}_R = \left[\frac{M}{r} - F\right]\hat{r} \tag{2.1.19}$$

2.1.4 General solution

Now that we know the different types of restoring forces, we can work on the differential equation to find the solution of an oscillatory system. To find the solution, we need to check if the differential equation has a theoretical solution. If it does not have it, then we use an approximation technique. If the equation has a solution, then we use the equation (2.1.2) to find it.

If we define the restitution acceleration as $a_R(r) = \frac{F_R(r)}{m}$. To simplify the calculus, we are going to work with only one dimension, but it is simple to expand it to the other two dimensions. Then, the equation can be rewritten as:

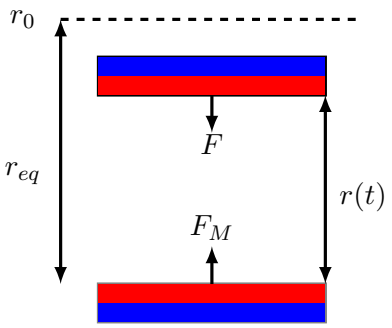


Figure 2.5: Graphic configuration of the magnetic system that uses the magnetic repulsive force to generate a magnetic restoring force.

$$\frac{d^2r(t)}{dt^2} + a_R(r) = 0. \tag{2.1.20}$$

We define the restitution acceleration function as the product of the square of the natural frequency of restitution (ω_0) and the position ($a_R(r) = \omega_0^2 r(t)$). Then, our differential equation is:

$$\frac{d^2r(t)}{dt^2} + \omega_0^2 r(t) = 0. \tag{2.1.21}$$

Now, to solve the equation we need to find the characteristic polynomial of the system. To find it, we use the technique to change a differential equation to an algebraic equation ($s^n = \frac{d^n r}{dt^n}$).

$$\frac{d^2r(t)}{dt^2} + \omega_0^2 r(t) = 0 \implies s^2 + \omega_0^2 = 0 \implies s^2 = -\omega_0^2 \implies \boxed{s_{1,2} = \pm i\omega_0}\,, \tag{2.1.22}$$

Introducing the solution of the characteristic polynomial in the general solution, we obtain.

$$r(t) = Ae^{s_1 t} + Be^{s_2 t} \implies \boxed{r(t) = Ae^{i\omega_0 t} + Be^{-i\omega_0 t}}\,, \tag{2.1.23}$$

we can transform our complex functions to real functions using the complex exponential definition ($e^{ix} = \cos(x) + i\sin(x)$).

$$\begin{aligned} r(t) &= Ae^{i\omega_0 t} + Be^{-i\omega_0 t} \\ &= A\left[\cos(\omega_0 t) + i\sin(\omega_0 t)\right] + B\left[\cos(\omega_0 t) - i\sin(\omega_0 t)\right] \\ &= \underbrace{(A+B)}_{C}\cos(\omega_0 t) + \underbrace{i\,(A-B)}_{D}\sin(\omega_0 t) \end{aligned} \tag{2.1.24}$$

$$\therefore \boxed{r(t) = C\cos(\omega_0 t) + D\sin(\omega_0 t)}\,,$$

where C and D are the amplitudes of the oscillations.

Now, to find the behavior of the system, we need to find the speed function. Then, we need to find the first change of the position function in time.

$$
\begin{aligned}
v &= \frac{dr(t)}{dt} \\
&= \frac{d}{dt}\left(C\cos\left(\omega_0 t\right) + D\sin\left(\omega_0 t\right)\right) \\
&= C\frac{d}{dt}\cos\left(\omega_0 t\right) + D\frac{d}{dt}\sin\left(\omega_0 t\right) \\
\therefore \boxed{v(t) &= -C\omega_0\sin\left(\omega_0 t\right) + D\omega_0\cos\left(\omega_0 t\right)}.
\end{aligned}
\tag{2.1.25}
$$

With the position and speed functions, we can plot a phase plane to know the behavior of a harmonic oscillator system.

2.1.5 Amplitude

The amplitude of the system is obtained using the Pythagorean theorem:

$$
A = \sqrt{C^2 + D^2}. \tag{2.1.26}
$$

It is important to say that the amplitude will be obtained after we used the initial conditions that are going to help us to find the values of C and D.

If the initial conditions are $r(0) = r_0$ and $v(0) = v_0$, then

$$
C = r(0) = r_0, \qquad D = \frac{v(0)}{\omega_0} = \frac{v_0}{\omega_0}.
$$

Therefore, the amplitude in terms of initial data is

$$
\boxed{R = \sqrt{r_0^2 + \left(\frac{v_0}{\omega_0}\right)^2}.}
$$

2.1.6 Energy

One of the most important things in a physical system is energy. For this reason, we need to find the expression of the energy present in our system.

As we know, the total energy of the system is composed of the kinetic and potential energies.

$$
E = T + V \implies E = \frac{1}{2}mv^2 + \frac{1}{2}kx^2 \implies E = \frac{m}{2}\left(v^2 + \frac{k}{m}x^2\right) \implies E = \frac{m}{2}\left(v^2 + \omega_0^2 x^2\right), \tag{2.1.27}
$$

where $\omega_0 \equiv \sqrt{\frac{k}{m}}$. If we replace the speed as the change of the position on time, we can rewrite the energy expression as.

$$
\begin{aligned}
E =& \frac{m}{2}\left[\left(\frac{dx(t)}{dt}\right)^2 + \omega_0^2 x^2(t)\right] \\
=& \frac{m}{2}\left[(-C\omega_0 \sin(\omega_0 t) + D\omega_0 \cos(\omega_0 t))^2 + \omega_0^2 (C\cos(\omega_0 t) + D\sin(\omega_0 t))^2\right] \\
=& \frac{m\omega_0^2}{2}\left[C^2 \sin^2(\omega_0 t) - 2CD\sin(\omega_0 t)\cos(\omega_0 t) + D^2\cos^2(\omega_0 t)\right] \\
&+ \frac{m\omega_0^2}{2}\left[C^2\cos^2(\omega_0 t) + 2CD\sin(\omega_0 t)\cos(\omega_0 t) + D^2\sin^2(\omega_0 t)\right] \\
=& \frac{m\omega_0^2}{2}\left[C^2\left(\sin^2(\omega_0 t) + \cos^2(\omega_0 t)\right) + D^2\left(\sin^2(\omega_0 t) + \cos^2(\omega_0 t)\right)\right] \\
\therefore & \boxed{E = \frac{m\omega_0^2}{2}\left(C^2 + D^2\right) = \frac{m\omega_0^2}{2}A^2}\,.
\end{aligned}
\tag{2.1.28}
$$

This result lets us know that the energy depends on the mass of the oscillator, the frequency, and the amplitude of the movement.

2.2 Damped oscillation

When an oscillator system presents a dissipative force, that system is called a ***damped oscillator***. In this system, the oscillation frequency is perturbed by friction, causing the oscillation to become slightly slower than in the beginning.

The dissipative force is the friction force of the environment and is proportional to the velocity. Then, the dissipative force is expressed as:

$$\vec{F}_d = \beta\vec{v}, \tag{2.2.1}$$

where β is the friction coefficient and $\vec{v}$ is the velocity of the oscillation.

Now, the forces in our system are three: the driving force (F), the friction force of the environment (F_d), and the restitution force (ω_R). These forces interact until the system arrives at equilibrium. Then, the sum of forces is:

$$\vec{F} + \vec{F}_d + \vec{F}_R = 0, \tag{2.2.2}$$

using the definitions of $\vec{F}_d$ and ω_R, let us know the set of differential equations that govern the motion in every direction.

$$m\frac{d^2\vec{r}}{dt^2} + \beta\frac{d\vec{r}}{dt} + \vec{F}_R(\vec{r}) = 0 \implies \sum_{i=1}^{3}\left(m\frac{d^2 r_i}{dt^2} + \beta\frac{dr_i}{dt} + F_{R\,i}(r_i)\right)\hat{e}_i = 0, \tag{2.2.3}$$

finding the equation of motion and dividing it over the mass, the differential equation is...

$$\sum_{i=1}^{3}\left(\frac{d^2 r_i}{dt^2} + \frac{\beta}{m}\frac{dr_i}{dt} + \frac{F_{R\,i}(r_i)}{m}\right)\hat{e}_i = 0 \implies \sum_{i=1}^{3}\left(\frac{d^2 r_i}{dt^2} + 2\gamma\frac{dr_i}{dt} + a_{R\,i}(r_i)\right)\hat{e}_i = 0, \tag{2.2.4}$$

where $\gamma = \frac{\beta}{2m}$ represents the half of the resistance frequency of the environment and $a_R = \frac{F_{R\,i}}{m}$ the restitution acceleration function.

If we define $s^n \equiv \frac{d^{(n)}r}{dt^n}$, we can rewrite our differential equation as a power equation, letting us find the characteristic equation of our homogenous system. For the non differential term ($r = \frac{d^{(0)}r}{dt^0} = s^0$) we can define $\omega_R(s^0) = \frac{a_R(r)}{r}$ as the frequency of the restitution forces.

Then, the characteristic equation in general form is:

$$s^2 + 2\gamma s + \omega_R^2(s^0) = 0 \implies s = -\gamma \pm \sqrt{\gamma^2 - \omega_R^2} = -\gamma \pm \eth, \tag{2.2.5}$$

where $\eth = \sqrt{\gamma^2 - \omega_R^2}$ is the discriminant of the system. Then, the solution of the system is:

$$\boxed{r(t) = e^{-\gamma t}\left[Ae^{\eth t} + Be^{-\eth t}\right]}, \tag{2.2.6}$$

this equation lets us know the amplitude of the oscillation at the time t. It is important to see that the discriminant value ($\eth$) defines the behavior of the system; there are three different values and three possible behaviors.

1. When $\eth < 0$ the oscillation is called ***underdamped***. In this type of oscillation, the amplitude of the oscillation decays gradually over time.
2. When $\eth = 0$ the oscillation is called ***critically damped***. In this kind of oscillation, the system returns quickly to the equilibrium point without oscillating.
3. When $\eth > 0$ the oscillation is called ***overdamped***. In this style of oscillation, the system does not oscillate but returns slowly to the equilibrium point.

2.2.1 Damped frequency

As we saw in the previous part, the only damped oscillation system where the system really has an oscillating behavior is the ***underdamped oscillator***. The underdamped oscillator requires that the discriminant be less than zero ($\eth < 0$), this causes:

$$\eth < 0 \implies \eth = \sqrt{\gamma^2 - \omega_R^2} < 0 \implies \omega_R > \gamma \therefore \eth \in \Im. \tag{2.2.7}$$

The previous solution lets us know that the discriminant is complex, so we can rewrite the discriminant in terms of a complex function.

$$\eth = i\sqrt{\omega_R^2 - \gamma^2} = i\omega_d, \tag{2.2.8}$$

where ω_d is the damped angular frequency of the system and represents how the frequency of restitution changes by the effect of the damping.

The damped frequency can be found using the relation between frequency and angular frequency:

$$f = \frac{\omega}{2\pi} \implies f_d = \frac{\omega_d}{2\pi} = \frac{\sqrt{\omega_R^2 - \gamma^2}}{2\pi}. \tag{2.2.9}$$

Now, we can find the solution of the system in terms of the damped angular frequency, as.

$$r(t) = e^{-\gamma t}\left[Ae^{i\omega_d t} + Be^{-i\omega_d t}\right]. \tag{2.2.10}$$

Using Euler's property by the complex numbers, we can simplify the solution as follows.

$$\begin{aligned} r(t) &= e^{-\gamma t}\left[Ae^{i\omega_d t} + Be^{-i\omega_d t}\right] \\ &= e^{-\gamma t}\left[A\left\{\cos(\omega_d t) + i\sin(\omega_d t)\right\} + B\left\{\cos(\omega_d t) - i\sin(\omega_d t)\right\}\right] \\ \therefore\ r(t) &= e^{-\gamma t}\left[C\cos(\omega_d t) + D\sin(\omega_d t)\right], \end{aligned} \tag{2.2.11}$$

where the constants are $C = A + B$ and $D = i(A - B)$. It is important to highlight that the C constant is a real number meanwhile the D constant is an imaginary number, so the solution of the system is a complex number.

Another important thing to see is that the oscillation decreases when the time increases, causing the term $e^{-\gamma t}$ to represent the dissipation.

2.2.2 Energy

Following the process of the subsection (2.1.6) we can find how the energy is expressed in the system. So finding the velocity and acceleration expressions.

We know that the velocity is :

$$\begin{aligned} v =& \frac{dr(t)}{dt} \\ =& \frac{d}{dt}\left\{e^{-\gamma t}\left[C\cos(\omega_d t) + D\sin(\omega_d t)\right]\right\} \\ =& \frac{d}{dt}\left\{e^{-\gamma t}C\cos(\omega_d t)\right\} + \frac{d}{dt}\left\{e^{-\gamma t}D\sin(\omega_d t)\right\} \\ =& \left[-\gamma Ce^{-\gamma t}\cos(\omega_d t) - \omega_d Ce^{-\gamma t}\sin(\omega_d t)\right] + \left[-\gamma De^{-\gamma t}\sin(\omega_d t) + \omega_d De^{-\gamma t}\cos(\omega_d t)\right] \\ =& e^{-\gamma t}\left[(-\gamma C + \omega_d D)\cos(\omega_d t) - (\omega_d C + \gamma D)\sin(\omega_d t)\right]. \end{aligned} \tag{2.2.12}$$

Now, use the expressions of the kinetic and the potential energy to find the total energy.

$$\begin{aligned} E = T + U \implies E &= \frac{1}{2}mv^2 + \frac{1}{2}kr^2 \\ \implies E = \frac{1}{2}\left[mv^2 + kr^2\right] \implies E &= \frac{m}{2}\left[v^2 + \frac{k}{m}r^2\right], \end{aligned} \tag{2.2.13}$$

using the natural frequency identity ($\omega_0 = \sqrt{\frac{k}{m}}$) we can rewrite the energy expression as:

$$E = \frac{m}{2}\left[v^2 + \omega_0^2 r^2\right], \tag{2.2.14}$$

if we introduce the x and v definitions in the last expression of the energy, we can obtain the following equation.

$$
\begin{aligned}
E &= \frac{m}{2}\left[v^2+\omega_0^2x^2\right] \\
&= \frac{m}{2}\left[e^{-2\gamma t}\left\{(-\gamma C+\omega_d D)\cos(\omega_d t)-(\omega_d C+\gamma D)\sin(\omega_d t)\right\}^2+\omega_0^2e^{-2\gamma t}\left\{C\cos(\omega_d t)+D\sin(\omega_d t)\right\}^2\right] \\
&= \frac{me^{-2\gamma t}}{2}\left[\left\{(-\gamma C+\omega_d D)\cos(\omega_d t)-(\omega_d C+\gamma D)\sin(\omega_d t)\right\}^2+\omega_0^2\left\{C\cos(\omega_d t)+D\sin(\omega_d t)\right\}^2\right],
\end{aligned}
\tag{2.2.15}
$$

from this result, we can see that the energy of the system decays to zero when the time increases, so we can be sure that in some time our system will stop its movement. The speed of decaying depends on the γ's value.

2.2.3 Note

In this project, we will work firstly with underdamped oscillations, but it is important to remember that de frequency ω_R depends on the nature of the force and the number of forces present on the system. So, a general form of the discriminant is:

$$
\eth = \sqrt{\gamma^2-\frac{1}{m^2}\left(\sum\frac{\vec{F}_{mec}(r_i)}{r_i}+\sum\frac{\vec{F}_{elec}(r_i)}{r_i}+\sum\frac{\vec{F}_{mag}(r_i)}{r_i}+\cdots\right)^2}
\tag{2.2.16}
$$

2.3 Forced oscillation

A ***forced oscillation*** is the system where an external force introduces energy into the oscillator with a specific frequency. This external force continuously drives the motion, compensating for the energy lost due to damping, and produces an oscillation that can reach a steady amplitude.

As we mentioned before, the external force acts periodically, so it can be expressed as

$$
\vec{F}_{ex} = \vec{F}\cos(\omega_{ex}t),
\tag{2.3.1}
$$

where $\vec{F}$ is the amplitude of the applied external force and ω_{ex} is the angular frequency of this force. It is important to note that the cosine function may be replaced by any other periodic function depending on the physical characteristics of the system.

Including the external excitation in the damped oscillator, the system is now subject to four forces:

$$
\vec{F}+\vec{F}_d+\vec{F}_R = \vec{F}_{ex}.
\tag{2.3.2}
$$

Using the definitions of each force, we obtain the differential equation that describes the motion of the system:

$$
m\frac{d^2\vec{r}}{dt^2}+\beta\frac{d\vec{r}}{dt}+\vec{F}_R = \vec{F}\cos(\omega_{ex}t) \implies \sum_{i=1}^{3}\left(m\frac{d^2r_i}{dt^2}+\beta\frac{dr_i}{dt}+F_{R\,i}(r_i)\right)\hat{e}_i = \sum_{i=1}^{3}F_i\cos(\omega_{ex}t)\hat{e}_i.
\tag{2.3.3}
$$

Now, dividing the whole expression by the mass and using Einstein's notation, we obtain

$$\frac{d^2r_i}{dt^2}+\frac{\beta}{m}\frac{dr_i}{dt}+\frac{F_{R\,i}(r_i)}{m}=\frac{F_i}{m}\cos(\omega_{ex}t) \implies \frac{d^2r_i}{dt^2}+2\gamma\frac{dr_i}{dt}+a_{R\,i}(r_i)=a_i\cos(\omega_{ex}t), \quad (2.3.4)$$

where $\gamma=\dfrac{\beta}{2m}$ represents half of the resistance frequency of the medium, $a_{R\,i}=\dfrac{F_{R\,i}}{m}$ is the restoring acceleration, and $a_i=\dfrac{F_i}{m}$ is the acceleration associated with the external excitation.

A forced oscillation is a non-homogeneous differential equation, which means that its solution must be divided into two parts: the homogeneous and the particular solutions.

$$\underbrace{\frac{d^2r_i}{dt^2}+2\gamma\frac{dr_i}{dt}+a_{R\,i}(r_i)}_{\text{homogeneous part}}=\underbrace{a_i\cos(\omega_{ex}t)}_{\text{non-homogeneous part}}. \quad (2.3.5)$$

The homogeneous part corresponds to the damped oscillation previously studied, while the particular solution represents the steady-state response caused by the external periodic force. The homogeneous solution can be written as

$$r_h(t)=Ae^{-\gamma t}\cos(\omega_d t+\phi), \quad (2.3.6)$$

where $\omega_d=\sqrt{\omega_0^2-\gamma^2}$ is the damped frequency, A is the amplitude determined by the initial conditions, and ϕ is the initial phase.

Now, we are going to find the particular solution of the system. As the forcing term is a cosine function, we propose a solution of the same form:

$$r_p(t)=C\cos(\omega_{ex}t-\delta), \quad (2.3.7)$$

where C is the amplitude of the steady oscillation and δ represents the phase difference between the external and internal oscillations.

Substituting this expression into the differential equation and simplifying, we obtain

$$C=\frac{a}{\sqrt{(\omega_0^2-\omega_{ex}^2)^2+(2\gamma\omega_{ex})^2}}, \qquad \tan\delta=\frac{2\gamma\omega_{ex}}{\omega_0^2-\omega_{ex}^2}. \quad (2.3.8)$$

Therefore, the complete solution for the forced oscillation is

$$\begin{aligned} r(t)&=r_h(t)+r_p(t)\\ &=Ae^{-\gamma t}\cos(\omega_d t+\phi)+\frac{a}{\sqrt{(\omega_0^2-\omega_{ex}^2)^2+(2\gamma\omega_{ex})^2}}\cos(\omega_{ex}t-\delta). \end{aligned} \quad (2.3.9)$$

The first term represents the *transient response*, which disappears after some time due to the exponential decay, while the second term corresponds to the *steady-state oscillation*, which remains as long as the external force acts on the system. When the external frequency ω_{ex} approaches the natural frequency ω_0, the amplitude reaches its maximum value. This phenomenon is known as **resonance**.

2.3.1 Energy

Now, we are going to obtain the expression of the energy in this system. As before, we need to find the velocity, which is obtained by differentiating $r(t)$:

$$\begin{aligned} v(t) &= \frac{dr}{dt} \\ &= -Ae^{-\gamma t}(\gamma \cos(\omega_d t + \phi) + \omega_d \sin(\omega_d t + \phi)) - C\omega_{ex} \sin(\omega_{ex} t - \delta). \end{aligned} \tag{2.3.10}$$

Using this result, we can write the total mechanical energy as

$$\begin{aligned} E = T + U \implies E &= \frac{1}{2}mv^2 + \frac{1}{2}kr^2 \\ \implies E &= \frac{m}{2}\left[\left(\frac{dr}{dt}\right)^2 + \omega_0^2 r^2\right]. \end{aligned} \tag{2.3.11}$$

In this case, the total energy of the system is not conserved because the external force continuously injects energy, while the damping force dissipates it. During the transient regime, the external source provides energy until the system reaches a dynamic equilibrium, where the energy supplied per cycle equals the energy lost due to damping. This balance produces a steady oscillation with constant amplitude, characterizing the behavior of a forced oscillator.

2.4 Resonance

Resonance represents one of the most remarkable phenomena in the study of oscillatory systems. It occurs when an oscillator interacts with an external periodic excitation, allowing the system to absorb energy efficiently from the external source. In this sense, resonance appears naturally in forced oscillation systems, where an external force continuously acts on the system.

Let us consider the general form of a damped, forced oscillator:

$$m\ddot{x} + b\dot{x} + kx = F_0 \cos(\omega_{ex} t), \tag{2.4.1}$$

where m is the mass of the oscillator, b is the damping coefficient, k is the elastic constant, F_0 is the amplitude of the external force, and ω_{ex} is its frequency.

In the steady-state regime, the particular solution of the system can be expressed as

$$x(t) = A(\omega_{ex}) \cos(\omega_{ex} t - \delta), \tag{2.4.2}$$

where $A(\omega_{ex})$ is the amplitude of the oscillation, which depends on the excitation frequency, and δ is the phase difference between the force and the displacement. The amplitude is given by

$$A(\omega_{ex}) = \frac{F_0/m}{\sqrt{(\omega_0^2 - \omega_{ex}^2)^2 + (2\beta\omega_{ex})^2}}, \tag{2.4.3}$$

where $\omega_0 = \sqrt{k/m}$ is the natural frequency of the system and $\beta = b/(2m)$ is the damping parameter.

From this expression, we can observe that the amplitude $A(\omega_{ex})$ reaches its maximum value when the denominator is minimum, which occurs close to the condition $\omega_{ex} = \omega_0$ (for small damping). This defines the ***resonance frequency*** of the system, given approximately by

$$\omega_{res} \approx \sqrt{\omega_0^2 - 2\beta^2}. \tag{2.4.4}$$

Under resonance, the amplitude of oscillation increases significantly, and the energy transfer between the external source and the oscillator becomes highly efficient. The energy stored in the oscillator is proportional to $A^2(\omega_{ex})$, which explains why even a small external force can produce large oscillations when the system is near resonance.

From an engineering perspective, resonance is a double-edged phenomenon. On one hand, it can lead to structural instabilities or even catastrophic failures if not properly controlled. A classical example is the shattering of a glass when exposed to a sound whose frequency matches its natural vibration mode. Similarly, during earthquakes, buildings whose natural frequencies coincide with those of the seismic waves experience amplified oscillations, sometimes leading to severe damage.

On the other hand, resonance can also be exploited advantageously. In this project, we aim to take advantage of resonance to extract energy from a fluid flow interacting with an oscillatory structure. By tuning the system parameters so that the excitation frequency of the flow matches the natural frequency of the structure, we can enhance the oscillatory response and maximize the energy conversion efficiency.

CHAPTER 3

Fluid-Structure interaction

> Fluid-structure interaction problems require the simultaneous solution of coupled fluid and structural equations with proper interface conditions.
> Farhat, Lesoinne & Le Tallec, 1998

The fluid-structure interaction is a phenomenon that is relevant in engineering, in particular for the engineering of bridges. This is because this interaction reveals how the flow of a fluid influences the dynamics of a structure.

The most famous example of fluid-structure interaction is the Tacoma Narrows Bridge. When a flow of air with a speed between 60 and 70 km caused the collapse of the bridge. This can be explained by the interaction of the air with the bridge. When air impacted the bridge, it generated some force over the bridge, but the bridge changed the direction of the flow, generating some vortices that induced vibrations in the structure. Those vibrations had the same frequency as the natural frequency of the bridge, causing resonance and, finally, the collapse of the bridge.

By the previous example, we can see that the fluid-structure interaction is so relevant for the building of bridges and some buildings. But the most important thing that we can highlight is that we can generate vibrations over structures with the flux of a fluid with a specific speed and density, with the intention of capturing the energy of the fluid and transforming it into another kind of energy.

To better understand this physical phenomenon, we will introduce the fundamental ideas of fluid dynamics and then analyze how the fluid affects an object and its dynamics.

3.1 Fluid dynamics

Matter can be classified into two groups: solids and fluids ([41]). As White said, to determine which group our object of study belongs to, we need to observe how it responds to a shear stress force. A substance is considered a solid if it resists deformation and maintains a fixed shape when subjected to a shear stress. In contrast, a substance is considered a fluid if it continuously deforms (flows) when subjected to a shear stress, no matter how small.

Fluids are classified into two groups: liquids and gases. And to determine which group our fluid belongs to depends on the cohesive force of the molecules. A liquid has a definite volume but takes the shape of its container, and this is because the cohesive forces are strong. In contrast, a gas has neither a definite shape nor volume and expands to fill its container; then the cohesive forces are weak.

When we

Another important thing that we need to consider when we are working with fluids is the dimensions. Previously, we knew that dimensional analysis is so important in physics and engineering, but this technique has a principal role in the physics of fluids. This is because by the principle of ***Dimensional Homogeneity*** lets us transform all the principles and laws of physics that we use in particles, to another kind of phenomenon like fluids. So, Newton's laws can be applied to fluids to find the dynamics that the fluid will follow, certainly with some changes in the expression of the force, which are characterized by its properties.

Now that we know the principal things about fluids, we are going to present the principal properties of fluids.

3.1.1 Properties of fluids

In this work, we will be using fluids, so it is essential to understand the elements that play a crucial role in the physics of the system. Then, we need to understand the properties that characterize fluids. The most important properties that a fluid has are density, pressure, viscosity, compressibility, and Reynolds number, so we are going to expand our knowledge of these properties.

Density (ρ)

Density is a measure of how matter is distributed in a given space. Another way to define density is as mass per unit of space. Then, we can write the mathematical expression as.

$$\rho = \frac{m}{V}, \tag{3.1.1}$$

where m is the mass of the fluid and V is the volume that the fluid occupies. It is important to see that in the definition we said space, then we can use another space that has fewer dimensions, then we have three different densities, and each one responds to the number of dimensions that the space has. These densities are:

1. Linear (λ): This density is when we analyze a unidimensional system, and is commonly used in cables or lines. Its units are $\frac{kg}{m}$ and its are dimensions $\left[\frac{M}{L}\right]$.
2. Surface (σ): This kind of density is applied when our space has two dimensions, length and height. Its units are $\frac{kg}{m^2}$ and its are dimensions $\left[\frac{M}{L^2}\right]$.
3. Volumetric (ρ): We use it when we have a space with three dimensions, length, height, and width; and it is the most common density because we live in a three-dimensional world. Its units are $\frac{kg}{m^3}$ and its dimensions are $\left[\frac{M}{L^3}\right]$.

Density plays an important role in the dynamics of a fluid because it affects buoyancy, low rate, and pressure distribution. And it is used to define another property, the compressibility.

Density appears naturally in the dynamics equation, then using Newton's second law.

$$F = ma \implies F = \rho V a \tag{3.1.2}$$

Pressure (p)

Forces in physics, like density, attend to dimensions and there are three different kind of forces, and they are:

1. Linear force (F_L): It is a force that is applied in one direction.
2. Surface force (F_S): It is the force that is applied to a surface.
3. Body force (F_B): It is the force that acts on all the mass particles of a body at the same time. An example of this kind of force is the gravitational force.

Then, the net force of a system can be expressed in the following form.

$$F = F_L + F_S + F_B \tag{3.1.3}$$

Pressure, as Pascal's law expresses, is a surface force. Then, pressure is the force that a fluid applies to the surface of a body. It is important to highlight that if we put the origin of our coordinate system in the center of the body, the pressure of the fluid is opposite to the resistance force. Sometimes, the pressure of a system can change by the position, or in other words, it is a function of some variable. So, the most complete form for expressing the surface force is:

$$d\vec{F}_S = -pd\vec{S}, \tag{3.1.4}$$

where p is the pressure and $\vec{S}$ is the normal vector to the surface. If we analyze the force in a specific direction (i direction), we obtain.

$$d\vec{F}_S = -pdS\hat{e}_i. \tag{3.1.5}$$

If the force is applied in i direction, the surface is placed in the other directions (j and k). So, if $dS = djdk$, then our force can be expressed as.

$$d\vec{F}_S = [p_i\hat{e}_i]\, djdk, \tag{3.1.6}$$

the force is how the pressure changes at every point of the surface, then we can rewrite the force as follows.

$$d\vec{F}_S = [p(i) - p(i+di)]\, djdk\hat{e}_i, \tag{3.1.7}$$

Using Taylor's series at first order in the i direction.

$$p(i+di) = p(i) + \frac{\partial p}{\partial i}di, \tag{3.1.8}$$

substituting in our force (equation (3.1.7)).

$$dF_{Si} = -\frac{\partial p}{\partial i} didjdk = -\frac{\partial p}{\partial i} dV. \tag{3.1.9}$$

Then, expand the definition of force to every direction.

$$d\vec{F}_S = -\frac{\partial p}{\partial i} dV \hat{e}_i = -\vec{\nabla} p dV, \tag{3.1.10}$$

where $\vec{\nabla}$ is the gradient operator and is defined as $\vec{\nabla} = \partial_i \hat{e}_i$. With the previous definition of surface force and we only have a body force and a surface, we can rewrite the force as:

$$d\vec{F} = dF_B + dF_S \implies d\vec{F} = \rho \vec{a} dV - \vec{\nabla} p dV \therefore \boxed{d\vec{F} = \left(\rho \vec{a} - \vec{\nabla} p\right) dV}. \tag{3.1.11}$$

We can define the force as the way the density force is applied on a volume, which can be expressed in a mathematical form as $d\vec{F} = \vec{f} dV$. Therefore, we can rewrite all in terms of density.

$$\boxed{d\vec{f} = \rho \vec{a} - \vec{\nabla} p}. \tag{3.1.12}$$

By the previous equation, we can appreciate that the density force depends on how the acceleration field interacts with the pressure field, and this defines the motion of the fluid particles.

Viscosity (μ)

Viscosity, as Landau, is the internal friction that a fluid presents. Viscosity affects the flow rate and the ease with which a fluid moves, as White ([41]) expressed, higher viscosity corresponds to a slower flow rate, while lower viscosity corresponds to a more easily flowing fluid.

Viscosity, in a Newtonian fluid, determines the fluid strain rate that is generated by a given shear stress ([41]). It can be interpreted as the rate at which the angle between fluid elements changes over time, or as the rate at which the velocity of the fluid changes with distance.

$$\tau = \mu \frac{d\theta}{dt} = \mu \frac{du}{dy}, \tag{3.1.13}$$

where μ is the viscosity coefficient, θ is the angle, t is the time, u is the velocity of the fluid, y is the direction of the motion and $\frac{du}{dy}$ is the velocity gradient perpendicular to the direction of flow. The viscosity coefficient has dimensions of stress-time ($[\frac{FT}{L^2}]$ or $[\frac{M}{LT}]$).

Something important to highlight is that if the fluid is Newtonian, then viscosity is a thermodynamic property because it depends on temperature and pressure.

Compressibility

Compressibility is the property of particles in a fluid to be more compact under some pressure at some temperature. If the particles are compact, this implies that more particles are in some volume, changing the density of particles. Then, we can define compressibility

as the change of density when we change the pressure at a constant temperature. Therefore, the mathematical expression for this property is.

$$\kappa = -\frac{1}{\rho}\left(\frac{\partial \rho}{\partial p}\right)_T \tag{3.1.14}$$

If we analyze the previous equation, we can see that the differentiation could take two different values, so we can conclude that fluids can be divided into two categories: compressible and incompressible fluids.

- Compressible: When the value of $\kappa \neq 0$, the most common compressible fluid is air.
- Incompressible: It is the fluid that has $\kappa = 0$, the most common incompressible fluid is water.

This property is so important because it determines how a fluid responds to changes in pressure. But if we work in a controllable environment, where the pressure does not change, then we can assume that a fluid is incompressible.

Reynolds number (Re)

This number is dimensionless, and it is used to correlate the behavior of the fluid with its viscosity. Reynolds number is defined as:

$$Re = \frac{\rho u L}{\mu} = \frac{uL}{\nu}, \tag{3.1.15}$$

where u is the velocity scale, L is the length scale of the flow. In the second form ν is the kinematic viscosity defined as.

$$\nu = \frac{\mu}{\rho}. \tag{3.1.16}$$

It is called kinematic because the mass units are canceled, and we only maintain the dimensions of length and time.

The Reynolds number is so important in fluid dynamics that it is very common to estimate it first because knowing the value of the Reynolds number lets us know what kind of flow we are working with.

Now, we are going to classify the flow of the fluid in terms of the Reynolds number.

- Laminar ($Re < 2000$): In this flow, particles move smoothly, in streamlines, and viscous forces dominate.
- Transitional ($2000 < Re < 4000$): In this flow, particles move in an unstable trajectory and can act like turbulent or laminar.
- Turbulent ($Re > 4000$): In this flow, particles have a chaotic movement, with eddies and vortices, and the inertial forces dominate.

Many other properties can be used, but in this work we will not utilize them.

Now, we are going to present the equations that govern the motion of fluids.

3.1.2 Static of fluid

When we are studying physics, we learn that there are two situations in dynamics: static and dynamic. The static in fluids occurs when the particles of the fluid do not have a relative motion between them, which can be the case when the fluid is at rest or when the fluid has a uniform acceleration.

If the force is zero ($f = 0$) the equation (5.3.43)

$$\rho\vec{a} - \vec{\nabla}p = 0 \implies \vec{\nabla}p = \rho\vec{a}. \tag{3.1.17}$$

The previous equation is a vector equation and to solve it, we need to solve the equation in any dimension. To solve it, we are going to use the tensor notation. Then, our equation is now.

$$\begin{aligned} \frac{\partial p}{\partial x_i}\hat{e}_i = \rho a_j \hat{e}_j \implies \frac{\partial p}{\partial x_i}\hat{e}_i(\partial x_i \hat{e}_i) = \rho a_j \hat{e}_j(\partial x_i \hat{e}_i) \\ \implies \partial p = \rho a_j \partial x_i \delta_{ij} \implies \partial p = \rho a_i \partial x_i = \rho\vec{a}\cdot\partial\vec{x}. \end{aligned} \tag{3.1.18}$$

Now, before solving our differential equation, we need to know if our fluid is compressible or incompressible. So, we are going to present the analyses of both kinds of fluids.

Incompressible fluid

As we know from previous sections, an incompressible fluid has a constant density. Then, equation (3.1.18) only depends on the form of the acceleration,

$$p = \rho\int \vec{a}\cdot\partial\vec{x}, \tag{3.1.19}$$

but if we are in a constant acceleration field, we can simplify the equation and obtain.

$$\int \partial p = \rho a_i \int \partial x_i \implies p = \rho a_i x_i + c. \tag{3.1.20}$$

This is important because we can express this in terms of the three dimensions as.

$$p(x, y, z) = \rho(a_x x + a_y y + a_z z) + c = \rho(\vec{a}\cdot\vec{x}) + c. \tag{3.1.21}$$

Compressible fluid

For a compressible fluid, we need to consider that the density is a function of the pressure. Therefore, before calculating the pressure, we need to find the expression for density.

If we suppose that we are working with an ideal gas, we can use the ideal gas law to find the expression of density.

$$\rho = \frac{m}{V} \text{ and } m = nM \implies \rho = \frac{nM}{V}, \tag{3.1.22}$$

where M is the molecular mass, n is the number of moles, and V is the volume of our gas. Substituting in the ideal gas law, we obtain.

$$pV = nRT \implies p = \frac{nRT}{V} \implies p = \frac{\rho RT}{M} \therefore \rho(p) = \frac{pM}{RT}. \tag{3.1.23}$$

Now, using the previous result in the static equation, our equation of behavior is.

$$\vec{\nabla} p = \rho \vec{a} = \frac{M\vec{a}}{RT} p. \tag{3.1.24}$$

It is important to note that our partial differential equation depends on the expression of the components of the acceleration vector. If we consider the acceleration as a constant field in one direction, the solution of our differential equation is as follows.

$$\begin{aligned} \frac{dp}{dx} = \frac{Ma}{RT} p \implies \frac{dp}{p} = \frac{Ma}{RT} dx \\ \implies \int \frac{dp}{p} = \int \frac{Ma}{RT} dx \implies ln(p) = \frac{Max}{RT} + c \\ p(x) = e^{\frac{Max}{RT} + c} \implies \boxed{p(x) = Ce^{\frac{Ma}{RT} x}}. \end{aligned} \tag{3.1.25}$$

As we can see from the expression of pressure in both cases (incompressible and compressible), density plays a crucial role in the pressure of our system and indirectly in the evolution of the system.

3.2 Fluid Dynamics

In order to describe the interaction between a fluid flow and an oscillatory structure, it is essential to recall the fundamental principles that govern the motion of fluids. In this section, we introduce the basic mathematical tools of fluid mechanics, including the material derivative, the conservation laws, and the equations that characterize fluid motion. These tools will be crucial in understanding how vortices are formed and how they later interact with structures.

3.2.1 Kinematics of a Fluid Flow

A fluid velocity field is denoted by $\vec{u}(\vec{x}, t)$, where $\vec{x}$ is the spatial coordinate and t is time. A fluid particle follows a trajectory $\vec{x}(t)$ determined by

$$\frac{d\vec{x}}{dt} = \vec{u}(\vec{x}(t), t). \tag{3.2.1}$$

To compute how a quantity changes as we move with a fluid particle, we introduce the **material derivative**. Given a scalar field $\phi(\vec{x}, t)$, its material derivative is defined as

$$\frac{D\phi}{Dt} = \frac{\partial \phi}{\partial t} + \vec{u} \cdot \vec{\nabla} \phi, \tag{3.2.2}$$

which represents the total temporal rate of change experienced by a particle. When applied to the velocity field we obtain the particle acceleration,

$$\frac{D\vec{u}}{Dt} = \frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \vec{\nabla})\vec{u}. \tag{3.2.3}$$

3.2.2 Conservation of Mass

The conservation of mass can be derived from the Reynolds Transport Theorem applied to a material volume. Expressing that the total mass does not change over time, we obtain

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{u}) = 0. \tag{3.2.4}$$

When the density is constant, the fluid is incompressible and the continuity equation simply becomes

$$\vec{\nabla} \cdot \vec{u} = 0. \tag{3.2.5}$$

3.2.3 Momentum Balance: Euler and Navier–Stokes Equations

Applying Newton's second law to a fluid particle yields

$$\rho \frac{D\vec{u}}{Dt} = \vec{f} + \vec{\nabla} \cdot \boldsymbol{\sigma}, \tag{3.2.6}$$

where $\vec{f}$ are body forces and $\boldsymbol{\sigma}$ is the stress tensor. For an ideal fluid the stress tensor reduces to a pressure term,

$$\boldsymbol{\sigma} = -p\, \vec{I}, \tag{3.2.7}$$

leading to the **Euler equations**,

$$\rho \frac{D\vec{u}}{Dt} = -\vec{\nabla} p + \vec{f}. \tag{3.2.8}$$

If viscosity is included, we assume a Newtonian model where

$$\boldsymbol{\sigma} = -p\, \vec{I} + \mu\big(\vec{\nabla}\vec{u} + \vec{\nabla}\vec{u}^{T}\big), \tag{3.2.9}$$

leading to the **Navier–Stokes equations**,

$$\rho \frac{D\vec{u}}{Dt} = -\vec{\nabla} p + \mu \vec{\nabla}^2 \vec{u} + \vec{f}. \tag{3.2.10}$$

3.2.4 Bernoulli's Principle

Under the assumptions of inviscid, incompressible, steady flow along a streamline, the Euler equation simplifies to

$$\vec{\nabla} \left(p + \frac{1}{2}\rho |\vec{u}|^2 + \rho g z \right) = 0, \tag{3.2.11}$$

which implies that

$$p + \frac{1}{2}\rho |\vec{u}|^2 + \rho g z = \text{constant along a streamline.} \tag{3.2.12}$$

This relation connects velocity, pressure, and elevation, and it plays a central role in understanding how vortex formation is initiated around a bluff body.

3.2.5 Streamlines

Streamlines are curves tangent to the velocity field,

$$\frac{d\vec{x}}{ds} = \vec{u}(\vec{x}, t), \tag{3.2.13}$$

and for steady flows they coincide with particle paths. The structure of streamlines around an obstacle determines where the flow separates and where recirculation regions appear. These regions are the seeds of periodic vortex formation, a phenomenon central to vortex-induced vibrations.

3.3 Fluid–Structure Interaction

When a fluid interacts with a solid body, the motion of the fluid modifies the motion of the structure, and the motion of the structure modifies the motion of the fluid. This mutual influence is what we call **fluid–structure interaction** (FSI). In this project, the solid body is an oscillatory system, so the fluid forces can play the role of an external excitation, damping, or even an added inertia. For this reason, understanding the mathematical form of these interactions is fundamental to predict how the structure behaves when it is surrounded by a moving fluid.

3.3.1 Interface kinematics and dynamics

Since both systems are physically in contact, the first condition that we must impose is *kinematic compatibility*. In simple words, the fluid that touches the structure must move exactly with it. Mathematically, this condition appears as

$$\vec{v}_f|_{\Gamma_{fs}} = \dot{\vec{u}}_s|_{\Gamma_{fs}}, \tag{3.3.1}$$

where Γ_{fs} is the interface between the fluid and the structure. This expression ensures that no fluid particle crosses the surface of the structure and that the surface does not "slip" relative to the fluid.

The second condition is *dynamic compatibility*. The stresses acting on the fluid and on the structure must balance each other at the interface:

$$\boldsymbol{\sigma}_f \vec{n}_f + \boldsymbol{\sigma}_s \vec{n}_s = \vec{0}. \tag{3.3.2}$$

Because the normals satisfy $\vec{n}_f = -\vec{n}_s$, the force exerted by the fluid on the structure is

$$\vec{F}_{\text{fluid}}(t) = \int_{\Gamma_{fs}} \boldsymbol{\sigma}_f \vec{n}\, dS = -\int_{\Gamma_{fs}} p\vec{n}\, dS + \int_{\Gamma_{fs}} \boldsymbol{\tau}_f \vec{n}\, dS, \tag{3.3.3}$$

where the first term is the pressure contribution and the second comes from viscosity. In many engineering applications the pressure term dominates.

3.3.2 Structural dynamics with fluid forcing

If the structure behaves approximately like a single-degree-of-freedom oscillator, then its motion is described by

$$m\ddot{y} + c\dot{y} + ky = F_{\text{fluid}}(t) + F_{\text{ext}}(t), \tag{3.3.4}$$

where $F_{\text{fluid}}(t)$ is the projection of the hydrodynamic force in the direction of motion and $F_{\text{ext}}(t)$ represents any other external excitation. At this point, the equation already shows that the fluid can input energy, extract energy, or modify the resonance properties of the oscillator.

3.3.3 Added mass: an inertial effect of the fluid

One of the most characteristic effects of FSI is the **added mass**. When the structure accelerates, it must push some of the surrounding fluid. This extra effort appears mathematically as if the mass of the structure were larger.

Under the assumptions of inviscid and irrotational flow, the velocity field can be written as $\vec{v} = \nabla\phi$, where ϕ satisfies

$$\nabla^2\phi = 0, \qquad \frac{\partial\phi}{\partial n} = \vec{V}_b \cdot \vec{n} \quad \text{on the surface.} \tag{3.3.5}$$

For small body motions the potential becomes a linear combination of elementary potentials:

$$\phi(\vec{x}, t) = \sum_j \Phi_j(\vec{x})\, V_{b,j}(t).$$

Inserting this expression into the kinetic energy of the fluid gives

$$T_f = \frac{\rho}{2}\int_{\Omega_f} |\nabla\phi|^2\, dV = \frac{1}{2}\sum_{i,j} M_{ij}^{\text{add}}\, V_{b,i} V_{b,j}, \tag{3.3.6}$$

where the coefficients

$$M_{ij}^{\text{add}} = \rho\int_{\Omega_f} \nabla\Phi_i \cdot \nabla\Phi_j\, dV$$

form the so-called **added mass matrix**. This quantity modifies the inertial term of the structure, producing the effective equation

$$(M + M^{\text{add}})\ddot{y} + C\dot{y} + Ky = F_{\text{ext}} + F_{\text{vortex}}. \tag{3.3.7}$$

For a circular cylinder in potential flow, the classical result gives

$$M^{\text{add}} = \rho\frac{\pi D^2}{4},$$

which shows that a non-negligible portion of fluid must always be accelerated together with the body.

3.3.4 Coupled system: fluid and structure

The complete FSI problem is the combination of the Navier–Stokes equations and the structural equation of motion, linked by the interface conditions previously described.

Fluid: Navier–Stokes equations

$$\rho_f \left(\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\vec{v} \right) = -\nabla p + \mu \nabla^2 \vec{v}, \tag{3.3.8}$$

$$\nabla \cdot \vec{v} = 0, \tag{3.3.9}$$

with the condition $\vec{v} = \dot{\vec{u}}_s$ at the fluid–structure interface.

Structure: oscillator with added mass

$$(M + M^{\text{add}})\ddot{y} + C\dot{y} + Ky = \int_{\Gamma_{fs}} (\boldsymbol{\sigma}_f \vec{n}) \cdot \hat{e}_y \, dS + F_{\text{ext}}. \tag{3.3.10}$$

Solving this system requires numerical methods in almost every practical scenario. However, the analytical understanding of added mass, hydrodynamic forcing, and interface constraints provides the conceptual tools needed to analyze more complex fluid–structure phenomena such as vortex-induced vibrations.

3.4 Vortex-Induced Vibrations

Among the most typical and important phenomena in fluid–structure interaction is Vortex-Induced Vibrations (VIV). When a fluid flows around a bluff body (e.g., cylinder, prism, or any object that does not keep the flow attached to it) a periodic and alternating detachment of vortices occurs bottom stream, forming a pattern called the Kármán vortex street, which creates an oscillating lift force on the structure. The lift force acting on the structure is oscillating in time and therefore not constant, which causes the structure to oscillate in response to the alternating vortices.

The main point is that the rate of vortex shedding is not random. For a particular flow rate U and some characteristic length D, the shedding frequency f_s is generally given by the Strouhal number,

$$St = \frac{f_s D}{U}, \tag{3.4.1}$$

where St remains approximately constant for a wide range of Reynolds numbers when dealing with bluff bodies. Therefore, knowing or controlling the flow speed means that we implicitly know the forcing frequency set up by the vortices.

The relevance of VIV develops when the shedding frequency f_s gets close to the natural frequency of the structure f_n. In this flow regime, the force exerted by the fluid motion has a strong coupling with the natural dynamics of the system, generating an effect that is similar to the mechanical resonance. Then, the amplitude of this damped oscillation could increase significantly, even with the external forcing (i.e., the flow) remaining constant.

Mathematically, we often model structural displacement $y(t)$ using the same type of linear oscillator that appears throughout this chapter:

$$m\ddot{y} + c\dot{y} + ky = F_L(t), \tag{3.4.2}$$

where $F_L(t)$ is the unsteady lift force generated by the periodic vortex shedding. A common phenomenological approximation assumes a harmonic lift force,

$$F_L(t) \approx F_0 \sin(2\pi f_s t), \tag{3.4.3}$$

which makes explicit the connection between VIV and forced oscillations.

However, VIV is more subtle than a simple forced harmonic response. When the oscillation amplitude grows, the structure itself modifies the flow, altering the vortex shedding pattern. This makes the problem inherently nonlinear and coupled: the fluid affects the structure, and the structure affects the fluid. One of the clearest manifestations of this coupling is the so-called "lock-in" or "synchronization" phenomenon. In this regime, the shedding frequency f_s no longer follows the Strouhal relation but becomes synchronized with the structural natural frequency f_n. As a result, the oscillation amplitude can remain large over a wide range of flow velocities.

This behavior has profound implications in engineering. On one hand, VIV may compromise structural integrity. Offshore risers, bridge cables, building components, and other slender structures can experience fatigue or catastrophic failure due to long-term VIV. On the other hand, this same phenomenon can be exploited. Because VIV converts kinetic energy from the passing flow into mechanical oscillations, one can design systems that deliberately enter the lock-in regime to maximize energy extraction. In this project, our goal is precisely to leverage VIV to harvest fluid-flow energy through a controlled oscillatory mechanism. Understanding the underlying dynamics is therefore essential to achieve efficient, stable, and predictable energy capture.

3.5 Example

The problem addressed in this document is to find the solution to the equation of motion for a cylinder submerged in a fluid, with the additional condition that the cylinder is attached to springs.

Below is a diagram representing the layout of the system to be solved.

3.5.1 Physical Formulation

We start from the equation of the forced harmonic oscillator, given by:

$$m\ddot{x} + \beta\dot{x} + kx = F\cos(\omega t), \tag{3.5.1}$$

where m is the mass of the cylinder, β is the damping coefficient, k is the elastic restoring constant, F is the external force acting on the system, and ω is the frequency of that external force.

Factoring out the mass of the cylinder leads to:

$$m\left(\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x\right) = F\cos(\omega t), \tag{3.5.2}$$

where $2\gamma = \frac{\beta}{m}$ and $\omega_0 = \sqrt{\frac{k}{m}}$. Since the system is set up for vertical oscillation, we replace x with y and cos with sin:

$$m\left(\ddot{y} + 2\gamma\dot{y} + \omega_0^2 y\right) = F_y\sin(\omega t), \tag{3.5.3}$$

Using the lift coefficient definition $C_L = \frac{F}{\frac{1}{2}\rho u^2 A}$, the equation becomes:

$$m\left(\ddot{y} + 2\gamma\dot{y} + \omega_0^2 y\right) = \frac{1}{2}\rho u^2 A C_L\sin(\omega t), \tag{3.5.4}$$

where A is the area of the cylinder interacting with the fluid, u is the fluid velocity, and ρ is the fluid density. Given that $A = \pi DL$, and introducing a phase shift ϕ in the sinusoid to account for any time instant, the equation of motion is:

$$\boxed{m\left(\ddot{y} + 2\gamma\dot{y} + \omega_0^2 y\right) = \frac{1}{2}\pi\rho u^2 DLC_L\sin(\omega t + \phi)} \tag{3.5.5}$$

3.5.2 Solution of the Differential Equation

To solve differential equation (3.5.5), we first simplify by dividing by the mass:

$$\ddot{y} + 2\gamma\dot{y} + \omega_0^2 y = \frac{1}{2m}\rho u^2 DLC_L\sin(\omega t + \phi) \tag{3.5.6}$$

We seek both the homogeneous and particular solutions.

Homogeneous Solution

The homogeneous part is:

$$\ddot{y} + 2\gamma\dot{y} + \omega_0^2 y = 0 \tag{3.5.7}$$

The associated characteristic polynomial is:

$$m^2 + 2\gamma m + \omega_0^2 = 0 \tag{3.5.8}$$

Its roots are:

$$m_1 = -\gamma + \sqrt{\gamma^2 - \omega_0^2}, \tag{3.5.9}$$

$$m_2 = -\gamma - \sqrt{\gamma^2 - \omega_0^2} \tag{3.5.10}$$

Thus the homogeneous solution is:

$$
\begin{aligned}
y_h(t) &= Ae^{m_1 t} + Be^{m_2 t} \\
&= Ae^{(-\gamma+\sqrt{\gamma^2-\omega_0^2})t} + Be^{(-\gamma-\sqrt{\gamma^2-\omega_0^2})t} \\
\therefore y_h(t) &= e^{-\gamma t}\left(Ae^{\sqrt{\gamma^2-\omega_0^2}\,t} + Be^{-\sqrt{\gamma^2-\omega_0^2}\,t}\right)
\end{aligned} \tag{3.5.11}
$$

Particular Solution

We propose a particular solution of the form:

$$
y_p(t) = C\cos(\omega t) + D\sin(\omega t) \tag{3.5.12}
$$

Its derivatives are:

$$
\dot{y}_p = -\omega C\sin(\omega t) + \omega D\cos(\omega t), \tag{3.5.13}
$$

$$
\ddot{y}_p = -\omega^2 C\cos(\omega t) - \omega^2 D\sin(\omega t) \tag{3.5.14}
$$

Substituting into the differential equation yields:

$$
\left[C\cos(\omega t) + D\sin(\omega t)\right]\left(\omega_0^2 - \omega^2\right) + 2\gamma\omega\left(D\cos(\omega t) - C\sin(\omega t)\right) = \frac{F}{m}\sin(\omega t) \tag{3.5.15}
$$

Separating coefficients leads to:

$$
D\left(\omega_0^2 - \omega^2\right) - 2C\gamma\omega = \frac{F}{m}, \tag{3.5.16}
$$

$$
C\left(\omega_0^2 - \omega^2\right) + 2D\gamma\omega = 0 \tag{3.5.17}
$$

Solving for C and D gives:

$$
C = -\frac{2\gamma\omega F}{m\left((\omega_0^2-\omega^2)^2 + 4\gamma^2\omega^2\right)}, \quad D = \frac{F(\omega_0^2-\omega^2)}{m\left((\omega_0^2-\omega^2)^2 + 4\gamma^2\omega^2\right)} \tag{3.5.18}
$$

Thus:

$$
y_p(t) = -\frac{2\gamma\omega F}{m\left((\omega_0^2-\omega^2)^2 + 4\gamma^2\omega^2\right)}\cos(\omega t) + \frac{F(\omega_0^2-\omega^2)}{m\left((\omega_0^2-\omega^2)^2 + 4\gamma^2\omega^2\right)}\sin(\omega t) \tag{3.5.19}
$$

The complete solution is $y(t) = y_h(t) + y_p(t)$:

$$
y(t) = e^{-\gamma t}\left(Ae^{\sqrt{\gamma^2-\omega_0^2}\,t} + Be^{-\sqrt{\gamma^2-\omega_0^2}\,t}\right) + \frac{F\left[(\omega_0^2-\omega^2)\sin(\omega t) - 2\gamma\omega\cos(\omega t)\right]}{m\left((\omega_0^2-\omega^2)^2 + 4\gamma^2\omega^2\right)} \tag{3.5.20}
$$

Substituting the physical values, we get:

$$y(t) = e^{-\gamma t}\left(Ae^{\sqrt{\gamma^2-\omega_0^2}\,t} + Be^{-\sqrt{\gamma^2-\omega_0^2}\,t}\right) + \frac{\pi\rho u^2 DLC_L\left[(\omega_0^2-\omega^2)\sin(\omega t) - 2\gamma\omega\cos(\omega t)\right]}{2m\left((\omega_0^2-\omega^2)^2 + 4\gamma^2\omega^2\right)}, \tag{3.5.21}$$

where γ is the damping, ω the forcing frequency, ω_0 the natural frequency, m the mass, ρ the fluid density, u the fluid velocity, D the diameter, L the length, and C_L the lift coefficient. Constants A and B depend on initial conditions.

At resonance ($\omega_0 = \omega$), the maximum amplitude is:

$$y(t) = e^{-\gamma t}\left(Ae^{\sqrt{\gamma^2-\omega_0^2}\,t} + Be^{-\sqrt{\gamma^2-\omega_0^2}\,t}\right) - \frac{\pi\rho u^2 DLC_L\cos(\omega_0 t)}{4m\gamma\omega_0}, \tag{3.5.22}$$

Thus, to model the system, we need ρ, u, D, L, m, ω_0, γ, and C_L.

CHAPTER 4

Magneto-gravitational oscillator (Interaction magnet-magnet with gravity presence)

> The long and constant persuasion that all the forces of nature are mutually dependent, having one common origin, or rather being different manifestations of one major power, has made me often think upon the possibility of establishing, by experiment, a connection between gravity and electricity...
>
> Michael Faraday, *"Experimental Researches in Electricity"*, 1850

This physical system is composed of two magnets; one of the magnets is fixed on the ground, and the other is free to move vertically over the fixed magnet. The free magnet is initially displaced from the equilibrium position (x_0) to some amplitude (A). For the gravity presence, the free magnet falls, decreasing the distance between the magnets until the magnetic repulsion force increases enough to push the free magnet, increasing the distance and generating an oscillation.

The magnetic repulsion force F_M is expressed in this problem as a relation between the magnetic energy (M) and the distance between the magnets (x). So the mathematical form of the magnetic force is:

$$F_M = \frac{M}{x}. \tag{4.0.1}$$

As initial conditions for this problem, we have that at the time $t = 0$ the distance between the magnets is a displacement from the equilibrium position and the speed of the free magnet in that initial position is zero. Then, our initial conditions mathematically are:

$$x(0) = x_{eq} + A \quad \wedge \quad \dot{x}(0) = 0. \tag{4.0.2}$$

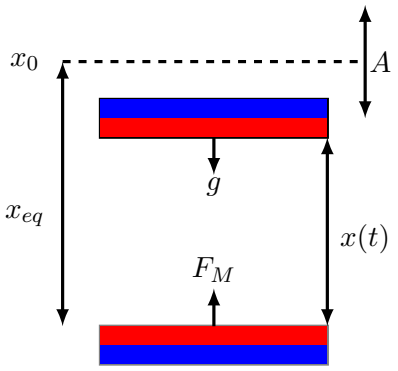


Figure 4.1: Graphic representation of the magnetic gravitational oscillator, where x_0 represents the equilibrium position, x_{eq} the equilibrium distance, A the oscillation amplitude and $x(t)$ the distance between of the two magnets.

The domain of the distance x goes from 0 to ∞, because the free magnet can be too far from the fixed magnet or so close but it can not touch it, so the domain is:

$$\text{Dom}(x) = (0, \infty), \tag{4.0.3}$$

If we use the domain of x and the initial condition of the position, we can find the domain of the amplitude.

$$A = x(0) - x_{eq} \implies A = \begin{cases} -x_{eq} & \text{if } x(0) = 0 \\ \infty & \text{if } x(0) = \infty \end{cases} \implies \text{Dom}(A) = (-x_{eq}, \infty), \tag{4.0.4}$$

this result lets us know that the initial amplitude can go from zero to infinity.

4.1 Ideal system

For the ideal case, the dissipative and external forces do not interact with the physical system, so they are zero. With this, the forces in the system are:

$$F = -F_g + F_M = -mg + \frac{M}{x}, \tag{4.1.1}$$

where F is the force, m is the mass of the free magnet, g the gravitational acceleration, x the distance between the two magnets and M is the constant of the magnetic energy. If the system is in equilibrium, then the net force is zero ($F = 0$) and we can find the equilibrium distance.

$$\frac{M}{x} - mg = 0 \implies \frac{M}{x} = mg \implies \boxed{x_{eq} = \frac{M}{mg}} \quad , \tag{4.1.2}$$

as we can see, the equilibrium distance compares the magnetic energy with the gravitational force; then the increase or decrease in some of these magnitudes affects the equilibrium distance of the system, this lets us modulate the equilibrium distance.

4.1.1 Lagrangian analysis

The relevance of the lagrangian analysis is that we can study the dynamic of the system in terms of its coordinates and degrees of freedom. The lagrangian is described as the sum of the kinetic energies minus the potential energies present in the system.

$$L = T - V \implies L = \frac{1}{2}\sum_i m_i \dot{q_i}^2 - V(q_1, q_2, \ldots, q_i), \tag{4.1.3}$$

where m_i is the ith mass of the system, $\dot{q_i}$ is the ith speed of the system, V is the potential energy function and $q_1, q_2, \ldots, q_i$ are the coordinates of the system.

Our system only have a degree of freedom, so the dynamic of the system is described by one coordinate (x), then our lagrangian has the following expression.

$$L = \frac{1}{2}m\dot{x}^2 - V(x), \tag{4.1.4}$$

where $V(x)$ is the potential energy in our system. To find the potential energy, we are going to use the relation between the work and the potential energy when the system is conservative.

$$\begin{aligned} -\frac{d}{dx}V(x) = F \implies -V(x) = \int F dx \implies -V(x) = -\int mg dx + \int \frac{M}{x} dx \\ \implies -V(x) = -mg\int dx + M\int \frac{dx}{x} \implies -V(x) = -mgx + M\ln(x) \\ \implies -V(x) = -\left[mgx - M\ln(x)\right] \therefore \boxed{V(x) = mgx - M\ln(x)}. \end{aligned} \tag{4.1.5}$$

Substituting this result in our lagrangian function (equation (4.1.4)), we obtain that our lagrangian is:

$$\boxed{L = \frac{1}{2}m\dot{x}^2 - mgx + M\ln(x)}. \tag{4.1.6}$$

With our lagrangian, we can find the energy of the system and the equation of motion of the system.

First, we going to find the energy of the system. The energy of the syste is define as.

$$E = \sum_i \dot{q_i}\frac{\partial L}{\partial \dot{q_i}} - L \implies E = \dot{x}\frac{\partial L}{\partial \dot{x}} - L \quad \therefore \quad \boxed{E = \frac{1}{2}m\dot{x}^2 + mgx - M\ln(x)}. \tag{4.1.7}$$

Now, using the Euler-Lagrange equation we can find the equation of motion of our system. As we know, the external forces are null ($Q = 0$). So the equation of motion for the coordinate x is.

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = Q \implies m\ddot{x} + mg - \frac{M}{x} = 0$$
$$\implies \ddot{x} + g - \frac{M}{mx} = 0 \quad \therefore \quad \boxed{\frac{d^2x}{dt^2} + g - \frac{M}{mx} = 0}. \tag{4.1.8}$$

This equation of motion is relevant because it lets us know the accelerations that generate the movement in our system. It's important to remark that the differential equation does not have an analytical solution.

4.1.2 Hamiltonian analysis

Now that we know the equation of motion, it is important to see if the system really oscillates. To do this we going to use the hamiltonian analysis. To use this analysis we need to construct the hamiltonian of our system.

$$H = E = \frac{1}{2}m\dot{x}^2 + mgx - M\ln(x)$$
$$\boxed{\therefore\ H = \frac{p^2}{2m} + mgx - M\ln(x)}, \tag{4.1.9}$$

where H is the hamiltonian of the system and p is the momentum.

Now, using the Hamilton's equations we can find the dynamic of the system. The Hamilton's equations are:

$$\dot{H} = \begin{pmatrix} \dot{x} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} \dfrac{p}{m} \\ \dfrac{M}{x} - mg \end{pmatrix} \tag{4.1.10}$$

In the Hamilton's equations we can see that the change of momentum depends of position and vice versa. If we plot the solution of the Hamilton's equations we obtain a phase portrait of our system to see the behavior of it.

As we can see in the figure (4.2), the system follows a closed cyclical trajectory, so we can confirm that the system oscillates.

4.1.3 Dimensional analysis

Once we know the behavior of the system, it is important to analyse how. To do that we use de dimensional analysis. To use this analysis, we need to know how many variables have a relevant role in the dynamic of the system. For our systeme we have six variables ($n = 6$), and that variables are: the mass (m), the position (x), the time (t), gravity (g), the amplitude of the oscillation (A) and the magnetic energy (M).

The nature of our system lets us know that there are three physical dimensions ($j = 3$) in which the system acts. The physical dimensionss are: the space, the time and the mass.

Applying the Buckingham's Pi theorem, we can find the dimensionless groups that describe the system. For our problem, we know that we have three dimensionless groups ($k = n - j = 3$).

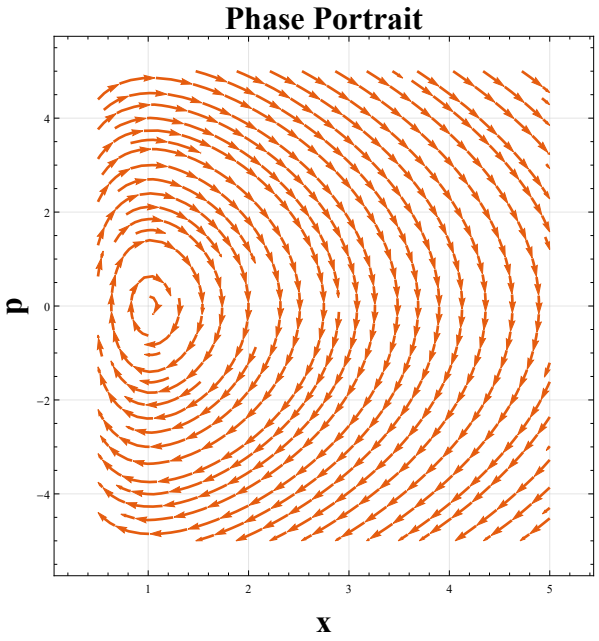


Figure 4.2: Phase portrait with $M = 10$, $g = 9.81$ and $m = 1$

To dimensionless our system, we need to do a dimensional analysis to every variable, so we obtain:

$$[x] = L, [A] = L, [t] = T, [m] = M, [g] = LT^{-2}, [M] = ML^2T^{-2} \tag{4.1.11}$$

If we select the variables m, g y M to dimensionless, we can find that every ith dimensionless set is the product of the powers of our variables that we use to dimensionless. Then, we have:

$$\Pi_i = \delta m^{\alpha} g^{\beta} M^{\lambda}, \tag{4.1.12}$$

where δ is the variable that we want to dimensioless, for our system we need to dimensionless the variables x, t y A; α, β and λ are the powers of the variables (m, g and M, in that order) that dimensionless the ith group.

We can write an equation system that lets us know how can be the powers (α, β and λ) that dimensionless every Pi group.

$$\begin{pmatrix} L \\ M \\ T \end{pmatrix} + \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ 0 & -2 & -2 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \lambda \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \implies \begin{pmatrix} \alpha \\ \beta \\ \lambda \end{pmatrix} = \begin{pmatrix} L + \frac{T}{2} - M \\ L + T \\ -L - \frac{T}{2}. \end{pmatrix} \tag{4.1.13}$$

It is necesary to clarify that every dimensionless group can be reduct to a fraction. The fraction is the relation between the variable of the system and its scale dimensionless factor. For that reason, the dimensionless groups can be write as:

$$\Pi_i = \frac{\delta}{\delta_s} = \tilde{\delta}, \tag{4.1.14}$$

where δ_s is the scale factor of our system's variable and $\tilde{\delta}$ is the dimensionless variable of the system.

Now, we can proceed to analyse the variables (x, t y A) of the system, with the intention to dimensionless the properties and the differential equation of the system.

Dimensionless group of the position:

As we see previously, x has only the space dimension ($L = 1$, $T = 0$, $M = 0$). Then, using the resul of the equation (4.1.13) we can find the powers of our dimensionless variables and the dimensionless group with the scale factor.

$$\begin{pmatrix}\alpha\\ \beta\\ \lambda\end{pmatrix} = \begin{pmatrix}1\\ 1\\ -1\end{pmatrix} \quad \therefore \quad \boxed{\Pi_1 = \frac{x}{x_s} = \frac{xmg}{M} = \frac{x}{x_{eq}} \implies x_s = \frac{M}{mg} = x_{eq}} \quad . \tag{4.1.15}$$

With the previous result we can see that the scale factor of the position is the equilibrium distance. Then, the position can be expressed as the product of the position's scale factor and the dimensionless position, as we can see in the following equation.

$$x = x_s\tilde{x} \tag{4.1.16}$$

If we differentiate the position, the differentiation applies only to the dimensionless position, because the scale factor is a constant. Then, we can express this as:

$$dx = d(x_s\tilde{x}) = x_s d\tilde{x} \implies d^2x = x_s d^2\tilde{x}. \tag{4.1.17}$$

This result is important because we going to use it to dimensionless the differential equation (the equation of motion obtained by the lagrangian analysis).

Dimensionless group of the time:

For the time, we have that the only dimension that affect it is the homologeous ($L = 0$, $T = 1$, $M = 0$). Then, substituting the values of the dimensions in the equation (4.1.13) we obtain the the Pi group and the time's scale factor.

$$\begin{pmatrix}\alpha\\ \beta\\ \lambda\end{pmatrix} = \begin{pmatrix}\frac{1}{2}\\ 1\\ -\frac{1}{2}\end{pmatrix} \quad \therefore \quad \boxed{\Pi_2 = \frac{t}{t_s} = tg\sqrt{\frac{m}{M}} \implies t_s = \frac{1}{g}\sqrt{\frac{M}{m}} = \sqrt{\frac{x_{eq}}{g}}} \quad . \tag{4.1.18}$$

For the previous solution, we can find the expressions for the scale factors of the frequency and the period of oscillation, that are:

$$\omega_s = \frac{1}{t_s} = \sqrt{\frac{g}{x_{eq}}} \quad \wedge \quad T_s = \frac{2\pi}{\omega_s} = 2\pi\sqrt{\frac{x_{eq}}{g}}. \tag{4.1.19}$$

The most important of these expressions is that they are equivalent to the frequency and the period of a pendulum, so we can conclude that this system is analogous to a pendulum.

We obtain this now by using the relation between the time variable with its scale factor and its dimensionless form.

$$t = t_s\tau \implies dt = t_s d\tau \implies dt^2 = t_s^2 d\tau^2 \tag{4.1.20}$$

Dimensionless group of the amplitude:

As the same as the position (x), the amplitude has the same dimensions ($L = 1, T = 0, M = 0$). Then the Pi group of the amplitude and its scale factor are:

$$\begin{pmatrix}\alpha\\ \beta\\ \lambda\end{pmatrix} = \begin{pmatrix}1\\ 1\\ -1\end{pmatrix} \quad \therefore \quad \boxed{\Pi_3 = \frac{A}{A_s} = \frac{Amg}{M} = \frac{A}{x_{eq}} \implies A_s = \frac{M}{mg} = x_{eq}} \quad . \tag{4.1.21}$$

From the above result, we can see that the amplitud, as the position, has a proportional relation with the equilibrium distance. Since the amplitude does not play a principal role in the equation of motion, we are going to use its dimensionless group as the correction factor (ϵ), then:

$$\epsilon = \frac{A}{x_{eq}} = \tilde{A}. \tag{4.1.22}$$

4.1.4 Dimensionless system

With all the dimensionless groups that we obtained previously, we can dimensionless the variables of our equation of motion obtained by the lagrangian (4.1.8). Then, our dimensionless equation of motion is:

$$\begin{aligned}
\frac{d^2x}{dt^2} + g - \frac{M}{mx} = 0 &\implies \frac{d^2(x_s\tilde{x})}{d(t_s\tau)^2} + g - \frac{M}{m(x_s\tilde{x})} = 0,\\
\frac{x_s}{t_s^2}\frac{d^2\tilde{x}}{d\tau^2} + g - \frac{M}{mx_s\tilde{x}} = 0 &\implies \frac{d^2\tilde{x}}{d\tau^2} + \frac{gt_s^2}{x_s} - \frac{Mt_s^2}{mx_s^2\tilde{x}} = 0,\\
\frac{d^2\tilde{x}}{d\tau^2} + \frac{gx_{eq}}{x_{eq}g} - \frac{Mx_{eq}}{mgx_{eq}^2\tilde{x}} = 0 &\implies \frac{d^2\tilde{x}}{d\tau^2} + 1 - \frac{M}{mgx_{eq}\tilde{x}} = 0,\\
\frac{d^2\tilde{x}}{d\tau^2} + 1 - \frac{x_{eq}}{x_{eq}\tilde{x}} = 0 &\implies \boxed{\frac{d^2\tilde{x}}{d\tau^2} + 1 - \frac{1}{\tilde{x}} = 0}.
\end{aligned} \tag{4.1.23}$$

If we multiply every term in our dimensionless differential equation to $\tilde{x}$, we can rewrite it as:

$$\frac{d^2\tilde{x}}{d\tau^2} + 1 - \frac{1}{\tilde{x}} = 0 \implies \boxed{\tilde{x}\frac{d^2\tilde{x}}{d\tau^2} + \tilde{x} - 1 = 0}. \tag{4.1.24}$$

It is relevant to see that our dimensionless differential equation is not homogeneous and non lineal, for this reason we can not solve it directly. Then, we are going to use an approximated solution. The approximated solution that we are going to applied is the asymptotic approximation, using ϵ to perturbate the original solution.

Once that we have our dimensionless differential equation, we are going to dimensionless the initial conditions of the system. Then, applying the dimensionless group of x in the initial condition of the position, we obtain:

$$x(0) = x_{eq} + A \implies x_s\tilde{x}(0) = x_{eq} + A \implies \tilde{x}(0) = \frac{x_{eq} + A}{x_s} \therefore \boxed{\tilde{x}(0) = 1 + \epsilon}. \quad (4.1.25)$$

Doing the same in the domains of x and A, the dimensionless domains are:

$$\begin{aligned} \text{Dom}(x) = (0, \infty) = \text{Dom}(x_s\tilde{x}) = x_{eq}\text{Dom}(\tilde{x}) &\implies \text{Dom}(\tilde{x}) = (0, \infty) \\ \text{Dom}(A) = (-x_{eq}, \infty) = \text{Dom}(A_s\tilde{A}) = x_{eq}\text{Dom}(\epsilon) &\implies \text{Dom}(\epsilon) = (-1, \infty) \end{aligned} \quad (4.1.26)$$

Now, we apply the differential property of x on the equation (4.1.17) to find the dimensionless form of the speed's initial condition.

$$\dot{x}(0) = 0 \implies x_s\dot{\tilde{x}} = 0 \therefore \boxed{\dot{\tilde{x}} = 0}. \quad (4.1.27)$$

4.1.5 Asymptotic approximation

As we have seen previously, the asymptotic approximation expresses the solution of a differential equation as a series. The approximated series or perturbed series is a sum of terms made up of a function (x_n) and a power of the control parameter (ϵ^n), and the objective is to balance the contribution of every term with the target to fit the perturbated series to the analytic solution. Then, the perturbated series is:

$$x \approx x_0 + x_1\epsilon + x_2\epsilon^2 + \cdots = \sum_{n=0}^{N} x_n\epsilon^n, \quad (4.1.28)$$

it is important to highlight that the approximation improves when the number of terms we use in the series is high. Another relevant observation is that we need to expand every function (x) with a series.

Using the perturbated series in our dimensionless equation of motion

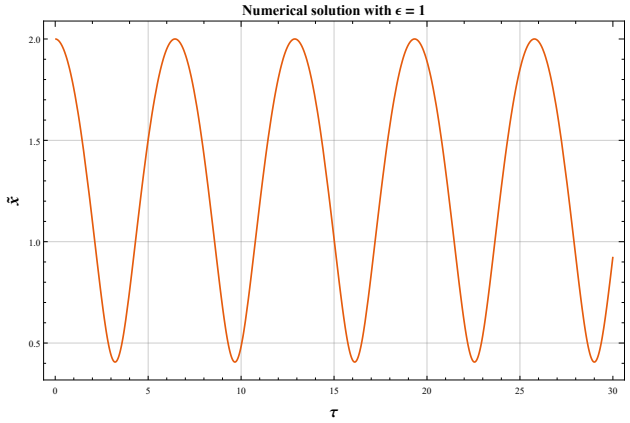
$$\sum_{n=0}^{N}\sum_{m=0}^{N} \epsilon^{n+m}\tilde{x}_n\frac{d^2\tilde{x}_m}{d\tau^2} = 1 - \sum_{j=0}^{N}\tilde{x}_j\epsilon^j. \quad (4.1.29)$$

4.1.6 Numerical solution

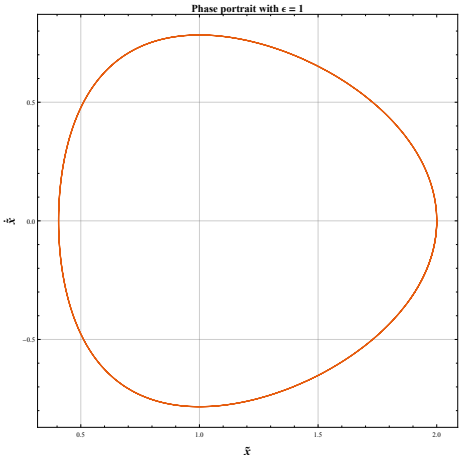
We know that our system does not have an analytical solution. It is for that reason that we need to use a numerical method to find a solution for the system and to contrast it with our perturbation solution.

We are going to use the Runge-Kutta method to find the numerical solution; and to do this, we are going to use the command ***NDSolve*** of the software ***Wolfram Mathematica***.

If we utilize $\epsilon = 1$ in our approximated solution, we get the plot that expresses the evolution of the system by time, and the parametric plot communicates the behavior of the system (this can be seen in figure 4.3).



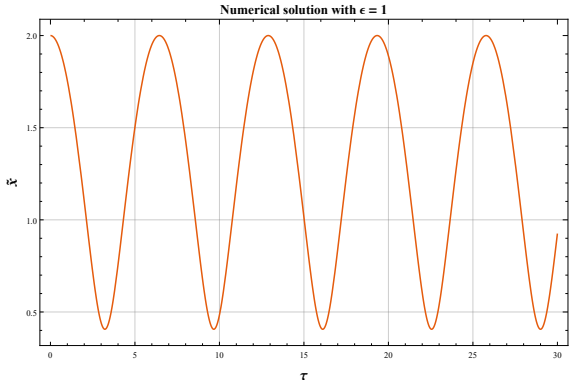
((a)) Time evolution



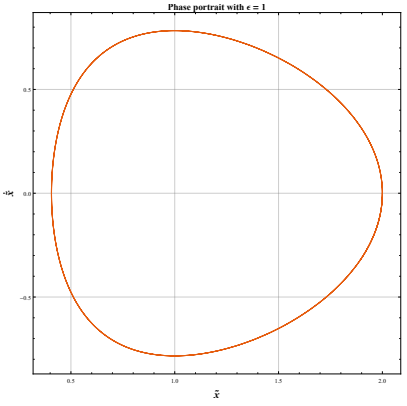
((b)) Phase portrait

Figure 4.3: Plot and parametric plot of the numerical solution with $\epsilon = 1$

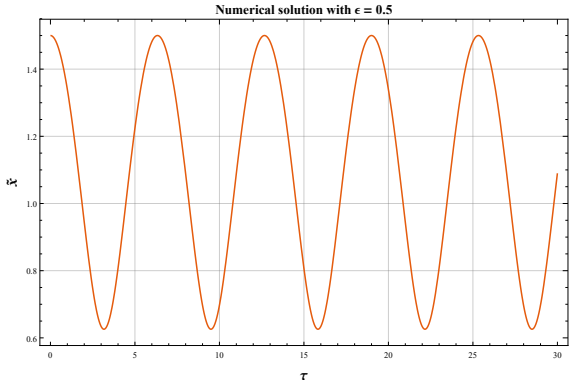
Something pertinent to highlight is how the system changes when we vary the value of ϵ. In figures (4.4 and 4.5), we can see these changes.



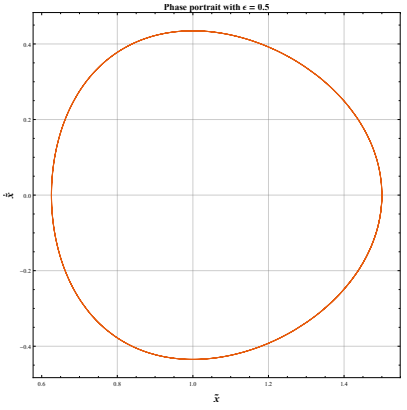
((a)) Time evolution ($\epsilon = 1$)



((b)) Phase portrait ($\epsilon = 1$)



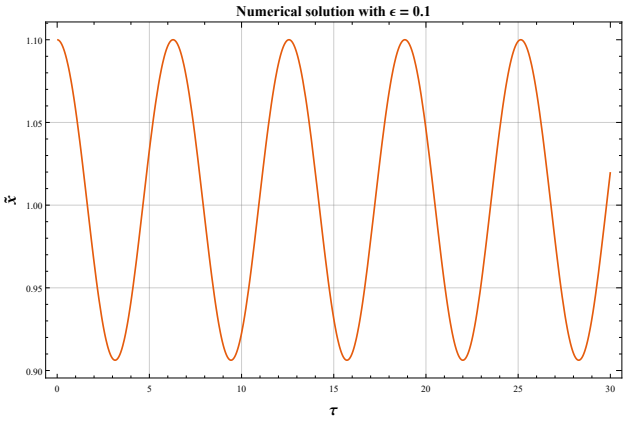
((c)) Time evolution ($\epsilon = 0.5$)



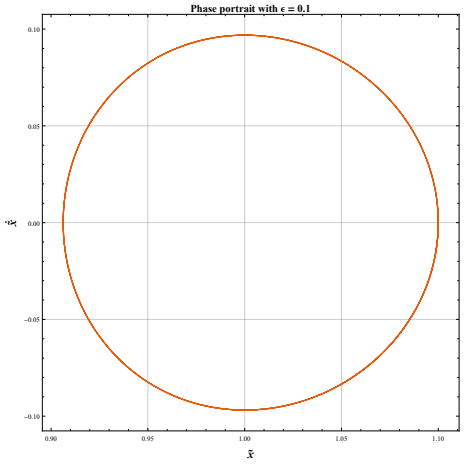
((d)) Phase portrait ($\epsilon = 0.5$)

Figure 4.4: System evolution and phase portraits for $\epsilon = 1$ and $\epsilon = 0.5$.

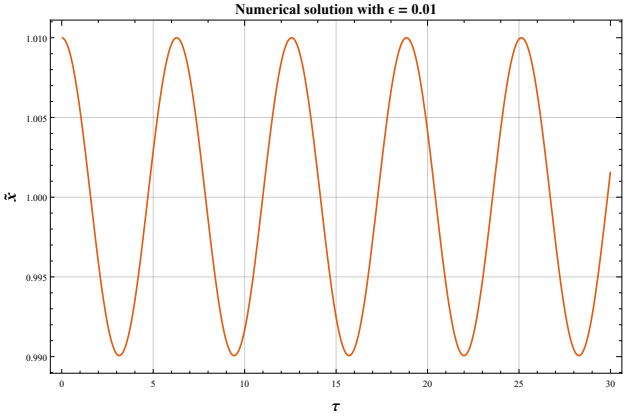
To previously figures named (figure (4.4) and figure (4.5)), we can see that when $\epsilon \approx 0$ is easier to fit a cyclical function, a cosine function.



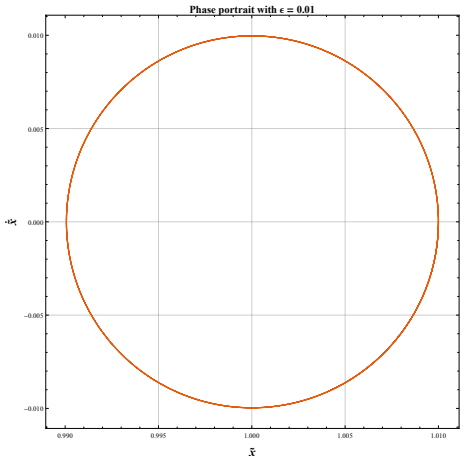
((a)) Time evolution ($\epsilon = 0.1$)



((b)) Phase portrait ($\epsilon = 0.1$)



((c)) Time evolution ($\epsilon = 0.01$)



((d)) Phase portrait ($\epsilon = 0.01$)

Figure 4.5: System evolution and phase portraits for $\epsilon = 0.1$ and $\epsilon = 0.01$.

4.1.7 Approximated solution

Zero order solution

The first step in our asymptotic approximation is to find the constant term of our series, to do this we need to solve the problem at zero order, that implies $\epsilon = 0$. Then the differential equation is:

$$\tilde{x}_0 \frac{d^2}{d\tau^2}\tilde{x}_0 = 1 - \tilde{x}_0 = 0 \implies \boxed{\tilde{x}_0 = 1}\,. \tag{4.1.30}$$

This result lets us know when the system is static, the position of the magnet is in the equilibrium distance.

First order solution

Now that we have the zero-order solution, we are going to find the first-order solution, e.i. , we need all the terms that are multiplied by ϵ. Then, the expression that we need to solve is:

$$\tilde{x}_0 \frac{d^2\tilde{x}_1}{d\tau^2} + \tilde{x}_1 \underbrace{\frac{d^2\tilde{x}_0}{d\tau^2}}_{0} = -\tilde{x}_1 \implies \frac{d^2\tilde{x}_1}{d\tau^2} = -\tilde{x}_1, \tag{4.1.31}$$

something important to highlight is that the natural frequency of the system is one ($\omega_0 = 1$). Solving the last differential equation we find that $\tilde{x}_1 = B_1 \cos(\tau) + B_2 \sin(\tau)$, therefore our series at first order is:

$$\tilde{x}(\tau) \approx 1 + \epsilon(B_1 \cos(\tau) + B_2 \sin(\tau)) \tag{4.1.32}$$

Using the initial conditions we can determinate the values of B_1 and B_2.

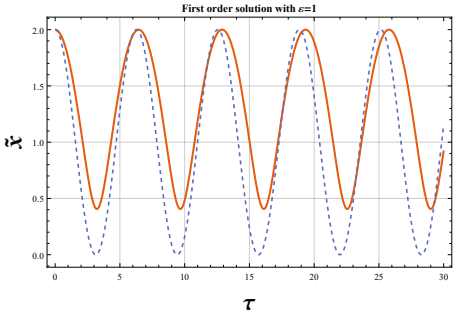
$$\tilde{x}(0) = 1 + \epsilon = 1 + \epsilon(B_1 \cos(0) + B_2 \sin(0)) \implies B_1 = 1, \tag{4.1.33}$$

$$\dot{\tilde{x}}(0) = \epsilon(-B_1 \sin(0) + B_2 \cos(0)) = 0 \implies B_2 = 0. \tag{4.1.34}$$

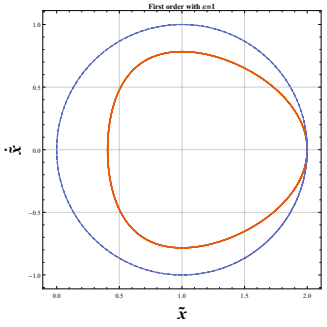
Then, our series at first order has the following form:

$$\tilde{x}(\tau) \approx 1 + \epsilon \cos(\tau). \tag{4.1.35}$$

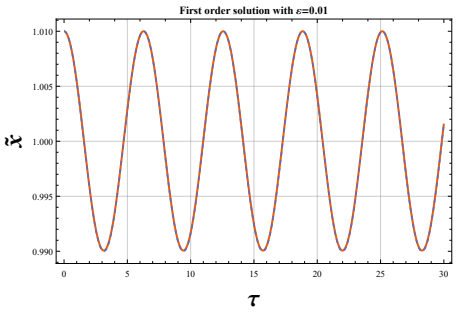
To see how approximate is our solution to the numerical solution, we have plotted both solutions with different values of ϵ to contrast their difference.



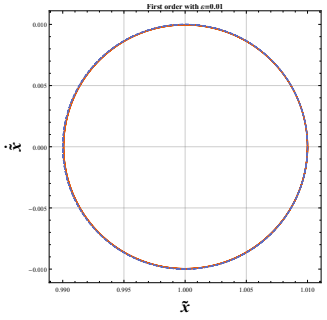
((a)) Time evolution contrast ($\epsilon = 1$)



((b)) Phase portrait contrast ($\epsilon = 1$)



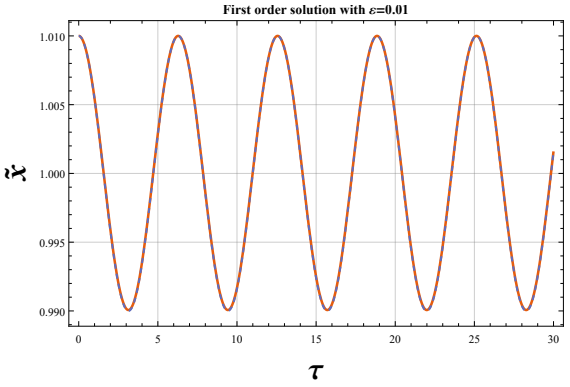
((c)) Time evolution ($\epsilon = 0.01$)



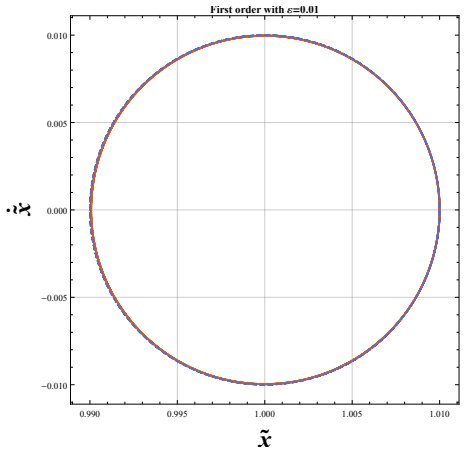
((d)) Phase portrait ($\epsilon = 0.01$)

Figure 4.6: Approximation of the solution to many values of ϵ.

If we analyze the previosly mentioned figure (figure 4.6), we can see when ϵ decrease, our perturbated series fits better with the numerical solution. Then, the first order of the series is a good approximation to a small perturbation ($|\epsilon| \ll 1$).



((a)) Time evolution ($\epsilon = 0.01$)



((b)) Phase portrait ($\epsilon = 0.01$)

Figure 4.7: Contrast of the numerical solution and the first order of the asymptotic solution with $\epsilon = 0.01$.

Second order solution

Once we have the first order solution, we can obtain the term of the second order. To do this we require every term that has a ϵ^2. Then, the dimensionless differential equation that we need to solve is:

$$\tilde{x}_0\frac{d^2\tilde{x}_2}{d\tau^2} + \tilde{x}_1\underbrace{\frac{d^2\tilde{x}_1}{d\tau^2}}_{-\tilde{x}_1} + \tilde{x}_2\underbrace{\frac{d^2\tilde{x}_0}{d\tau^2}}_{0} = -\tilde{x}_2 \implies \frac{d^2\tilde{x}_2}{d\tau^2} = \tilde{x}_1^2 - \tilde{x}_2 = \cos(\tau)^2 - \tilde{x}_2, \tag{4.1.36}$$

solving the differential equation we obtain $\tilde{x}_2(\tau) = \frac{1}{2} - \frac{1}{6}\cos(2\tau) + C\cos(\tau) + D\sin(\tau)$. Therefore, our solution series at second order is:

$$\tilde{x}(\tau) \approx 1 + \epsilon\cos(\tau) + \epsilon^2\left(\frac{1}{2} - \frac{1}{6}\cos(2\tau) + C\cos(\tau) + D\sin(\tau)\right) \tag{4.1.37}$$

Using the initial conditions we can find the values of the constants C and D:

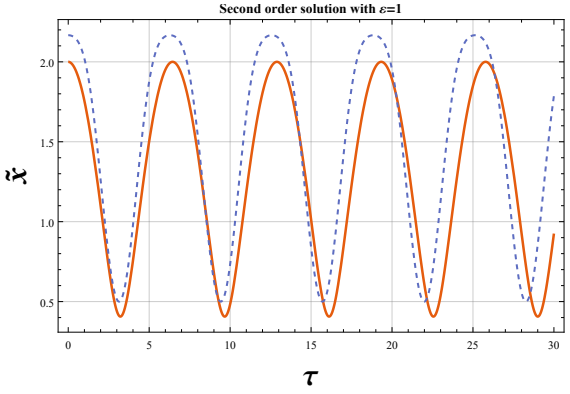
$$\begin{aligned}
\tilde{x}(0) = 1 + \epsilon\cos(0) + \epsilon^2\left(\frac{1}{2} - \frac{1}{6}\cos(0) + C\cos(0) + D\sin(0)\right) &= 1 + \epsilon, \\
1 + \epsilon + \epsilon^2\left(\frac{1}{2} - \frac{1}{6} + C\right) &= 1 + \epsilon, \\
\epsilon^2\left(\frac{1}{2} - \frac{1}{6} + C\right) &= 0, \\
\therefore C &= -\frac{1}{3}.
\end{aligned} \tag{4.1.38}$$

$$\begin{aligned}\dot{\tilde{x}}(0) = -\epsilon \sin(0) + \epsilon^2 \left(\frac{2}{6}\sin(0) - C\sin(0) + D\cos(0)\right) = 0 \\ D\epsilon^2 = 0 \\ \therefore D = 0.\end{aligned} \tag{4.1.39}$$

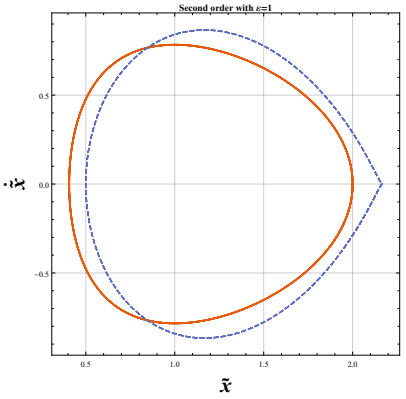
With the previous results the solution at second order is:

$$\tilde{x}(\tau) \approx 1 + \epsilon\cos(\tau) + \frac{\epsilon^2}{6}\left(3 - \cos(2\tau) - 2\cos(\tau)\right). \tag{4.1.40}$$

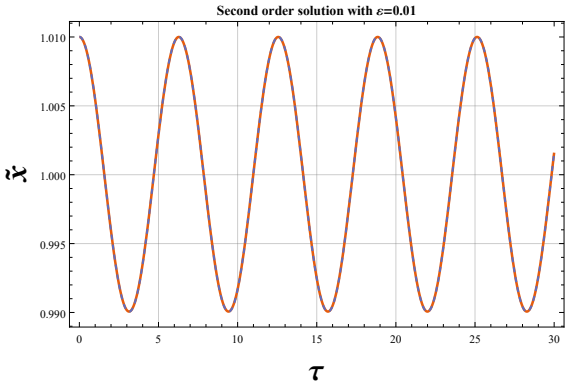
Now, we are going to compare the numerical and asymptotic solutions. To do that, we are going to plot both solutions. Those plots can be seen in figure (4.8). In that figure, we can see that the second order solution is more precise than the first order solution.



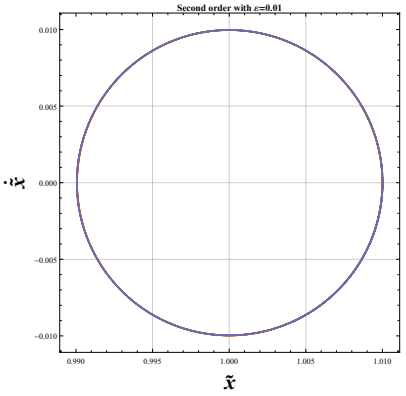
((a)) Time evolution contrast ($\epsilon = 1$)



((b)) Phase portrait contrast ($\epsilon = 1$)



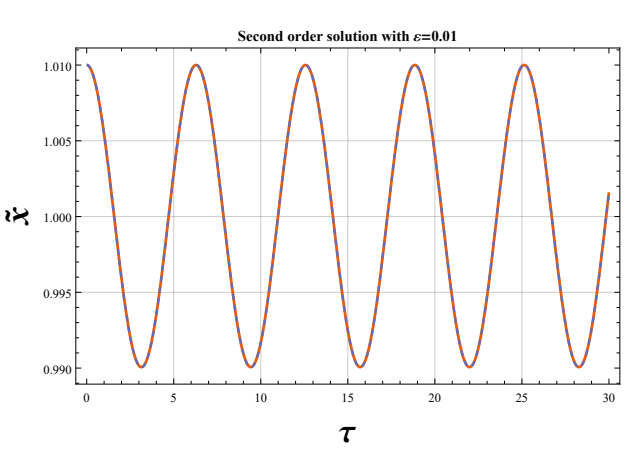
((c)) Time evolution ($\epsilon = 0.01$)



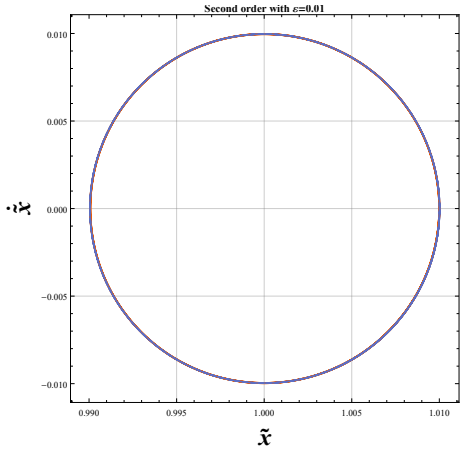
((d)) Phase portrait ($\epsilon = 0.01$)

Figure 4.8: Approximated solution at second order to the numerical solution with different values of ϵ.

We can see when ϵ decrease, the second order series fits better than the first order. The plotting of the second order series with $\epsilon = 0.01$ is:



((a)) Time evolution ($\epsilon = 0.01$)



((b)) Phase portrait ($\epsilon = 0.01$)

Figure 4.9: Comparison of the numerical solution with the second order solution and their parametric plots, with $\epsilon = 0.01$.

If we compare the figures (4.6) and (4.8), we can conclude that the second order solution is more precise than the first order to big values of ϵ. Therefore, to big values of ϵ we need to use more terms in our solution.

4.2 Damped system

For this case, we are going to consider the presence of a dissipative force and the absence of external forces. With these considerations, we have the following system of forces:

$$F = -F_g + F_m - F_f = -mg + \frac{M}{x} - \beta v, \tag{4.2.1}$$

where F is the net force, m is the mass of the free magnet, g is the gravitational acceleration, x the distance between the magnets, β is the mass flow rate coefficient, v the velocity and M is the constant of the magnetic energy.

If we consider the static case, where the net force is null ($F = 0$), we can find the equilibrium distance.

$$\frac{M}{x} - mg - \beta v = 0 \implies \frac{M}{x} = mg + \beta v \implies \boxed{x_{eq} = \frac{M}{mg + \beta v}}, \tag{4.2.2}$$

for the previous result, we can see that the equilibrium distance changes in the function of the velocity. This is because the damping will decrease the velocity of the system, causing the equilibrium point to change until the value of the velocity is zero and we recover the equilibrium position of the ideal system (equation (4.1.2)).

It is important to remark that the domain of the position (x) and the amplitude (A), and the initial conditions of the system are the same of the ideal system.

4.2.1 Lagrangian analysis

As we know, the lagrangian is a functional that acts under the supposition that the system is conservative. For this reason, we can not introduce in it the dissipative term in an

explicit form. To do this, we are going to separate the force's system into two parts, in conservative and in unconservative parts.

$$\underbrace{F + mg - \frac{M}{x}}_{\text{conservative}} + \underbrace{\beta v}_{\text{unconservative}} = 0. \tag{4.2.3}$$

Now that we have separated the force's system, we are going to apply the lagrangian analysis to the conservative part. If we do this the lagrangian of the conservative part is the lagrangian of the ideal system (equation (4.1.6)).

Now, to find the equation of motion of our damping system, we need to union the equation of motion of both parts (conservative and unconservative parts) as the sum of both. We know that the equation of motion of the conservative part is described by the Euler-Lagrange equation and the motion of the unconservative part is described by its differential form. Then, the equation of motion of our damping system is:

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} + \beta v = 0 \implies \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} + \beta \frac{dx}{dt} = 0 \\ \implies m\ddot{x} + mg - \frac{M}{x} + \beta \frac{dx}{dt} = 0 \implies \ddot{x} + g - \frac{M}{mx} + \frac{\beta}{m}\frac{dx}{dt} = 0, \end{aligned} \tag{4.2.4}$$

if we define the β coefficient as $\beta = 2m\phi$, where ϕ is the friction coefficient. Then, the equation of motion has the following form.

$$\ddot{x} + g - \frac{M}{mx} + 2\phi\frac{dx}{dt} = 0 \implies \boxed{\frac{d^2x}{dt^2} + 2\phi\frac{dx}{dt} + g - \frac{M}{mx} = 0}. \tag{4.2.5}$$

4.2.2 Hamiltonian

However, when we introduce damping, the total mechanical energy is no longer conserved, and this breaks the traditional Hamiltonian structure. It is important to remember that a Hamiltonian system must preserve energy unless it depends explicitly on time. For this reason, we cannot construct a time-independent Hamiltonian for a damped oscillator in the usual way.

To solve this problem, we introduce the **Caldirola–Kanai formalism**. This method modifies the Lagrangian with an exponential factor that grows in time. At first sight this may look artificial, but its purpose is simple: the exponential factor "restores" the conservation structure inside the variational formulation, allowing the Euler–Lagrange equations to recover the correct damping term.

Caldirola-Kanai Lagrangian

We start from the equation of motion of the damped oscillator:

$$m\ddot{x} + \beta m\,\dot{x} + V'(x) = 0, \tag{4.2.6}$$

where ϕ represents the damping coefficient. The Caldirola-Kanai proposal consists in defining the Lagrangian

$$L_{CK}(x, \dot{x}, t) = e^{\beta t}\left(\frac{1}{2}m\dot{x}^2 - V(x)\right) = L(x, \dot{x}, t). \tag{4.2.7}$$

Now, the Euler–Lagrange equation naturally generates the correct dissipative term. The exponential factor is the key that allows us to keep the action principle while still describing a system that loses energy.

Canonical momentum

The next step is to compute the canonical momentum associated with the coordinate x:

$$p = \frac{\partial L}{\partial \dot{x}} = m e^{\beta t} \dot{x}. \tag{4.2.8}$$

Here it is important to clarify something: the canonical momentum p is not the same as the physical momentum mv. Instead, both are related by

$$p = mv\, e^{\beta t}. \tag{4.2.9}$$

Solving for the velocity gives

$$\dot{x} = \frac{p}{m} e^{-\beta t}. \tag{4.2.10}$$

Construction of the Hamiltonian

Once we know the canonical momentum, we can construct the Hamiltonian using the definition

$$H(x, p, t) = p\dot{x} - L. \tag{4.2.11}$$

To do this, we first rewrite the Lagrangian in terms of x and p. Using $\dot{x} = \frac{p}{m} e^{-\beta t}$ we obtain

$$L = \frac{p^2}{2m} e^{-\beta t} - V(x) e^{\beta t}. \tag{4.2.12}$$

Substituting into the Legendre transform, we get

$$\begin{aligned} H &= p\left(\frac{p}{m} e^{-\beta t}\right) - \left(\frac{p^2}{2m} e^{-\beta t} - V(x) e^{\beta t}\right) \\ &= \frac{p^2}{2m} e^{-\beta t} + V(x) e^{\beta t}. \end{aligned} \tag{4.2.13}$$

We know that our potential energy $V(x) = mgx - M \ln(x)$. Therefore, the Hamiltonian of the damped oscillator is

$$\boxed{H(x, p, t) = \frac{p^2}{2m} e^{-\beta t} + \left(mgx - M \ln(x)\right) e^{\beta t}.} \tag{4.2.14}$$

Notice that this Hamiltonian depends explicitly on time. This is exactly what we expect in a system that loses mechanical energy.

Hamilton's equations and the phase portrait

Now we apply Hamilton's equations. The first one is

$$\dot{x} = \frac{\partial H}{\partial p} = \frac{p}{m} e^{-\beta t}. \tag{4.2.15}$$

The second one is

$$\dot{p} = -\frac{\partial H}{\partial x} = -V'(x)\, e^{\beta t}. \tag{4.2.16}$$

Therefore, the phase portrait of the Caldirola–Kanai oscillator is determined by the system

$$\boxed{\begin{aligned} \dot{x} &= \frac{p}{m} e^{-\beta t}, \\ \dot{p} &= -\left(mg - \frac{M}{x} \right) e^{\beta t}. \end{aligned}} \tag{4.2.17}$$

Something important to highlight is that this Hamiltonian describes the conservative system and does not reflect the physical behavior of the system, but lets us know how the energy depends on time.

If we want to see the behavior of the system by a phase portrait, we can use our Lagrangian. Then, our phase portrait is.

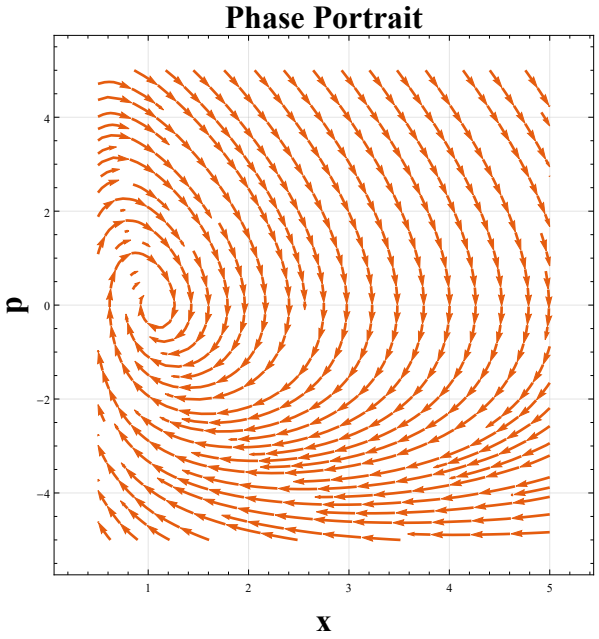


Figure 4.10: Phase portrait with $M = 10$, $g = 9.81$, $\phi = 0.75$ and $m = 1$

4.2.3 Dimensional analysis

We are going to use the same dimensionless groups that we find in the ideal system (section 4.1.3), but we only need to find the dimensionless group of the friction number.

As in the ideal case, we use the variables m, g, and M to dimensionless our system. Doing the dimensional analysis of these variables and the friction number.

$$[\phi] = T^{-1}, [m] = M, [g] = LT^{-2}, [M] = ML^2T^{-2} \tag{4.2.18}$$

With this, we can find the dimensionless groups of the friction number as the product of the powers of our variables that we use to dimensionless. Then, we have:

$$\Pi_i = \delta m^{\alpha} g^{\beta} M^{\lambda}, \tag{4.2.19}$$

where δ is the variable that we want to make dimensionless. Now, using the equation (4.1.13), we can find the powers of our variables to build our dimensionless groups. This equation is:

$$\begin{pmatrix} \alpha \\ \beta \\ \lambda \end{pmatrix} = \begin{pmatrix} L + \frac{T}{2} - M \\ L + T \\ -L - \frac{T}{2}. \end{pmatrix} \tag{4.2.20}$$

Now, we can proceed to analyze our new variable (ϕ) of the system.

Friction group

Now, we are going to work with the friction number (ϕ) that only works with one dimension ($L = 0$, $T = -1$, $M = 0$). Then, using our equation to find the powers (equation (4.3.8)), we obtain that our dimensionless group and its scale factor are.

$$\begin{pmatrix} \alpha \\ \beta \\ \lambda \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ -1 \\ \frac{1}{2} \end{pmatrix} \quad \therefore \quad \boxed{\Pi_4 = \frac{\phi}{\phi_s} = \frac{\phi}{g}\sqrt{\frac{M}{m}} \implies \phi_s = \frac{1}{t_s} = g\sqrt{\frac{m}{M}} = \sqrt{\frac{g}{x_{eq}}}} \quad . \tag{4.2.21}$$

This result is particular because it is the frequency of the pendulum. Then, the frequency is a factor of damping. This could be explained like the friction acts more when the frequency is higher.

Now, we define the dimensionless friction number. To do this, we use the dimensionless group of the frequency ($\Pi_4 = \Phi$). Now, we can express the frequency in terms of the dimensionless frequency, as we can see in the following expression.

$$\Pi_4 = \frac{\phi}{\phi_s} \implies \phi = \phi_s \Phi = \frac{1}{t_s}\Phi \tag{4.2.22}$$

4.2.4 Dimensionless system

Now, using the dimensional groups, we are going to dimensionless the equation of motion (4.2.5). Then, we obtain:

$$\begin{aligned}
\frac{d^2x}{dt^2} + 2\phi\frac{dx}{dt} + g - \frac{M}{mx} &= 0 \\
\implies \frac{d^2(x_s\tilde{x})}{d(t_s\tau)^2} + 2\phi\frac{d(x_s\tilde{x})}{d(t_s\tau)} + g - \frac{M}{mx_s\tilde{x}} &= 0 \\
\implies \frac{x_s}{t_s^2}\frac{d^2\tilde{x}}{d\tau^2} + \frac{2(\phi_s\Phi)x_s}{t_s}\frac{d\tilde{x}}{d\tau} + g - \frac{M}{mx_s\tilde{x}} &= 0 \\
\implies \frac{d^2\tilde{x}}{d\tau^2} + 2\Phi\frac{d\tilde{x}}{d\tau} + \frac{gt_s^2}{x_s} - \frac{Mt_s^2}{mx_s^2\tilde{x}} &= 0 \\
\implies \frac{d^2\tilde{x}}{d\tau^2} + 2\Phi\frac{d\tilde{x}}{d\tau} + 1 - \frac{M}{mgx_{eq}\tilde{x}} &= 0 \\
\implies \frac{d^2\tilde{x}}{d\tau^2} + 2\Phi\frac{d\tilde{x}}{d\tau} + 1 - \frac{1}{\tilde{x}} &= 0.
\end{aligned} \tag{4.2.23}$$

Multiplying all the terms by $\tilde{x}$.

$$\frac{d^2\tilde{x}}{d\tau^2} + 2\Phi\frac{d\tilde{x}}{d\tau} + 1 - \frac{1}{\tilde{x}} = 0 \implies \tilde{x}\frac{d^2\tilde{x}}{d\tau^2} + 2\Phi\tilde{x}\frac{d\tilde{x}}{d\tau} + \tilde{x} - 1 = 0. \tag{4.2.24}$$

It is important to highlight that the domains of $\tilde{x}$ and ϵ and the initial conditions are the same of the ideal system.

4.2.5 Asymptotic solution

Using the asymptotic series (4.1.28) in our differential equation, we can find the perturbed series that allows us to approximate the solution.

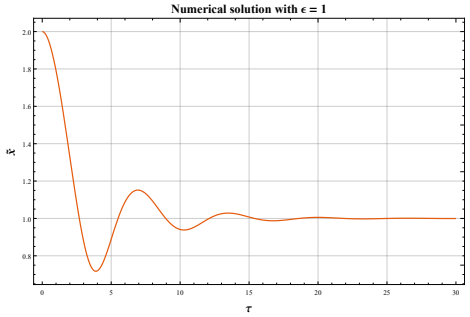
$$\begin{aligned}
&\sum_{i=0}^{N}\left(\epsilon^i\tilde{x}_i\right)\left(\sum_{j=0}^{N}\epsilon^j\frac{d^2\tilde{x}_j}{d\tau^2}\right) + 2\Phi\left(\sum_{k=0}^{N}\epsilon^k\tilde{x}_k\right)\left(\sum_{\ell=0}^{N}\epsilon^\ell\frac{d\tilde{x}_\ell}{d\tau}\right) + \sum_{m=0}^{N}\epsilon^m\tilde{x}_m - 1 = 0 \\
&\implies \sum_{i=0}^{N}\sum_{j=0}^{N}\epsilon^{i+j}\tilde{x}_i\frac{d^2\tilde{x}_j}{d\tau^2} + 2\Phi\sum_{k=0}^{N}\sum_{\ell=0}^{N}\epsilon^{k+\ell}\tilde{x}_k\frac{d\tilde{x}_\ell}{d\tau} + \sum_{m=0}^{N}\epsilon^m\tilde{x}_m - 1 = 0.
\end{aligned} \tag{4.2.25}$$

4.2.6 Numerical solution

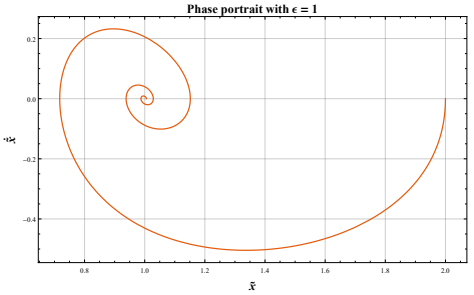
As we have seen in the previous section, we used the Runge-Kutta method to find the numerical solution; the only difference is that in our differential equation, we add the damping term.

As we can see, the damping term caused a decrease in the amplitude of the oscillation. For this numerical simulation, we are going to use $\Phi = \frac{1}{4}$.

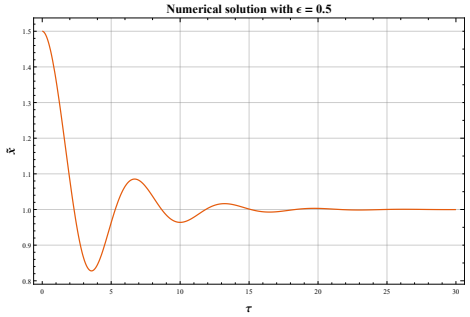
The same as in the ideal case, we will show the evolution of the system when ϵ changes its value. We are going to use the following values of ϵ: 1, 0.5, 0.1, and 0.01. This can be shown in figures 4.11 and 4.12



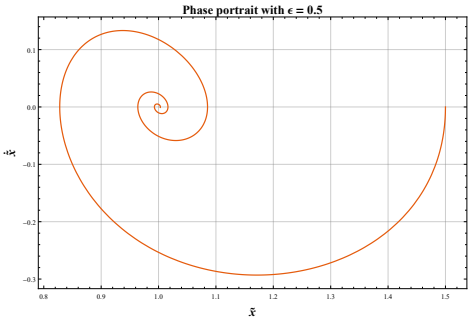
((a)) Time evolution ($\epsilon = 1$)



((b)) Phase portrait ($\epsilon = 1$)

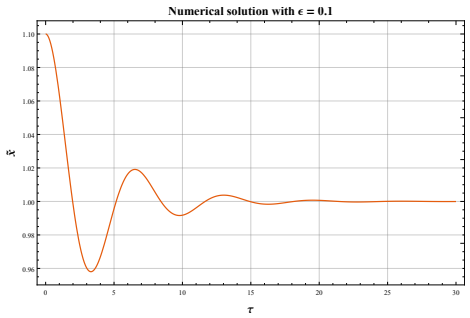


((c)) Time evolution ($\epsilon = 0.5$)

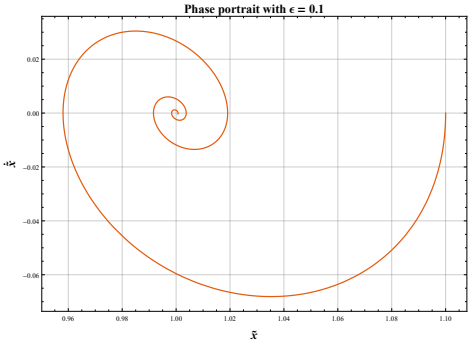


((d)) Phase portrait ($\epsilon = 0.5$)

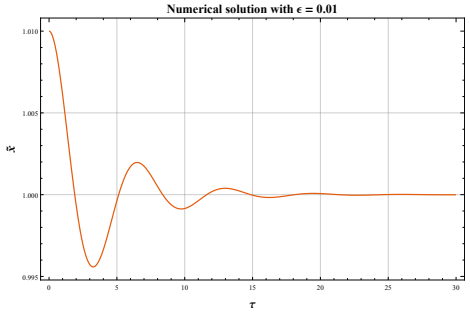
Figure 4.11: System evolution and phase portraits for $\epsilon = 1$ and $\epsilon = 0.5$.



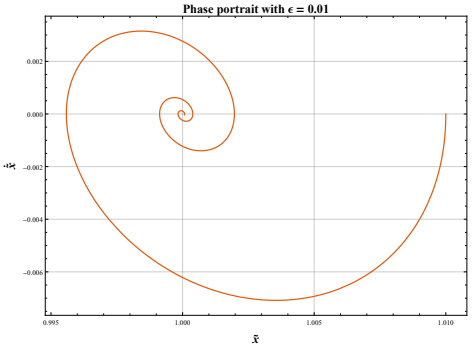
((a)) Time evolution ($\epsilon = 0.1$)



((b)) Phase portrait ($\epsilon = 0.1$)



((c)) Time evolution ($\epsilon = 0.01$)



((d)) Phase portrait ($\epsilon = 0.01$)

Figure 4.12: System evolution and phase portraits for $\epsilon = 0.1$ and $\epsilon = 0.01$.

4.2.7 Approximated solution

Zero order solution

To generate our approximation, we need to find the different order terms of our approximated series. So firstly, we are going to find the constant term, or in other words, the zero-order term. Then, we require using all the terms that have ϵ^0.

$$\begin{aligned} \epsilon^{0+0}\tilde{x}_0\frac{d^2\tilde{x}_0}{d\tau^2} + 2\Phi\epsilon^{0+0}\tilde{x}_0\frac{d\tilde{x}_0}{d\tau} + \epsilon^0\tilde{x}_0 - 1 = 0 \\ \tilde{x}_0\underbrace{\frac{d^2\tilde{x}_0}{d\tau^2}}_{0} + 2\Phi\tilde{x}_0\underbrace{\frac{d\tilde{x}_0}{d\tau}}_{0} + \tilde{x}_0 - 1 = 0 \\ \therefore \tilde{x}_0 = 1. \end{aligned} \tag{4.2.26}$$

Then, our series approximation at zero-order is:

$$\tilde{x}(t) \approx 1 \tag{4.2.27}$$

First order solution

Now, to build our first-order term of the series approximation, we only need the terms that have a multiple of ϵ^1 and solve the respective differential equation.

$$\begin{aligned} \sum_{i=0}^{1}\sum_{j=0}^{1}\epsilon^{i+j}\tilde{x}_i\frac{d^2\tilde{x}_j}{d\tau^2} + 2\Phi\sum_{k=0}^{1}\sum_{\ell=0}^{1}\epsilon^{k+\ell}\tilde{x}_k\frac{d\tilde{x}_\ell}{d\tau} + \sum_{m=0}^{1}\epsilon^m\tilde{x}_m - 1 = 0 \\ \epsilon^{1+0}\tilde{x}_1\frac{d^2\tilde{x}_0}{d\tau^2} + \epsilon^{0+1}\tilde{x}_0\frac{d^2\tilde{x}_1}{d\tau^2} + 2\Phi\epsilon^{1+0}\tilde{x}_1\frac{d\tilde{x}_0}{d\tau} + 2\Phi\epsilon^{0+1}\tilde{x}_0\frac{d\tilde{x}_1}{d\tau} + \epsilon^1\tilde{x}_1 = 0 \\ \tilde{x}_1\underbrace{\frac{d^2\tilde{x}_0}{d\tau^2}}_{0} + \underbrace{\tilde{x}_0}_{1}\frac{d^2\tilde{x}_1}{d\tau^2} + 2\Phi\left(\tilde{x}_1\underbrace{\frac{d\tilde{x}_0}{d\tau}}_{0} + \underbrace{\tilde{x}_0}_{1}\frac{d\tilde{x}_1}{d\tau}\right) + \tilde{x}_1 = 0 \\ \frac{d^2\tilde{x}_1}{d\tau^2} + 2\Phi\frac{d\tilde{x}_1}{d\tau} + \tilde{x}_1 = 0. \end{aligned} \tag{4.2.28}$$

The differential equation that we have obtained is the differential equation of a damped oscillator. To solve this differential equation, we need the characteristic equation.

$$s_{1,2} = \frac{-2\Phi \pm \sqrt{4\Phi^2 - 4}}{2} \implies s_{1,2} = -\Phi \pm \underbrace{\sqrt{\Phi^2 - 1}}_{\Delta}, \tag{4.2.29}$$

from the characteristic solutions, we can see that our determinant Δ defines the type of oscillator that we have. If we change the signs in our determinant, we can build a new determinant, so we obtain.

$$\Delta = \sqrt{\Phi^2 - 1} = \sqrt{-1(1 - \Phi^2)} = i\underbrace{\sqrt{1 - \Phi^2}}_{\eth} \implies \Delta = i\eth, \tag{4.2.30}$$

the new determinant $\eth$, depends of Φ. The oscillator is underdamped when $\Phi < 1$, is critically damped when $\Phi = 1$, and is overdamped when $\Phi > 1$. Then our characteristic solution can be expressed with a complex term as.

$$s_{1,2} = -\Phi \pm \Delta = -\Phi \pm i\eth \tag{4.2.31}$$

If we solve the differential equation, and we apply Euler's complex relation, we find that our solution is.

$$\begin{aligned} \tilde{x}_1(\tau) =& Ae^{s_1\tau} + Be^{s_2\tau} \\ =& e^{-\Phi\tau}\left[Ae^{i\eth\tau} + Be^{-i\eth\tau}\right] \\ \therefore \tilde{x}_1(\tau) =& e^{-\Phi\tau}\left[A\cos(\eth\tau) + B\sin(\eth\tau)\right]. \end{aligned} \tag{4.2.32}$$

Now, applying the initial conditions ($\tilde{x}_1(\tau) = 1$ and $\dot{\tilde{x}}_1(0) = 0$), we can find the values of our constants. Then, with the initial position ($\tilde{x}_1(\tau) = 1$), we have the folowing result.

$$\begin{aligned} \tilde{x}_1(0) =& 1 \\ e^0\left[A\cos(0) + B\sin(0)\right] =& 1 \\ 1\left[A + 0\right] =& 1 \\ A =& 1 \end{aligned} \tag{4.2.33}$$

Finding the speed expression.

$$\begin{aligned} \dot{\tilde{x}}_1(\tau) =& \frac{d}{d\tau}(\tilde{x}_1(\tau)) = \frac{d}{d\tau}\left\{e^{-\Phi\tau}\left[A\cos(\eth\tau) + B\sin(\eth\tau)\right]\right\} \\ =& -\Phi e^{-\Phi\tau}\left[A\cos(\eth\tau) + B\sin(\eth\tau)\right] + \eth e^{-\Phi\tau}\left[-A\sin(\eth\tau) + B\cos(\eth\tau)\right] \\ \therefore \dot{\tilde{x}}_1(\tau) =& e^{-\Phi\tau}\left[(\eth B - \Phi A)\cos(\eth\tau) - (\eth A + \Phi B)\sin(\eth\tau)\right] \end{aligned} \tag{4.2.34}$$

Using the initial speed condition ($\dot{\tilde{x}}_1(0) = 0$) we obtain.

$$\begin{aligned} \dot{\tilde{x}}_1(0) =& 0 \\ 0 =& e^0\left[(\eth B - \Phi A)\cos(0) - (\eth A + \Phi B)\sin(0)\right] \\ 0 =& \eth B - \Phi A \\ \therefore B =& \frac{\Phi}{\eth} = \frac{\Phi}{\sqrt{1-\Phi^2}}, \end{aligned} \tag{4.2.35}$$

something important to see of the B constant is that $\Phi \neq 1$, because if $\Phi = 1$ the solution is undefined. So we can not work with a critically damped oscillator.

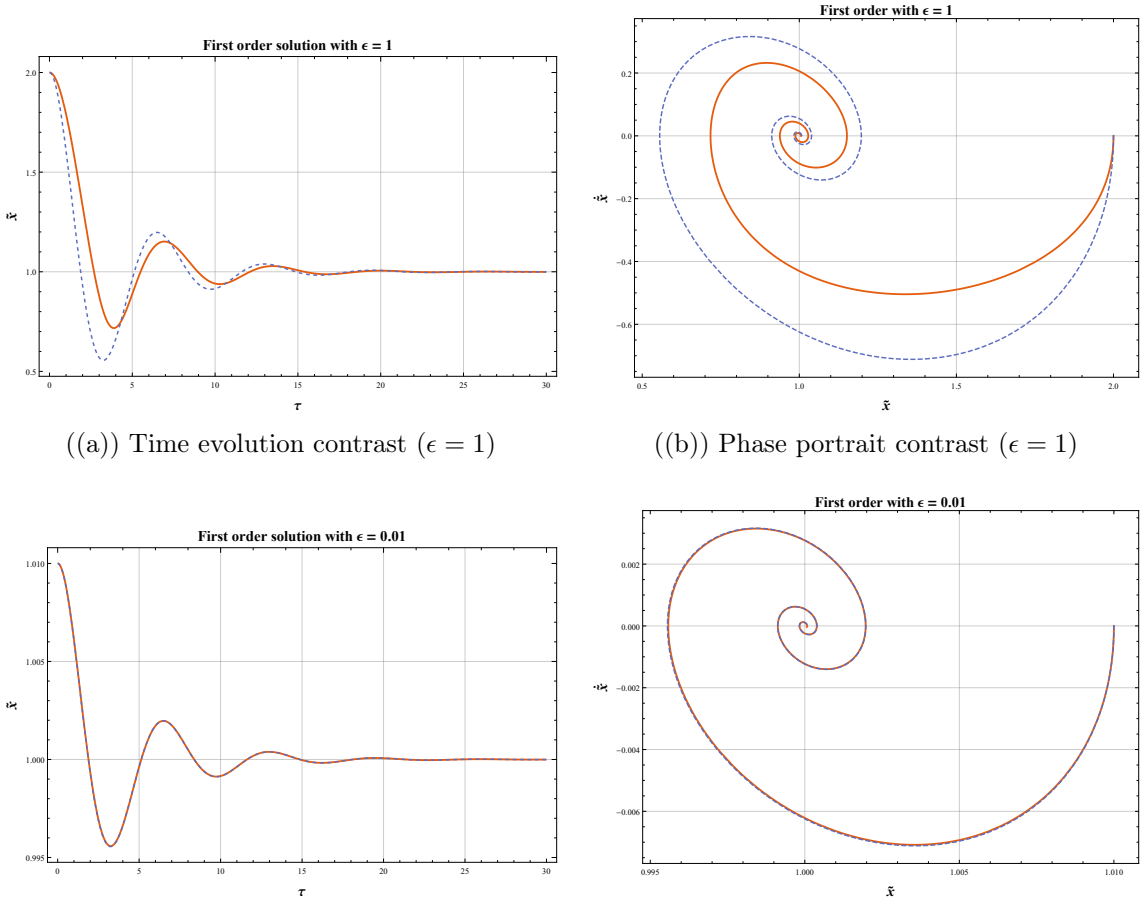
If we substitute the value of the B constant in the approximation of the first order term, we have it as.

$$\tilde{x}_1(\tau) = e^{-\Phi\tau}\left[\cos\left(\eth\tau\right) + \frac{\Phi}{\eth}\sin\left(\eth\tau\right)\right]. \tag{4.2.36}$$

And our approximated series solution at first order is:

$$\tilde{x} \approx \tilde{x}_0 + \epsilon\tilde{x}_1 = 1 + \epsilon e^{-\Phi\tau}\left[\cos\left(\eth\tau\right) + \frac{\Phi}{\eth}\sin\left(\eth\tau\right)\right]. \tag{4.2.37}$$

Now, we plot the numerical and our approximation to contrast the effectiveness of our approximation, we can see this in the following figure.



((a)) Time evolution contrast ($\epsilon = 1$)

((b)) Phase portrait contrast ($\epsilon = 1$)

((c)) Time evolution ($\epsilon = 0.01$)

((d)) Phase portrait ($\epsilon = 0.01$)

Figure 4.13: Approximation of the solution of $\epsilon = 1$ and $\epsilon = 0.01$.

Second-order solution

To find the second-order term of our approximated series, we only need all the terms that contain ϵ^2. So, checking the equation (4.2.25) we find the following terms.

$$\begin{aligned} 0 =& \epsilon^{2+0}\tilde{x}_2\frac{d^2\tilde{x}_0}{d\tau^2} + \epsilon^{1+1}\tilde{x}_1\frac{d^2\tilde{x}_1}{d\tau^2} + \epsilon^{0+2}\tilde{x}_0\frac{d^2\tilde{x}_2}{d\tau^2} \\ &+ 2\Phi\left[\epsilon^{2+0}\tilde{x}_2\frac{d\tilde{x}_0}{d\tau} + \epsilon^{1+1}\tilde{x}_1\frac{d\tilde{x}_1}{d\tau} + \epsilon^{0+2}\tilde{x}_0\frac{d\tilde{x}_2}{d\tau}\right] + \epsilon^2\tilde{x}_2. \end{aligned} \tag{4.2.38}$$

Using the definition of $\tilde{x}_0 = 1$, we can simplify our equation to.

$$\tilde{x}_1 \frac{d^2\tilde{x}_1}{d\tau^2} + \frac{d^2\tilde{x}_2}{d\tau^2} + 2\Phi \left[\tilde{x}_1 \frac{d\tilde{x}_1}{d\tau} + \frac{d\tilde{x}_2}{d\tau}\right] + \tilde{x}_2 = 0 \tag{4.2.39}$$

Now, using the definition of $\tilde{x}_1$ and its derivatives, we can rewrite our differential equation as.

$$\begin{aligned} \tilde{x}_1 \left[-\tilde{x}_1 - 2\Phi \frac{d\tilde{x}_1}{d\tau}\right] + \frac{d^2\tilde{x}_2}{d\tau^2} + 2\Phi \left[\tilde{x}_1 \frac{d\tilde{x}_1}{d\tau} + \frac{d\tilde{x}_2}{d\tau}\right] + \tilde{x}_2 = 0 \\ \implies -\tilde{x}_1^2 + \frac{d^2\tilde{x}_2}{d\tau^2} + 2\Phi \frac{d\tilde{x}_2}{d\tau} + \tilde{x}_2 = 0 \\ \therefore \frac{d^2\tilde{x}_2}{d\tau^2} + 2\Phi \frac{d\tilde{x}_2}{d\tau} + \tilde{x}_2 = \tilde{x}_1^2 \end{aligned} \tag{4.2.40}$$

Finding the characteristic equation of the homogeneous part.

$$\begin{aligned} 0 =& \frac{d^2\tilde{x}_2}{d\tau^2} + 2\Phi \frac{d\tilde{x}_2}{d\tau} + \tilde{x}_2 \implies s_{1,2} = -\Phi \pm \sqrt{\Phi^2 - 1} = -\Phi \pm \Delta \\ \therefore s_{1,2} =& -\Phi \pm \sqrt{(-1)(1-\Phi^2)} = -\Phi \pm i\eth \end{aligned} \tag{4.2.41}$$

Then, the solution of the homogeneous part is.

$$\begin{aligned} \tilde{x}_{2h} =& Ae^{s_1\tau} + Be^{s_2\tau} = Ae^{-\Phi\tau + i\eth\tau} + Be^{-\Phi\tau - i\eth\tau} = e^{-\Phi\tau}\left[Ae^{i\eth\tau} + Be^{-i\eth\tau}\right] \\ \therefore \tilde{x}_{2h} =& e^{-\Phi\tau}\left[A\cos(\eth\tau) + B\sin(\eth\tau)\right] \end{aligned} \tag{4.2.42}$$

Now, solving the particular solution of our system. Applying the method of undetermined coefficients, we solve for the steady-state particular response $\tilde{x}_{2p}(\tau)$. Utilizing the algebraic identity of the subamortized regime to simplify the denominators, we obtain a remarkably compact and physically meaningful particular solution:

$$\tilde{x}_{2p}(\tau) = e^{-2\Phi\tau}\left[\mathcal{C}_{\text{dc}} + \mathcal{C}_{\sin}\sin(2\eth\tau) + \mathcal{C}_{\cos}\cos(2\eth\tau)\right] \tag{4.2.43}$$

where the constant coefficients $\mathcal{C}_{\text{dc}}$, $\mathcal{C}_{\sin}$, and $\mathcal{C}_{\cos}$ are explicitly determined by the dimensionless friction number Φ and the frequency parameter $\eth$ as:

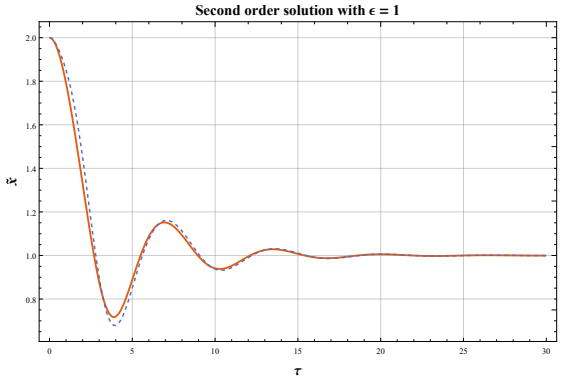
$$\mathcal{C}_{\text{dc}} = \frac{1}{2\eth^2} = \frac{1}{2(1-\Phi^2)} \tag{4.2.44}$$

$$\mathcal{C}_{\sin} = \frac{\Phi(8\Phi^2 - 5)}{\eth(9 - 8\Phi^2)} \tag{4.2.45}$$

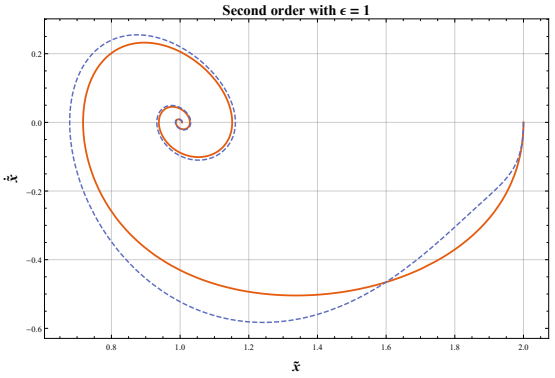
$$\mathcal{C}_{\cos} = \frac{-16\Phi^4 + 18\Phi^2 - 3}{2\eth^2(9 - 8\Phi^2)} \tag{4.2.46}$$

Here, $\mathcal{C}_{\text{dc}}$ dictates the non-oscillatory exponential decay of the transient mean position, while the harmonic terms $\mathcal{C}_{\sin}$ and $\mathcal{C}_{\cos}$ capture the energy transfer into the second harmonic ($2\eth$) generated by the quadratic non-linearity of the magnetic restoring force.

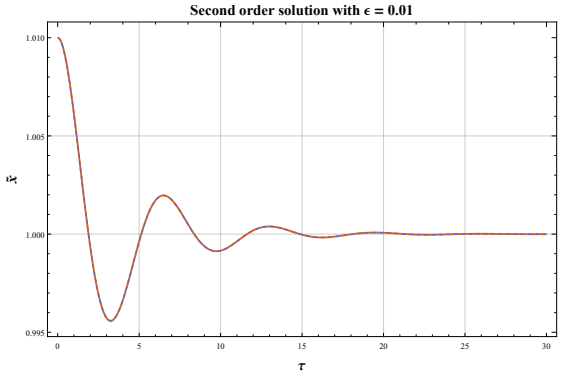
Plotting our solution and contrasting with the numerical solution we obtain the following figures.



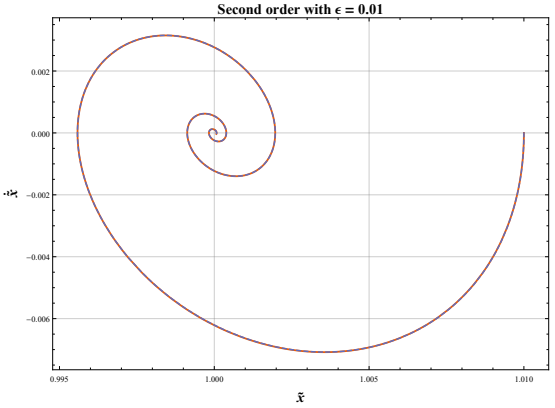
((a)) Time evolution contrast ($\epsilon = 1$)



((b)) Phase portrait contrast ($\epsilon = 1$)



((c)) Time evolution ($\epsilon = 0.01$)



((d)) Phase portrait ($\epsilon = 0.01$)

Figure 4.14: Approximation of the solution of $\epsilon = 1$ and $\epsilon = 0.01$.

4.3 Forcing system

In this case, we consider the presence of all kinds of forces (net force, dissipative force, and an external force, which is periodic). Then, our system of forces has the following aspect:

$$F = -F_g + F_m - F_f + F_{ex} = -mg + \frac{M}{x} - \beta v + F_0 \cos(\omega t), \tag{4.3.1}$$

where F is the net force, m is the mass of the free magnet, g is the gravitational acceleration, x the distance between the magnets, β is the mass flow rate coefficient, v the velocity, M is the constant of the magnetic energy, F_0 is the magnitude of the external force, and ω is the frequency of the external force.

If we consider the static case, where the net force is null ($F = 0$), we can find the equilibrium distance.

$$\frac{M}{x} - mg - \beta v + F_0 \cos(\omega t) = 0 \implies \frac{M}{x} = mg + \beta v - F_0 \cos(\omega t) \implies \boxed{x_{eq} = \frac{M}{mg + \beta v - F_0 \cos(\omega t)}}, \tag{4.3.2}$$

for this result, we can see that the equilibrium distance changes in the function of the velocity and time. In other terms, we have a dynamic equilibrium point. But at some time the oscillation will be stable, causing our system to obtain an equilibrium point like the ideal oscillator (equation (4.1.2)). This equilibrium point only appears when the system passes the region of instability.

This system has the same domain as the ideal and damping cases.

4.3.1 Lagrangian analysis

To find the Lagrangian of this case, we are going to use the definition of a general Lagrangian, which is expressed in equation (1.4.4). To work with it, we need to find the conservative forces (forces that come to a potential) and the external forces.

$$\underbrace{F + mg - \frac{M}{x}}_{\text{conservative}} + \underbrace{\beta v - F_0 \cos(\omega t)}_{\text{unconservative}} = 0 \implies Q = F_0 \cos(\omega t) - \beta v. \tag{4.3.3}$$

With this, using the potential that we found in the ideal case (equation (4.1.5)) lets us use the Lagrangian of the system (equation (4.1.6)).

Now, to find the equation of motion of our forced system, we are going to use the Euler-Lagrange equation (equation (1.4.4)):

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = Q \implies \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} + \beta \frac{dx}{dt} = F_0 \cos(\omega t) \\ \implies m\ddot{x} + mg - \frac{M}{x} + \beta \frac{dx}{dt} = F_0 \cos(\omega t) \implies \ddot{x} + g - \frac{M}{mx} + \frac{\beta}{m}\frac{dx}{dt} = \frac{F_0}{m}\cos(\omega t), \end{aligned} \tag{4.3.4}$$

if we define the β coefficient as $\beta = 2m\phi$, where ϕ is the friction coefficient. Then, the equation of motion has the following form.

$$\ddot{x} + g - \frac{M}{mx} + 2\phi\frac{dx}{dt} = \frac{F_0}{m}\cos(\omega t) \implies \boxed{\frac{d^2x}{dt^2} + 2\phi\frac{dx}{dt} + g - \frac{M}{mx} = \frac{F_0}{m}\cos(\omega t)} \tag{4.3.5}$$

4.3.2 Dimensional analysis

We are going to use the same dimensionless groups that we used in the ideal and damping cases, but we only need to find the dimensionless groups of the external force. The external force has two new elements, the magnitude of the force and the frequency.

As in the previous cases, we use the variables m, g, and M to dimensionless our system. So firstly, we need to do the dimensional analysis for these variables and the variables of the external are:

$$[F] = MLT^{-2}, [\omega] = T^{-1}, [m] = M, [g] = LT^{-2}, [M] = ML^2T^{-2} \tag{4.3.6}$$

With this, we can find the dimensionless groups of the external force variables as the product of the powers of our variables that we use to dimensionless. Then, we have:

$$\Pi_i = \delta m^{\alpha} g^{\beta} M^{\lambda}, \tag{4.3.7}$$

where δ is the variable that we want to make dimensionless. Now, using the equation (4.1.13) we can find the powers of our variables to build our dimensionless groups. This equation is:

$$\begin{pmatrix} \alpha \\ \beta \\ \lambda \end{pmatrix} = \begin{pmatrix} L + \frac{T}{2} - M \\ L + T \\ -L - \frac{T}{2}. \end{pmatrix} \tag{4.3.8}$$

Now, we can proceed to analyze the new variables (F and ω) of the system. We are going to start with the force.

Force group

As we see previously, F use the three dimension ($L = 1$, $T = -2$, $M = 1$). Then, using the result of the equation (4.3.8), we obtain that our dimensionless group and its scale factor are.

$$\begin{pmatrix} \alpha \\ \beta \\ \lambda \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix} \quad \therefore \quad \boxed{\Pi_5 = \frac{F_0}{F_s} = \frac{F_0}{mg} \implies F_s = mg} \quad . \tag{4.3.9}$$

This result is logical, because the principal force that initiates the movement is the gravitational force.

To use the dimensionless group of the Force in our equation, we need to define a symbol that lets us work easily, and this definition is $\Pi_5 = \tilde{F}$

$$\Pi_5 = \frac{F_0}{F_s} \implies F_0 = F_s \tilde{F} = mg\tilde{F} \tag{4.3.10}$$

Frequency group

Now, we are going to work with the frequency (ω) that only works with one dimension ($L = 0$, $T = -1$, $M = 0$). Then, using our equation to find the powers (equation (4.3.8)), we obtain that our dimensionless group and its scale factor are.

$$\begin{pmatrix} \alpha \\ \beta \\ \lambda \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ -1 \\ \frac{1}{2} \end{pmatrix} \quad \therefore \quad \boxed{\Pi_6 = \frac{\omega}{\omega_s} = \frac{\omega}{g}\sqrt{\frac{M}{m}} \implies \omega_s = \frac{1}{t_s} = g\sqrt{\frac{m}{M}} = \sqrt{\frac{g}{x_{eq}}}} \quad . \tag{4.3.11}$$

This result is so important because it is the frequency of the pendulum. So, in other words, our oscillator is analogous to a pendulum.

Now, we define the dimensionless frequency. To do this, we use the dimensionless group of the frequency ($\Pi_6 = \Omega$). Now, we can express the frequency in terms of the dimensionless frequency, as we can see in the following expression.

$$\Pi_6 = \frac{\omega}{\omega_s} \implies \omega = \omega_s \Omega = \frac{1}{t_s}\Omega \tag{4.3.12}$$

4.3.3 Dimensionless system

Using all the dimensional groups that we found previously, we are going to dimensionless our equation of motion (4.3.5). Then, we obtain:

$$\begin{aligned}
&\frac{d^2x}{dt^2} + 2\phi\frac{dx}{dt} + g - \frac{M}{mx} = \frac{F_0}{m}\cos(\omega t) \\
\implies &\frac{d^2(x_s\tilde{x})}{d(t_s\tau)^2} + 2(\phi_s\Phi)\frac{d(x_s\tilde{x})}{d(t_s\tau)} + g - \frac{M}{mx_s\tilde{x}} = \frac{(F_s\tilde{F})}{m}\cos((\omega_s\Omega)(t_s\tau)) \\
\implies &\frac{x_s}{t_s^2}\frac{d^2\tilde{x}}{d\tau^2} + \frac{2\Phi x_s}{t_s^2}\frac{d\tilde{x}}{d\tau} + g - \frac{M}{mx_s\tilde{x}} = \frac{F_s\tilde{F}}{m}\cos(\Omega\tau) \\
\implies &\frac{d^2\tilde{x}}{d\tau^2} + 2\Phi\frac{d\tilde{x}}{d\tau} + \frac{gt_s^2}{x_s} - \frac{Mt_s^2}{mx_s^2\tilde{x}} = \frac{t^2F_s\tilde{F}}{x_sm}\cos(\Omega\tau) \\
\implies &\frac{d^2\tilde{x}}{d\tau^2} + 2\Phi\frac{d\tilde{x}}{d\tau} + 1 - \frac{M}{mgx_{eq}\tilde{x}} = \tilde{F}\cos(\Omega\tau) \\
\implies &\frac{d^2\tilde{x}}{d\tau^2} + 2\Phi\frac{d\tilde{x}}{d\tau} + 1 - \frac{1}{\tilde{x}} = \tilde{F}\cos(\Omega\tau),
\end{aligned} \tag{4.3.13}$$

Multiply all terms of our differential equation by $\tilde{x}$.

$$\frac{d^2\tilde{x}}{d\tau^2} + 2\Phi\frac{d\tilde{x}}{d\tau} + 1 - \frac{1}{\tilde{x}} = \tilde{F}\cos(\Omega\tau) \implies \tilde{x}\frac{d^2\tilde{x}}{d\tau^2} + 2\Phi\tilde{x}\frac{d\tilde{x}}{d\tau} + \tilde{x} - 1 = \tilde{x}\tilde{F}\cos(\Omega\tau). \tag{4.3.14}$$

It is important to highlight that our differential equation does not have an analytical solution, so we only explore the behavior of the system through a numerical analysis.

4.3.4 Asymptotic solution

Before using the asymptotic procedure, we are going to simplify our differential equation. This could be expressed as it can show in the following expression.

$$\tilde{x}\frac{d^2\tilde{x}}{d\tau^2} + 2\Phi\tilde{x}\frac{d\tilde{x}}{d\tau} - \tilde{x}\tilde{F}\cos(\Omega\tau) + \tilde{x} - 1 = 0 \implies \tilde{x}\left(\frac{d^2\tilde{x}}{d\tau^2} + 2\Phi\frac{d\tilde{x}}{d\tau} - \tilde{F}\cos(\Omega\tau) + 1\right) - 1 = 0 \tag{4.3.15}$$

Now, using the asymptotic series (4.1.28), we can find the perturbed series that allows us to approximate the solution.

$$\left(\sum_{i=0}^{N}\epsilon^i\tilde{x}_i\right)\left(\sum_{j=0}^{N}\epsilon^j\frac{d^2\tilde{x}_j}{d\tau^2} + 2\Phi\sum_{k=0}^{N}\epsilon^k\frac{d\tilde{x}_k}{d\tau} - \sum_{\ell=0}^{N}\epsilon^\ell\tilde{x}_\ell\tilde{F}\cos(\Omega\tau) + 1\right) - 1 = 0. \tag{4.3.16}$$

4.3.5 Numerical solution

In the impossibility of finding an analytical solution, we use a numerical method to understand the behavior that our system will follow.

Dynamical system stability

To analyze the behavior of our system, we can use our techniques for differential equations. We can not use the phase portrait technique because our dynamical system depends directly on time. Then we are going to use the Poincaré map to find the stability of our system. It is important to highlight that the Poincaré map is very useful in the stroboscopy space, where $\tau = T = \frac{2\pi}{\Omega}$

Firstly, we need to reduce our second-order differential equation to a set of first-order differential equations. The forcing phase relation is $\tilde{\theta} = \Omega\tau$. These equations are:

$$\tilde{v} = \frac{d\tilde{x}}{d\tau} \tag{4.3.17}$$

$$\dot{\tilde{v}} = \tilde{F}\cos(\tilde{\theta}) - 2\Phi\tilde{v} - 1 + \frac{1}{\tilde{x}} \tag{4.3.18}$$

$$\Omega = \frac{d\tilde{\theta}}{d\tau} \tag{4.3.19}$$

These equations can be expressed as a vectorial system as:

$$\tilde{r} = \frac{d}{d\tau}\begin{pmatrix}\tilde{x}\\ \tilde{v}\\ \tilde{\theta}\end{pmatrix} = \begin{pmatrix}\tilde{v}\\ \tilde{F}\cos(\tilde{\theta}) - 2\Phi\tilde{v} - 1 + \frac{1}{\tilde{x}}\\ \Omega\end{pmatrix}, \tag{4.3.20}$$

where $\tilde{r}$ is the vector that models our system. If we want to know the stability of our system, we need the non-autonomous system, which implies that we do not need to attend to the time. Then our vector $\tilde{r}$ is really formed only by the first and second terms. To find the stability we need the Jacobian matrix of our non-autonomous system. Which is:

$$\mathbf{J}(\tilde{x},\tilde{v}) = \begin{pmatrix}\frac{\partial\tilde{r}_1}{\partial\tilde{x}} & \frac{\partial\tilde{r}_1}{\partial\tilde{v}}\\ \frac{\partial\tilde{r}_2}{\partial\tilde{x}} & \frac{\partial\tilde{r}_2}{\partial\tilde{v}}\end{pmatrix} \implies \mathbf{J}(\tilde{x},\tilde{v}) = \begin{pmatrix}0 & 1\\ -\frac{1}{\tilde{x}^2(\tau)} & -2\Phi\end{pmatrix} = \mathbf{J}(\tau). \tag{4.3.21}$$

With the Jacobian of the system we can find the Monodromy matrix (that is, the transformation of our Jacobian matrix to the Poincaré discrete time space), and this matrix follows the next:

$$\dot{\mathbf{M}}(\tau) = \mathbf{J}(\tau)\mathbf{M}(\tau), \quad \mathbf{M}(0) = \mathbf{I}. \tag{4.3.22}$$

This system is so complicated to solve analytically. Then we are going to use Liouville's formula:

$$\det(\mathbf{M}(\tau)) = \det\mathbf{M}(t_0)e^{\int_{t_0}^{\tau}\operatorname{tr}(\mathbf{J}(\tau))d\tau}. \tag{4.3.23}$$

Now, integrating from 0 to T, to find the. Then we obtain:

$$\begin{aligned} \det(\mathbf{M}(T)) &= \det(\mathbf{M}(0)) e^{\int_0^T \mathrm{tr}(\mathbf{J}(\tau)) d\tau} \\ \det(\mathbf{M}(T)) &= \det(\mathbf{I}) e^{\int_0^T -2\Phi d\tau} \\ \therefore \boxed{\det(\mathbf{M}(T)) = e^{-2\Phi T}} &. \end{aligned} \tag{4.3.24}$$

With this result, we can find the Floquet multipliers, which are the eigenvalues of the Monodromy matrix ($\mathbf{M}(T)$) and they are fundamental to determine the stability of the system. Using the definition of Floquet multipliers, we obtain.

$$\rho 1 \cdot \rho 2 = \det(\mathbf{M}(T)) = e^{-2\Phi T}, \tag{4.3.25}$$

where ρ_1 and ρ_2 are the Floquet multipliers. The Floque multipliers let us know if the trajectory that the system follows is stable or unstable. To know this, we have the following table:

Condition	**How many?**	**System behavior**
$\rho < 1$	All	Stable limit cycle (perturbations decay)
$\rho = 1$	At least one (others < 1)	Marginal stability (bifurcation point)
$\rho > 1$	At least one	Unstable limit cycle (perturbations grow)

Table 4.1: Different behaviors of the system in terms of the Floquet multipliers

If $\Phi > 0$, that is the case of a damped system, the product of the Floquet multipliers is always equal or less than one. That implies that the system has a limit cycle. To show that a limit cycle exists, we now make a Poincaré map.

To make our Poincaré map, we are going to use the Runge-Kutta method in our dynamical system, and we are going to save the values of $(\tilde{x}, \tilde{v})$ when we pass for a multiple of T. Doing this, we obtain the following Poincaré map

Lyapunov exponents

While Floquet multipliers characterize the stability of a periodic orbit, many systems, in particular the forced nonlinear oscillators, can exhibit chaotic behavior where no periodic orbit exists. In such cases, a more general tool is needed: the ***Lyapunov exponents***. The behavior of the system in terms of the Lyapunov exponents follows the next table:

Condition	**How many?**	**System behavior**
$\lambda < 0$	All	Stable dynamics
$\lambda = 0$	At least one (others < 0)	Marginal stability
$\lambda > 0$	At least one	Chaotic dynamics

Table 4.2: Different behaviors of the system in terms of the Lyapunov exponents

When a system is stable, there exists a relation between the Floquet multipliers and the Lyapunov exponents. That relation is:

$$\lambda_i = \frac{1}{T} \ln(\rho_i) \tag{4.3.26}$$

Now, using the Lyapunov exponents equation:

$$\lambda_1 + \lambda_2 = \frac{1}{T}\ln(\rho_1) + \frac{1}{T}\ln(\rho_2) = \frac{1}{T}\ln(\rho_1 \cdot \rho_2) \tag{4.3.27}$$

Using the value of our Floquet product (5.3.32)

$$\lambda_1 + \lambda_2 = \frac{1}{T}\ln\left(e^{-2\Phi T}\right) \implies \lambda_1 + \lambda_2 = -2\Phi \tag{4.3.28}$$

Using the condition of stability:

$$\lambda_1 + \lambda_2 = -2\Phi < 0 \implies \Phi > 0, \tag{4.3.29}$$

this result implies that if the friction number (Φ) increases, the stability increases too. But the amplitude of the oscillation decreases. Another important thing that we can find is the values of Lyapunov exponents that give us stable system

$$\lambda_1 + \lambda_2 = -2\Phi \implies \lambda_1 = -(2\Phi + \lambda_2), \tag{4.3.30}$$

we know that a stable system needs $\lambda_i < 0$, then:

$$\lambda_1 < 0 \implies -(2\Phi + \lambda_2) < 0 \implies 2\Phi + \lambda_2 > 0 \implies \lambda_2 > -2\Phi, \tag{4.3.31}$$

but both Lyapunov exponents need to be less than one; therefore:

$$-2\Phi < \lambda_2 < 0. \tag{4.3.32}$$

Now, we can find the limit for the first Lyapunov exponent domain for a stable system:

$$\begin{aligned} &\lambda_1 = -(2\Phi + 0) = -2\Phi \\ &\lambda_1 = -(2\Phi - 2\Phi) = 0 \\ &\therefore -2\Phi < \lambda_1 < 0. \end{aligned} \tag{4.3.33}$$

We can see that both Lyapunov exponents have the same domain if the system is stable, but it is important to highlight that they can have different values; the only thing that they need to follow is that the sum of both is -2Φ as is presented in the stability condition equation 5.3.36

The previous result indicates that chaos in the system is completely related to the force ($\tilde{F}$) and frequency ($\tilde{\Omega}$) parameters.

Now that we have our domain for the Lyapunov exponents, we can start to plot them and see under what kind of parameters the system is stable. It is important to remember that we use a numerical method to find the Lyapunov exponents, and they change as time increases.

For a particular value of $\tilde{F}$, Φ and Ω, we can find the value of time where the system begins to be stable or not. An example of this is the figure 4.15

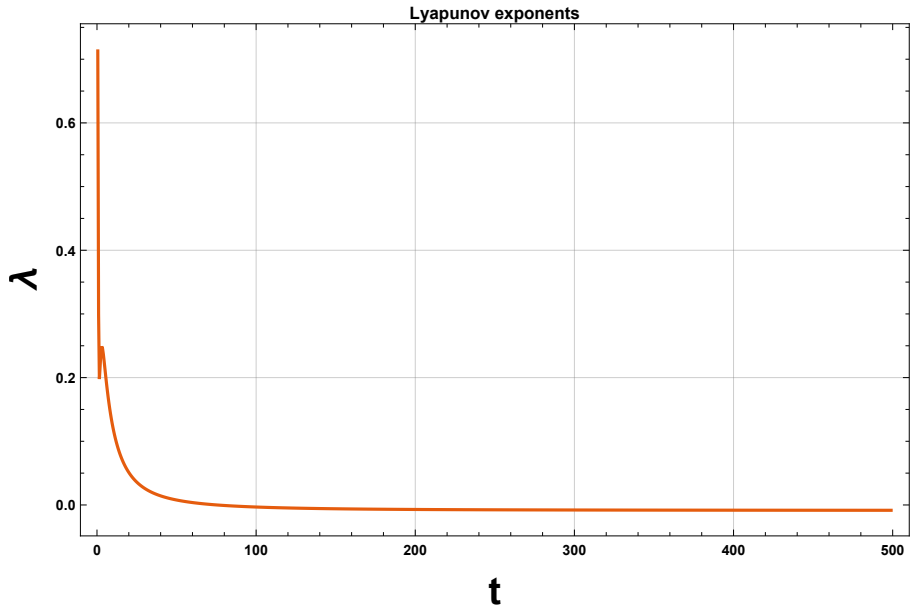


Figure 4.15: Change of the Lyapunov exponents in terms of time for: $\tilde{F} = 1$, $\Phi = 0.1$, and $\Omega = 1$

In the previous figure we can see how the system starts acting like a chaotic system, but with the pass of the time, the system begins to be stable. This tells us that our system with those specific parameters is close to the parameters that generate chaos in the system. So we need to see what kind of values of the force number can generate chaos.

So we decide to run our numerical method to see how Lyapunov exponents change when the value of the force increases. This can be shown in a plot as follows:

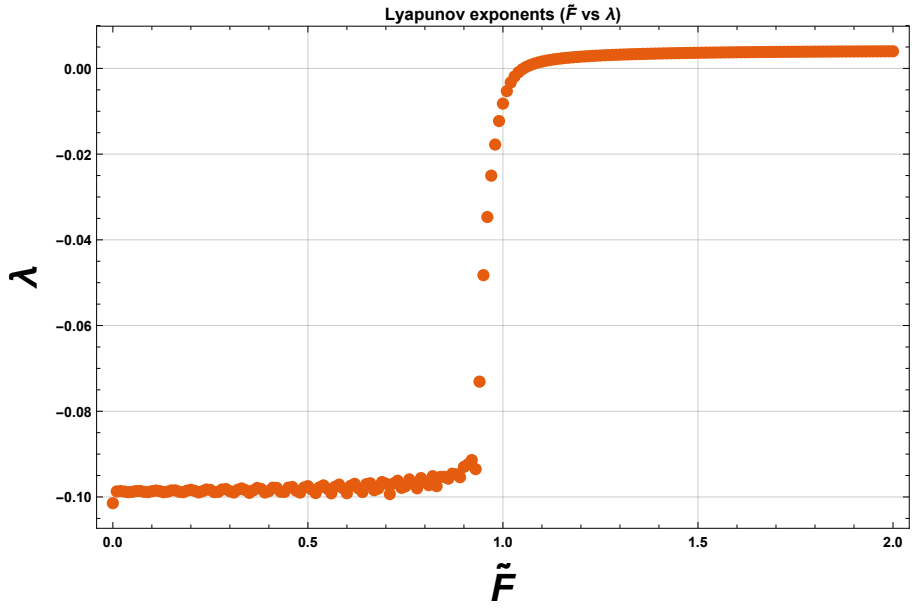


Figure 4.16: Value of Lyapunov exponents when $\tilde{F}$ change it value: $\Phi = 0.1$, and $\Omega = 1$

As we can see in the previous figure, for our system with $\Phi = 0.1$ and $\Omega = 1$, when the

force crosses the horizontal axis, the system begins to be chaotic. To see this in a better way, we do a bifurcation plot of our dynamical system. The bifurcation plot with these parameters is:

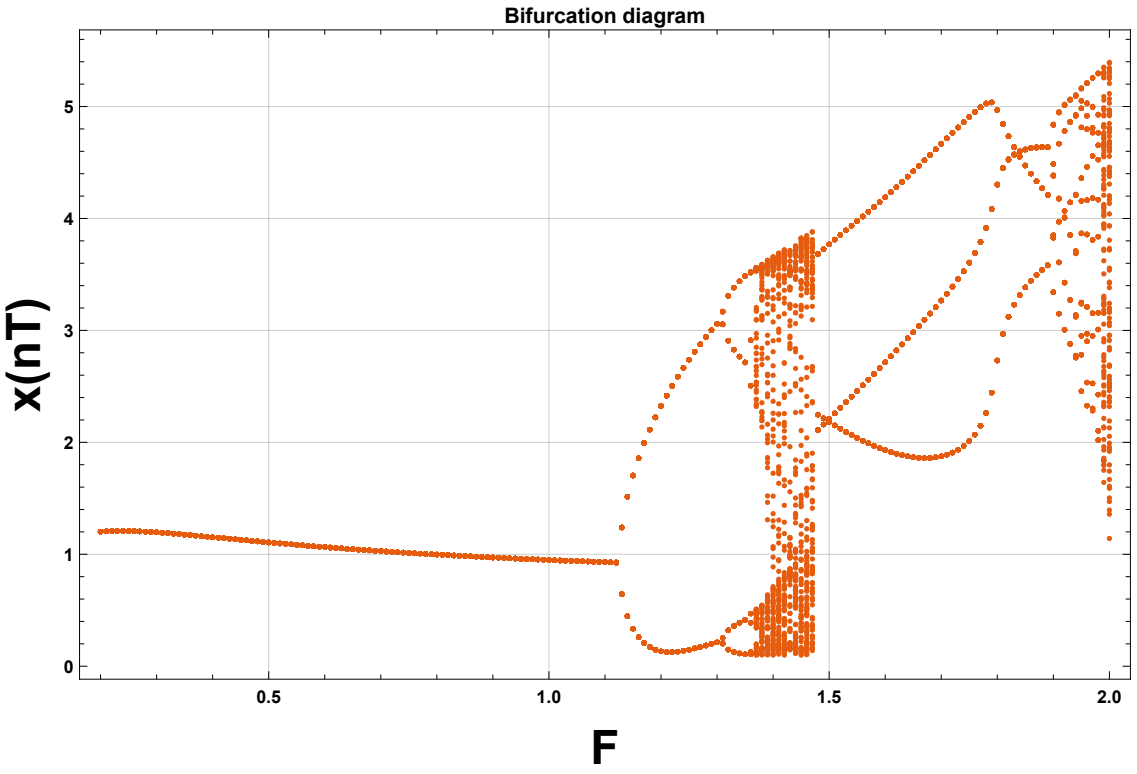
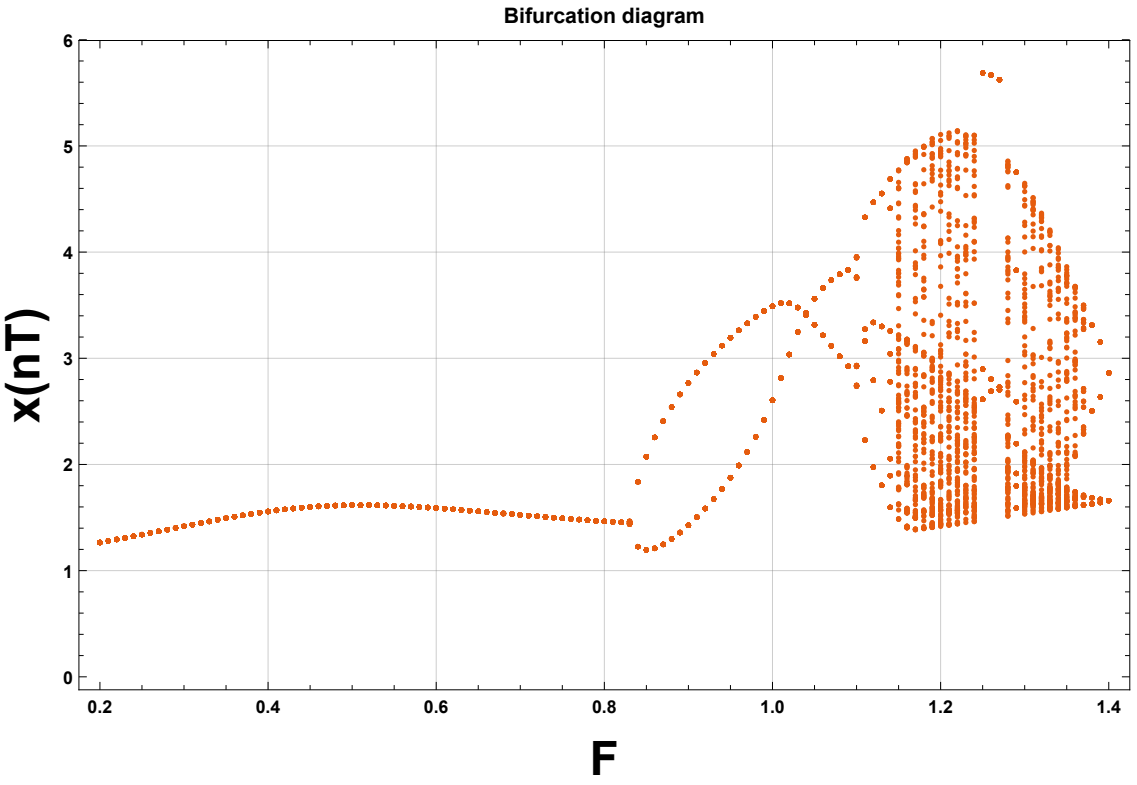


Figure 4.17: Bifurcation plot of the system when $\tilde{F}$ change it value: $\Phi = 0.1$, and $\Omega = 1$

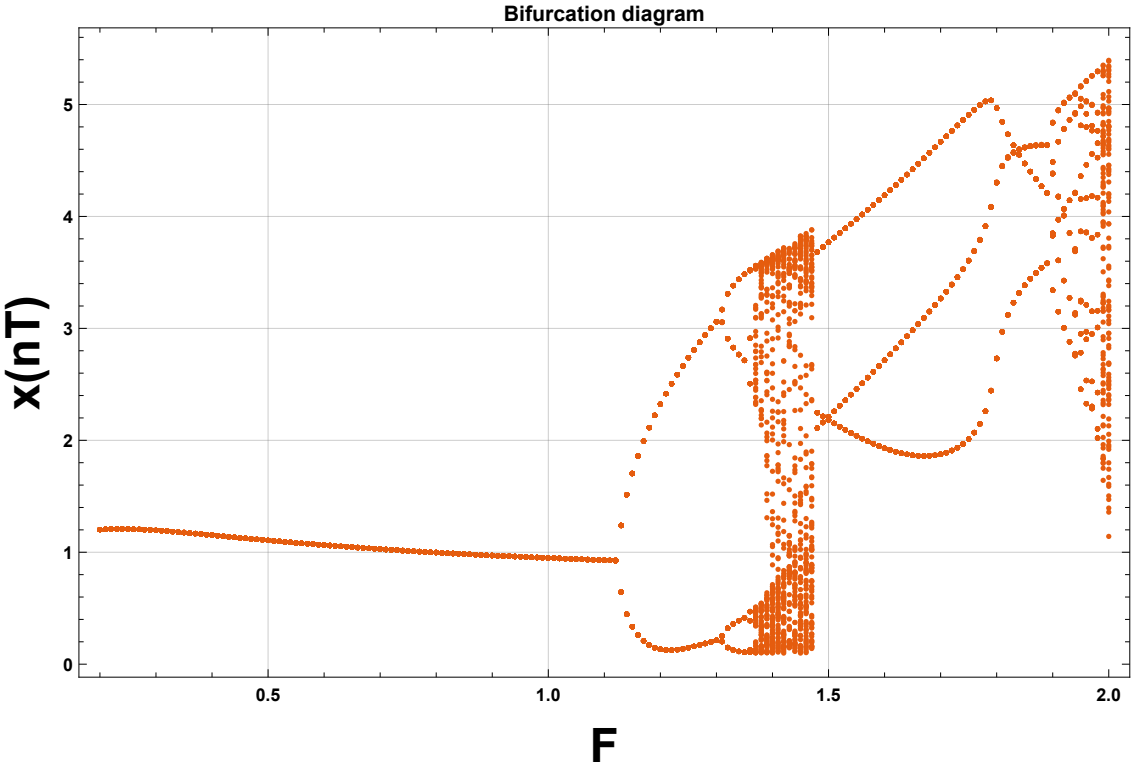
As it was shown in the figure 4.17 when the value of the force pass the axis ($\tilde{F} \approx 1.1$), the system begins chaotic, generating a bifurcation in our system.

Now, we can check how the system responds to different values in the friction number (Φ) and the frequency number (Ω). For bigger values of the friction number, we can speculate that the force number needs to increase to generate chaos, but with the frequency number, things change because it affects the magnitude of the force number.

Generating some bifurcation diagrams to see with which force number ($\tilde{F}$) the system Ω begins to be chaotic. This can be shown in the figures 4.18 and 4.19.

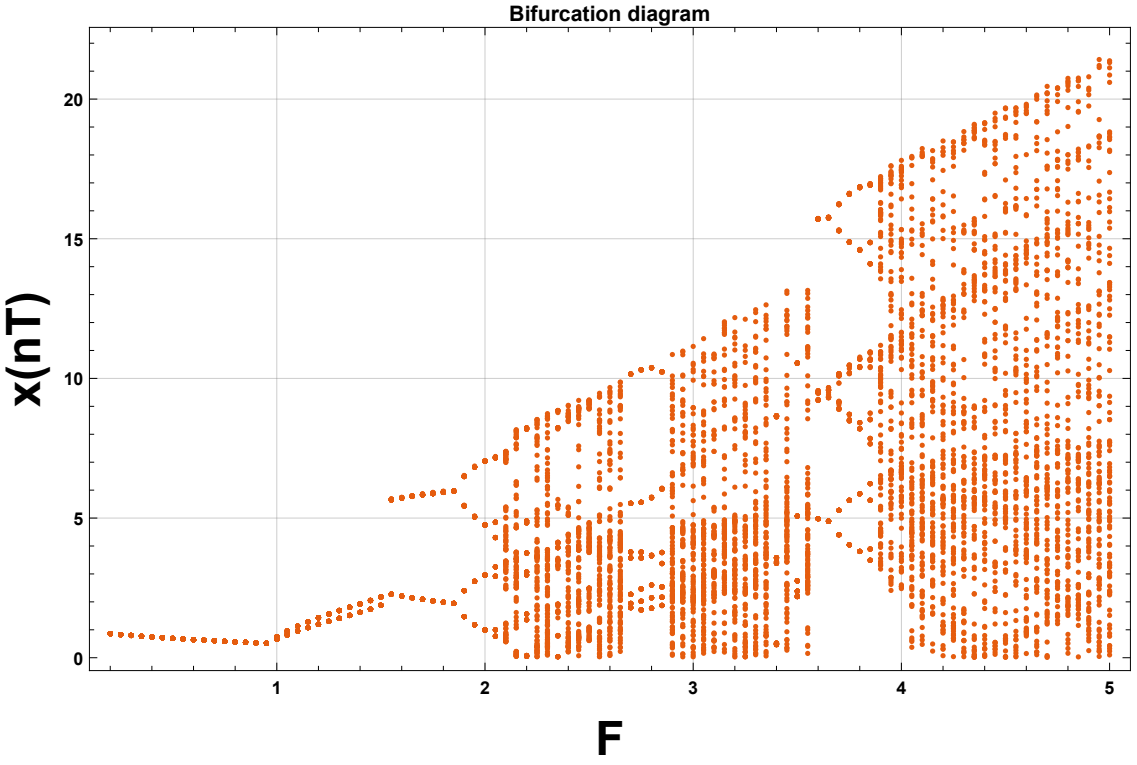


((a)) Bifurcation diagram of the position to different values of $\tilde{F}$ with $\Phi = 0.1$ and $\Omega = 0.5$

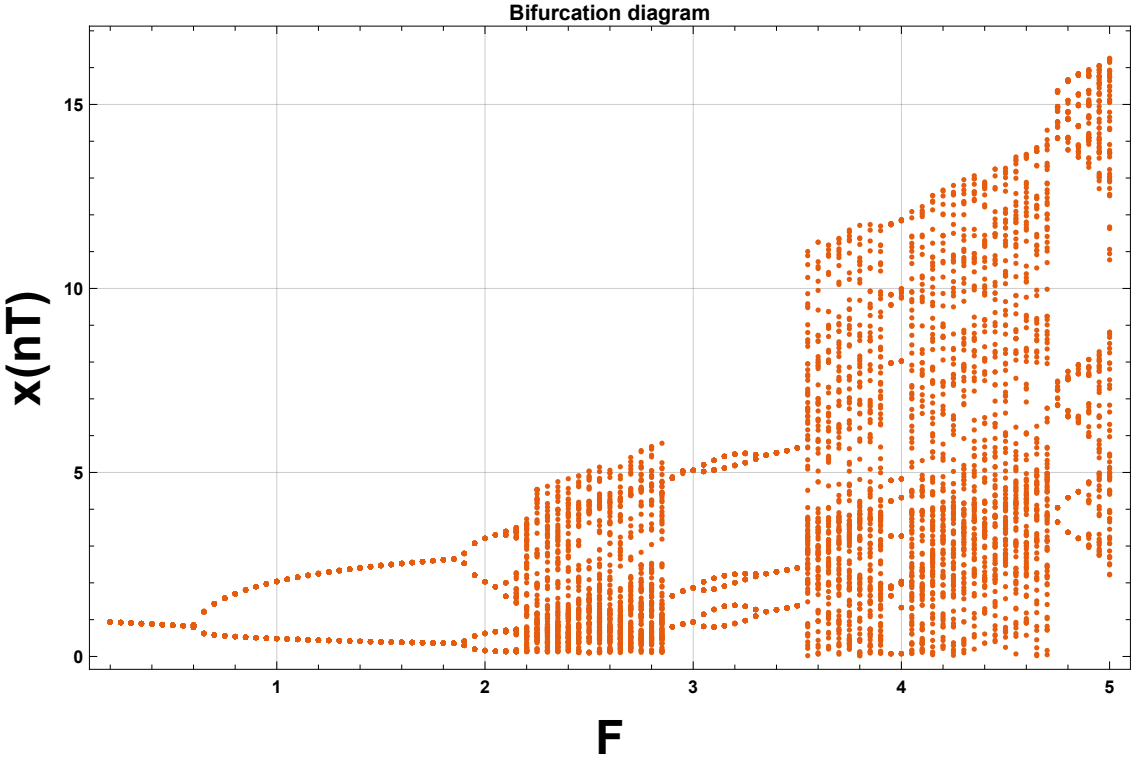


((b)) Bifurcation diagram of the position to different values of $\tilde{F}$ with $\Phi = 0.1$ and $\Omega = 1$

Figure 4.18: Numerical demonstration that the angle (Ω) plays a relevant role in the chaos of the system, and it defines the force value (F) that generates chaos in the system



((a)) Bifurcation diagram of the position to different values of $\tilde{F}$ with $\Phi = 0.1$ and $\Omega = 1.5$



((b)) Bifurcation diagram of the position to different values of $\tilde{F}$ with $\Phi = 0.1$ and $\Omega = 2$

Figure 4.19: Numerical demonstration that the angle (Ω) plays a relevant role in the chaos of the system, and it defines the force value (F) that generates chaos in the system

As we can see in the previous figures, the frequency number has an important role in the chaos, becuase when the value of the frequency increases, the needed force to generate chaos decreases. For this reason, it is important to know the frequency of our system.

Numerical analysis

For the nature of our differential equation, a nonhomogeneous second-order differential equation, we decided to use the Runge-Kutta method. Using the method, we can simulate the system. To know what values of the force are able to generate an stable oscillation, we firstly do the Lyapunov exponents plot. For this system, we are going to use the following values: $\Phi = 0.5$, and $\Omega = 0.9$.

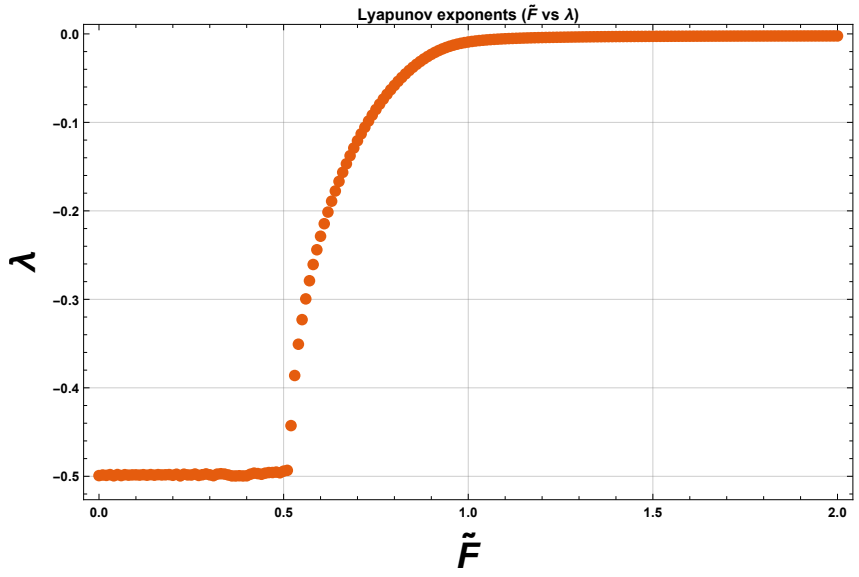


Figure 4.20: Value of Lyapunov exponents when $\tilde{F}$ change it value: $\Phi = 0.1$, and $\Omega = 1$

Now that we know the values of the force ($\tilde{F}$) that generates an stable oscillation, we can simulate our system using the following values: $\epsilon = 0.5$, $\tilde{F} = 1$, $\Phi = 0.5$, and $\Omega = 0.9$,they give us the following numerical solution.

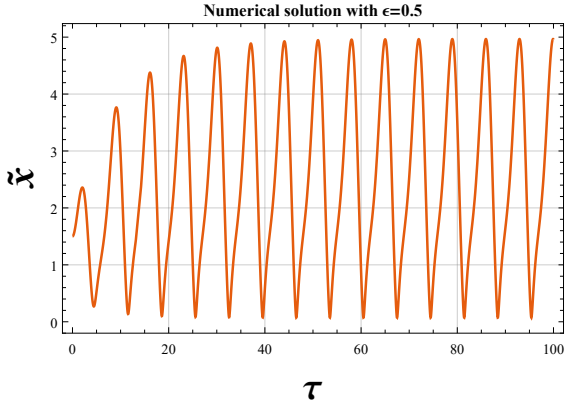


Figure 4.21: Numerical solution of our system with an external force, where the dimensionless parameters have the following values: $\tilde{F} = 1$, $\Phi = 0.5$, $\epsilon = 0.5$, and $\Omega = 0.9$

It is important to see that three dimensionless variables control the behavior of our system. These variables are the friction number (Φ), the magnitude of force ($\tilde{F}$), and the frequency (Ω).

The behavior of the system could be one of the following states, with a specific configuration of values of our system. These configurations can be shown in the table 4.3.

Condition	**System behavior**
$\Phi = 0$, $F = 0$	Conservative (harmonic) motion
$\Phi > 0$, $F = 0$	Damped oscillation to equilibrium
$\Phi > 0$, $F \neq 0$	Forced nonlinear response
$\Omega \approx \omega_0$	Resonance (maximum amplitude)

Table 4.3: Different behaviors of the system with different values of our dimensionless parameters

From the previous table (4.3) we can see that there are two variables that direct affect the amplitude of our system, they are the friction number (Φ) and the frequency (Ω). The principal reason that the dimensionless force is not consider like a direct parameter in the amplitude is because if the frequency of the external force is similar to the natural frequency, it can increase the action of the force until it will be a little.

To find the final amplitude of our forced system we decided to check the medium difference between the maximum amplitude value with the minimum amplitude value in the stable region. We can be obtain using the following equation.

$$A = \frac{max(x_{st}) - min(x_{st})}{2}, \tag{4.3.34}$$

where x_{st} is the position of our magnet in the stable region. Once we have the amplitude, we can contrast how the amplitude changes when we change the value of some dimensionless parameters.

Firstly, we are going to check how the amplitude changes when the frequency changes. To do that, we are going to consider a unitary force $\tilde{F} = 1$ and different values of the friction number (Φ).

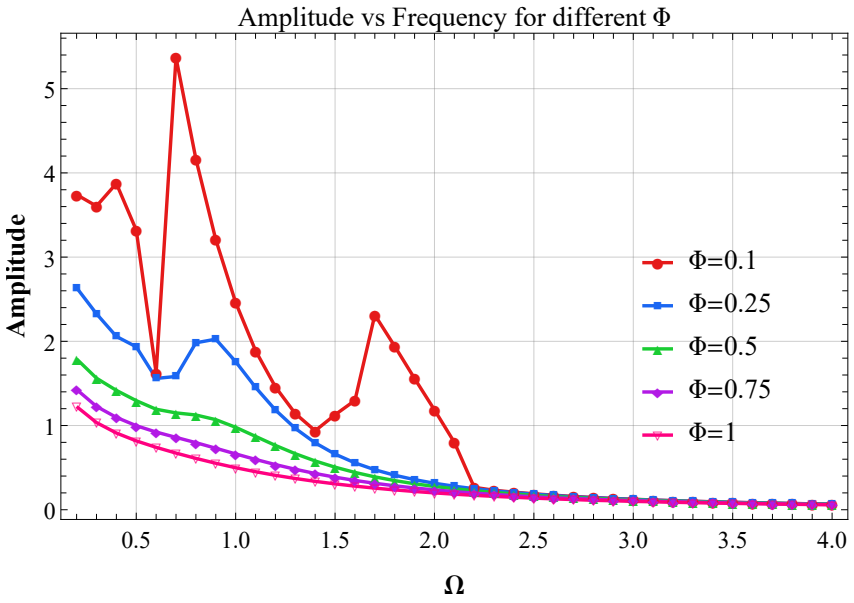


Figure 4.22: Change of the amplitude in terms of Φ, with different values of the frequency (Ω).

In the figure (4.22) we can see that the amplitude decreases when the frequency increases, and when we are close to 1, that is the natural frequency of the system, the amplitude increases a little, this is for the resonance.

Now, we can check how the amplitude changes as the friction number changes. As the same force is unitary, we try with a set of numbers of different frequencies.

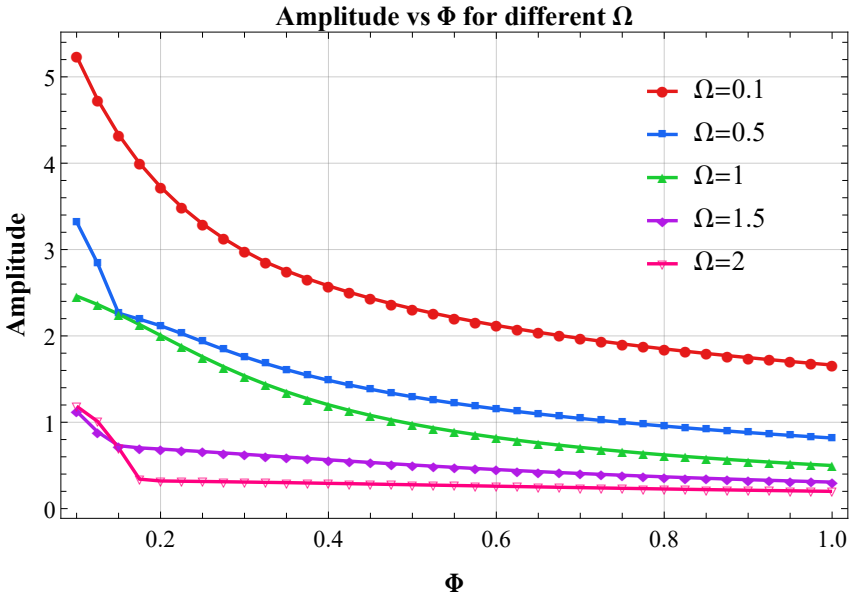


Figure 4.23: Change of the amplitude in terms of Ω, with different values of the friction number (Φ).

It is clear that as the friction number increases, the amplitude decreases. With the previous

analysis, we can generate a 3D plot that tells us what the amplitude of the stable region of our system is with a unitary force and with different frequencies and friction numbers.

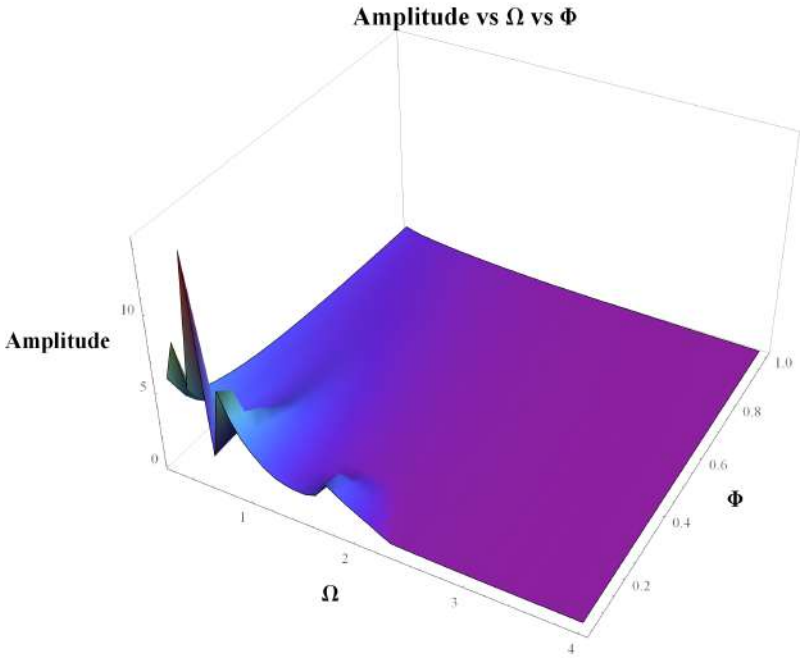


Figure 4.24: 3D graph of the amplitude of the system in terms of the frequency (Ω) and the friction number (Φ).

To this previous figure, we can see that our system needs a low friction number,, close to zero, and a frequency close to 1.

4.3.6 Fluid force

Once we see a general forced system, we are going to use a fluid force. This implies that everything in our previous treatment is right for our new system, with the difference in the force form.

We defined the Inertial–Gravitational Interaction Number Γ as the structural acceleration due to flow forcing relative to gravity acceleration, that is, $\Gamma = \frac{U}{\sqrt{gD}}$, where U is the speed of the fluid and D is the characteristic length, which in our system is $D = x_{eq}$.

The force of a fluid is :

$$F_f = \frac{1}{2}\rho U^2 dLC_L, \tag{4.3.35}$$

where L is the axial longitude of the free magnet, C_L is the lift coefficient, d is the width of our free magnet (diameter), and ρ is the density of our fluid. If the external force is the fluid force. Then, we can rewrite the force terms of the Gravity-inertia ratio as:

$$
\begin{aligned}
F_0 = \frac{1}{2}\rho U^2 dLC_L \implies \tilde{F}F_s = \frac{1}{2}\rho U^2 dLC_L \\
\tilde{F} = \frac{\rho U^2 dLC_L}{2F_s} \implies \tilde{F} = \frac{\rho U^2 dLC_L}{2mg} \\
\tilde{F} = \frac{\rho \Gamma^2 D dLC_L}{2m} \therefore \tilde{F} = \frac{1}{2m^*} C_L \Gamma^2.
\end{aligned}
\tag{4.3.36}
$$

We define m^* as the mass ratio between the mass of the structure and the mass of the fluid, that is, $m^* = \left(\frac{m}{\rho DS}\right)$. Something that is relevant to highlight is that the lift coefficient (C_L) could be a function of time, which would depend on the geometry, but in this specific example, we are going to work with a constant lift coefficient.

Then, substituting this in our dimensionless differential equation, we obtain

$$
\frac{d^2\tilde{x}}{d\tau^2} + 2\Phi\frac{d\tilde{x}}{d\tau} + 1 - \frac{1}{\tilde{x}} = \frac{1}{2m^*} C_L \Gamma^2 \cos(\Omega\tau) \tag{4.3.37}
$$

In this system, the frequency of the system is determined by the Strouhal number. Therefore, the shape of our magnet plays a crucial role in determining how the system responds to the force of the fluid. The Stouhal number definition is: $St = \frac{f_s D}{U}$. Then we can find a relation between the Strouhal and the frequency number. This relation is:

$$
\begin{aligned}
\omega = 2\pi f_s \implies \frac{\Omega}{t_s} = 2\pi\left(\frac{StU}{D}\right) \\
\implies \Omega = \frac{2\pi StU t_s}{D} \implies \Omega = \frac{2\pi StU}{Dg}\sqrt{\frac{M}{m}} \\
\implies \Omega = \frac{2\pi StU}{\sqrt{gD}}\sqrt{\frac{M}{mgD}} \implies \boxed{\Omega = 2\pi St\Gamma\sqrt{\frac{M}{mgD}} = \frac{2\pi StU t_s}{D}}.
\end{aligned}
\tag{4.3.38}
$$

Then, the frequency of the system is related to the Strouhal and Gravity-inertia ratios. If the characteristic length is equal to the equilibrium distance ($D = x_{eq} = \frac{M}{mg}$), the frequency number can be simplified as:

$$
\Omega = 2\pi St\Gamma. \tag{4.3.39}
$$

Then, we can rewrite our dimensionless differential equation as:

$$
\frac{d^2\tilde{x}}{d\tau^2} + 2\Phi\frac{d\tilde{x}}{d\tau} + 1 - \frac{1}{\tilde{x}} = \frac{1}{2m^*} C_L \Gamma^2 \cos(2\pi St\Gamma\tau). \tag{4.3.40}
$$

It is important to see that our system now depends on the values of the Strouhal (St) and Gravity-Inertia number (Γ) numbers and the friction number (Φ). In this system, the most important numbers are the Strouhal and Gravity-inertia ratios because they define the frequency of the oscillation.

New dimensionless equation

The equation of motion with a fluid force (equation (4.3.40)) is right, but the frequency of the vortices depends of two different values, but if we check the equation of the frequency (equation (4.3.38)), we can find that there is another possible dimensionless that help us to let the dimensionless frequency be only in terms of the Strouhal numbers. This new dimensionless uses fluid and structural properties. The new relation is.

$$\Omega = \frac{2\pi StUt_s}{D} = \Omega = 2\pi St \implies \frac{Ut_s}{D} = 1 \quad \therefore \quad \boxed{t_s = \frac{D}{U}}, \tag{4.3.41}$$

with this new time scale we are going to rewrite our differential equation. To rewrite it, we are going to use the dimensionless groups that we obtained previously. Starting the new dimensionless analysis in the equation of motion (4.3.13).

Applying the dimensionless groups in our differential equation, we can find the equation of motion.

$$\begin{aligned}
&\frac{d^2x}{dt^2} + 2\phi\frac{dx}{dt} + g - \frac{M}{mx} = \frac{F_0}{m}\cos(\omega t) \\
\implies &\frac{d^2(x_s\tilde{x})}{d(t_s\tau)^2} + 2(\phi_s\Phi)\frac{d(x_s\tilde{x})}{d(t_s\tau)} + g - \frac{M}{mx_s\tilde{x}} = \frac{(F_s\tilde{F})}{m}\cos((\omega_s\Omega)(t_s\tau)) \\
\implies &\frac{x_s}{t_s^2}\frac{d^2\tilde{x}}{d\tau^2} + \frac{2\Phi x_s}{t_s^2}\frac{d\tilde{x}}{d\tau} + g - \frac{M}{mx_s\tilde{x}} = \frac{F_s\tilde{F}}{m}\cos(\Omega\tau) \\
\implies &\frac{d^2\tilde{x}}{d\tau^2} + 2\Phi\frac{d\tilde{x}}{d\tau} + \frac{gt_s^2}{x_s} - \frac{Mt_s^2}{mx_s^2\tilde{x}} = \frac{t_s^2F_s\tilde{F}}{x_sm}\cos(\Omega\tau) \\
\implies &\frac{d^2\tilde{x}}{d\tau^2} + 2\Phi\frac{d\tilde{x}}{d\tau} + \frac{gD}{U^2} - \frac{M}{mU^2\tilde{x}} = \frac{M\tilde{F}}{U^2m}\cos(\Omega\tau)
\end{aligned} \tag{4.3.42}$$

Applying the definition of the equilibrium point ($x_{eq} = \frac{M}{mg}$), our differential equation is:

$$\frac{d^2\tilde{x}}{d\tau^2} + 2\Phi\frac{d\tilde{x}}{d\tau} + \frac{gD}{U^2} - \frac{x_{eq}g}{U^2\tilde{x}} = \frac{x_{eq}g\tilde{F}}{U^2}\cos(\Omega\tau). \tag{4.3.43}$$

If $D = x_{eq} = \frac{M}{mg}$ and using the definition of the Gravity-inertia ratio, then we can rewrite our equation:

$$\frac{d^2\tilde{x}}{d\tau^2} + 2\Phi\frac{d\tilde{x}}{d\tau} + \frac{1}{\Gamma^2} - \frac{1}{\Gamma^2\tilde{x}} = \frac{\tilde{F}}{\Gamma^2}\cos(\Omega\tau). \tag{4.3.44}$$

Now, using the dimensionless form of the fluid force (equation (4.3.36)), we can find the following equation.

$$\begin{aligned}
&\frac{d^2\tilde{x}}{d\tau^2} + 2\Phi\frac{d\tilde{x}}{d\tau} + \frac{1}{\Gamma^2} - \frac{1}{\Gamma^2\tilde{x}} = \frac{\tilde{F}}{\Gamma^2}\cos(\Omega\tau) \\
\implies &\frac{d^2\tilde{x}}{d\tau^2} + 2\Phi\frac{d\tilde{x}}{d\tau} + \frac{1}{\Gamma^2} - \frac{1}{\Gamma^2\tilde{x}} = \frac{\frac{1}{2m^*}C_L\Gamma^2}{\Gamma^2}\cos(\Omega\tau) \\
\therefore \quad &\frac{d^2\tilde{x}}{d\tau^2} + 2\Phi\frac{d\tilde{x}}{d\tau} + \frac{1}{\Gamma^2} - \frac{1}{\Gamma^2\tilde{x}} = \frac{1}{2m^*}C_L\cos(2\pi St\tau).
\end{aligned} \tag{4.3.45}$$

Now, the frequency of the external force only depends on the Strouhal number, or in other words, the vortex–shedding frequency of the von Kármán street. If we do some algebra, we can factorize the Gravity-inertia ratio. Obtaining the following equation.

$$\frac{d^2\tilde{x}}{d\tau^2}+2\Phi\frac{d\tilde{x}}{d\tau}+\frac{1}{\Gamma^2}-\frac{1}{\Gamma^2\tilde{x}}=\frac{1}{2m^*}C_L\cos\left(2\pi St\tau\right)\implies\frac{d^2\tilde{x}}{d\tau^2}+2\Phi\frac{d\tilde{x}}{d\tau}+\frac{1}{\Gamma^2}\left(1-\frac{1}{\tilde{x}}\right)=\frac{1}{2m^*}C_L\cos\left(2\pi St\tau\right) \tag{4.3.46}$$

Something important is that our system works when we know three dimensionless parameters, which are: the Strouhal (St), Gravity-Inertia ratio (Γ), and friction (Φ) numbers.

Something important to see is that if we work with small amplitudes, we can find an equation that has an analytical solution. Firstly, we are going to use an expansion at first order.

$$\tilde{x}(\tau)=1+\xi(\tau),\qquad|\xi|\ll 1. \tag{4.3.47}$$

Now, using a first order Taylor expanssion of the nonlinear term.

$$\frac{1}{\tilde{x}}=\frac{1}{1+\xi}=1-\xi+O(\xi^2). \tag{4.3.48}$$

Substituting $\tilde{x}=1+\xi$ in our differential equation.

$$\frac{d^2}{d\tau^2}(1+\xi)+2\Phi\frac{d}{d\tau}(1+\xi)+\frac{1}{\Gamma^2}-\frac{1}{\Gamma^2}(1-\xi)=\frac{1}{2m^*}C_L\cos(2\pi St\,\tau)+O(\xi^2). \tag{4.3.49}$$

We know that the constant term in $(1+\xi)$ is zero, and dropping $O(\xi^2)$ terms. Then, our differential equation is now.

$$\begin{aligned}&\frac{d^2\xi}{d\tau^2}+2\Phi\frac{d\xi}{d\tau}+\frac{1}{\Gamma^2}-\frac{1}{\Gamma^2}+\frac{1}{\Gamma^2}\xi=\frac{1}{2m^*}C_L\cos(2\pi St\,\tau)\\&\implies\frac{d^2\xi}{d\tau^2}+2\Phi\frac{d\xi}{d\tau}+\underbrace{\frac{1}{\Gamma^2}}_{\omega_0^2}\xi=\frac{1}{2m^*}C_L\cos(2\pi St\,\tau)\\&\implies\frac{d^2\xi}{d\tau^2}+2\Phi\frac{d\xi}{d\tau}+\omega_0^2\xi=\frac{1}{2m^*}C_L\cos(2\pi St\,\tau),\end{aligned} \tag{4.3.50}$$

where ω_0 is the natural frequency of the linearized system .Using the natural frequency of this small aplitudes system, we find the differential equation of a damped forced oscillator system.

The resonance of the system as we know, appears when the frequency of the external force is similar to the natural frequency ($\Omega\approx\omega_0=\frac{1}{\Gamma}$). But the frequency of the external force is defined by the Strouhal number, which reveals a clear relation between the fluid motion and the system's natural frequency. Then the Strouhal number that generates a resonance is:

$$2\pi St\approx\frac{1}{\Gamma}\quad\implies\quad\boxed{St\approx\frac{1}{2\pi\,\Gamma}.} \tag{4.3.51}$$

Another thing that we can obtain is the amplitude relation for our small-amplitude system, that is:

$$A(\Omega) \approx \frac{\frac{1}{2m^*}C_L}{\sqrt{(\omega_n^2 - \Omega^2)^2 + (2\Phi\Omega)^2}}. \tag{4.3.52}$$

4.4 Harnessing electric model

With the equation of motion of the system, we know the behavior of our dynamical system, and we are able to predict its motion. That lets us know where and when the system is near or far to a point. So, using this knowledge and the induction Faraday's law, we can induce a voltage in an open circuit. Then, introducing the Faraday's law:

$$\varepsilon = -n\frac{d\Phi_B}{dt}, \tag{4.4.1}$$

where n is the number of turns of a coil, Φ_B is the magnetic flux. Now, using the definition of the magnetic flux:

$$\Phi_B = BS\cos(\alpha), \tag{4.4.2}$$

where B is the magnetic field, S the surface area of the coil, and α is the angle between the magnetic field and the normal vector of the surface area. It is important to remark that in our system the magnet and the coil are always aligned with $\alpha = \pi$. Using this information our magnetic flux is:

$$\Phi_B = -BS. \tag{4.4.3}$$

If we use the magnetic field of a dipole magnet, we obtain that it depends of the distance. The magnetic field is.

$$B = \frac{\mu_0 \mathcal{M}}{2x^3}, \tag{4.4.4}$$

where μ_0 is the magnetic permeability of the fre space, $\mathcal{M}$ is the magnetic dipole moment,and x is the distance between the magnet and the coil. Applying the previous definition in the induction Faraday's law.

$$\varepsilon = n\frac{d}{dt}(BS) \implies \varepsilon = \frac{n\mu_0\mathcal{M}S}{2}\frac{d}{dt}\left(\frac{1}{x(t)^3}\right). \tag{4.4.5}$$

For simplicity we define $\zeta = \frac{n\mu_0\mathcal{M}S}{2}$. Then, our generated voltage is:

$$\varepsilon = \zeta\frac{-3x(t)^2\cdot\dot{x}(t)}{x(t)^6} = -\frac{3\zeta\dot{x}(t)}{x(t)^4}. \tag{4.4.6}$$

Now, we are going to dimensionless the position and time to relate to our previous solutions. Then, we obtain.

$$\varepsilon = -3\zeta\frac{\frac{d(x_s\tilde{x}(\tau))}{d(t_s\tau)}}{(x_s\tilde{x}(\tau))^4} = -\frac{3\zeta x_s}{t_s x_s^4}\cdot\frac{\dot{\tilde{x}}(\tau)}{\tilde{x}(\tau)^4} = -\frac{3\zeta}{t_s\tilde{x}(\tau)^3}\cdot\frac{\dot{\tilde{x}}(\tau)}{\tilde{x}(\tau)^4} \tag{4.4.7}$$

Using the definitions of the scale factors of our system, we obtain.

$$\varepsilon = -\frac{3\zeta U}{x_{eq}^4} \cdot \frac{\dot{\tilde{x}}(\tau)}{\tilde{x}(\tau)^4} = -\frac{3\zeta U m^4 g^4}{M^4} \cdot \frac{\dot{\tilde{x}}(\tau)}{\tilde{x}(\tau)^4} \tag{4.4.8}$$

.

Now, using the equation of the induced voltage and the numerical solution of the system, we can find the response of the voltage to the motion of our system. For an specific case we use in the system the following values: $\Phi = 0.5$, $x_0 = 1$, $\Omega_0 = 1$, $\tilde{F} = 1$. Then, the voltage generated in time is represented in the following figure.

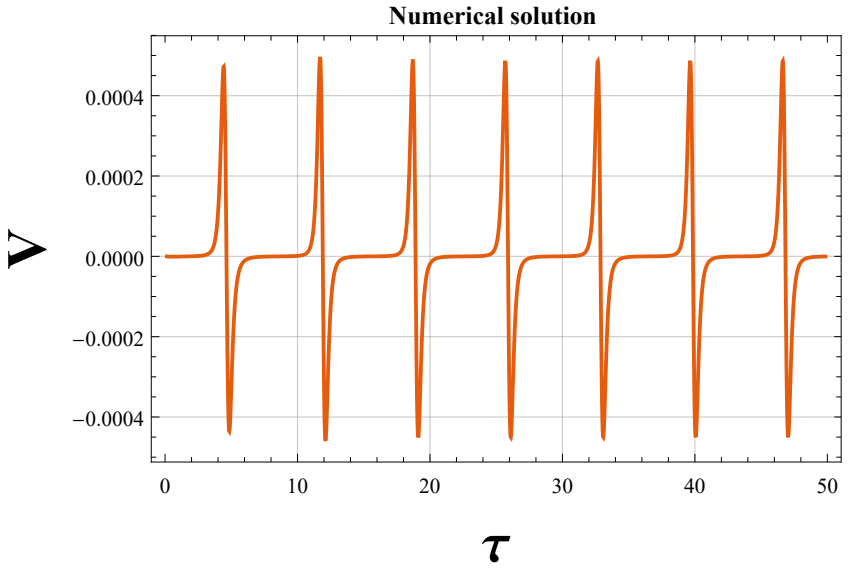


Figure 4.25: Voltage generated by the system with $\tilde{F}$, $\Omega_0 = 1$, $\Phi = 0.5$, $x_0 = 1$

Now, to see the dependence of the voltage on friction, we plot the voltage generated by different Φ values.

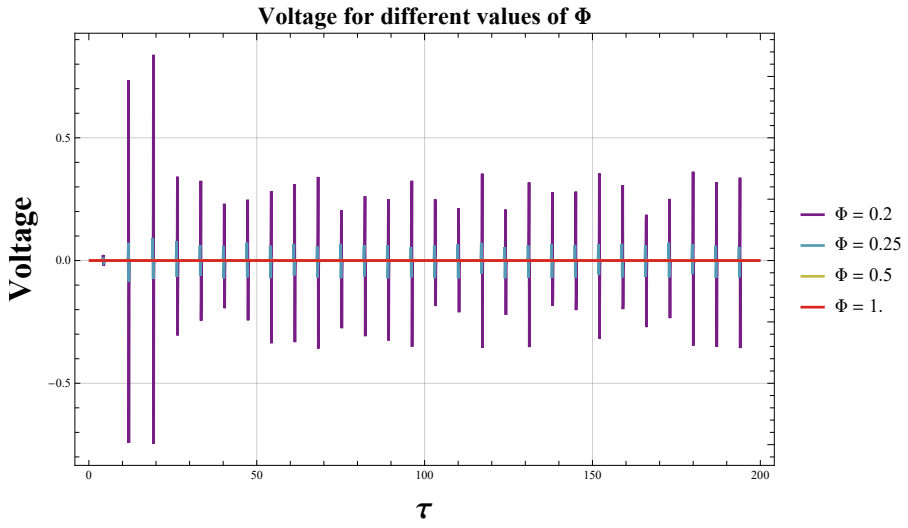


Figure 4.26: Reaction of the voltage generated to different values of damping.

For the previous figure, we can see that the voltage decreases when the damping increases, and this makes sense because more damping implies less movement, which directly affects the voltage's value. Another interesting case is how the voltage reacts to different frequency values. These frequencies are the frequencies of the vortices.

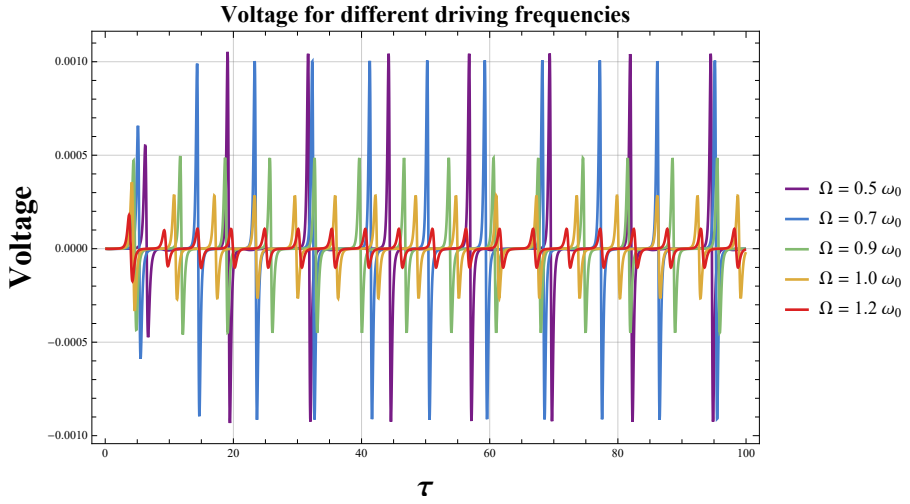


Figure 4.27: Reaction of the voltage generated to different frequencies.

As we can see in the previous figure, the magnitude of the voltage reduces when the frequency tends to the natural frequency of the system, but the voltage is more recurrent.

4.5 Conclusions

In this chapter, we confirm that our system can oscillate, and the interaction with a fluid in motion could be the source of energy that it needs to maintain the oscillation. The oscillation is asymmetric because in the lowest part, the system is repelled quickly, while in the highest part, the attraction is very slow.

In terms of the external force, we can see that the frequency of the vortices generated by the fluid-structure interaction are the responsable of the motion and the quantity of the voltage generated.

The amplitude of the oscillation depends mainly on the shape of the free magnet. For this reason, it becomes essential to study different geometries and choose the one that maximizes the lift coefficient. A better shape would increase the force that the fluid transfers to the magnet, allowing larger oscillations and a more efficient energy exchange.

We also observed that the fluid parameters have important roles. The Strouhal number sets the resonance condition that creates the oscillation, while the inertial gravity interaction number affects not only the resonance itself but also the stability and complexity of the dimensionless equation. This shows how sensitive the system is to the properties of the fluid and how these parameters can change the global behavior of the solution.

Finally, all the mathematical tools used throughout the chapter—Lagrangian and Hamiltonian formulations, dimensional analysis, and asymptotic approximations—were very useful to understand how the system reacts and why it behaves the way it does. These methods helped us find expressions that describe the dynamics more clearly and opened the possibility of using this kind of system in future projects related to clean energy generation.

The ideas developed here could be the base to design simple and stable oscillators powered only by natural fluid motion.

CHAPTER 5

Three magnets oscillation with gravity

> The energy of the magnetic field resides in the space between and around the magnets... This energy is constantly undergoing transformation, manifesting itself as a stress that exerts attractive or repulsive forces upon the bodies within its influence.
>
> James Clerk Maxwell, *"A Treatise on Electricity and Magnetism"*, 1873

The system of this problem is constituted by three magnets; two of them are fixed, one on the ground and the other is set some distance h over the first. The third magnet is free to move in the region between the two fixed magnets. The free magnet is initially displaced some distance A from the equilibrium position (x_0). The free magnet is affected by the gravitational force and the magnetic repulsion forces.

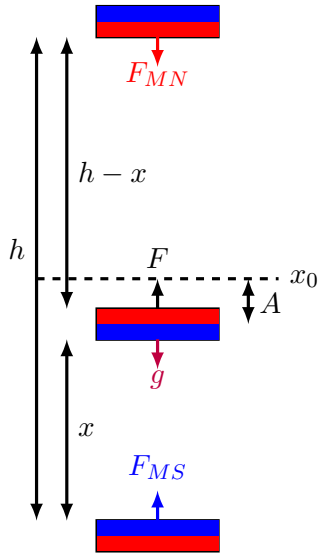


Figure 5.1: Graphic representation of the system, where F is the force, F_{rN} and F_{rS} are the magnetic repulsion force of each pole and g is the gravitational acceleration.

To simplify the system and the calculus we are going to consider that the two magnetic repulsion forces have the same form and are identically with the difference in the direction. The magnetic repulsion force F_M is expressed in this problem as a relation between the magnetic energy (M) and the distance between the magnets (x). So the mathematical form of the magnetic force is:

$$F_M = \frac{M}{x}. \tag{5.0.1}$$

As initial conditions for this problem we have that at the time $t = 0$ the distance between the magnets is a displacement from the equilibrium position and the speed of the free magnet in that initial position is cero. So our initial conditions mathematically are:

$$x(0) = x_{eq} + A \quad \wedge \quad \dot{x}(0) = 0. \tag{5.0.2}$$

The domain of the distance x goes from 0 to h, because the free magnet in the limit can be so close to any of the fixed magnets, but it can not touch any of them, so the domain is: $0 < x < h$. So the domain of the amplitude is:

$$\begin{aligned} 0 = x_{eq} + A &\implies A = -x_{eq} \\ h = x_{eq} + A &\implies A = h - x_{eq} \\ \therefore \quad &\boxed{-x_{eq} < A < h - x_{eq}} \end{aligned} \tag{5.0.3}$$

5.1 Ideal system

For the ideal case, the dissipative and external forces do not interact with the physical system. With this, the forces in the system are:

$$F = -F_g + F_M = -mg + \frac{M}{x} - \frac{M}{h-x}, \tag{5.1.1}$$

where F is the force, m is the mass of the free magnet, g the gravitational acceleration, x the distance between the two magnets and M is the magnetic energy constant. If the system is in equilibrium, then the net force is cero ($F = 0$) and we can find the equilibrium distance.

$$\begin{aligned} \frac{M}{x} - mg - \frac{M}{h-x} = 0 &\implies M(h-x) = mgx(h-x) + Mx \\ &\therefore \boxed{x_{eq} = \frac{2M + mgh \pm \sqrt{4M^2 + m^2g^2h^2}}{2mg}} \quad , \end{aligned} \tag{5.1.2}$$

for the previous solution, we have two options for the equilibrium distance, so we need to find what is the distance that belongs to the x domain. So we going to compare the positive solution to h.

$$\frac{2M+mgh+\sqrt{4M^2+m^2g^2h^2}}{2mg} < h \implies 2M+mgh+\sqrt{4M^2+m^2g^2h^2} < 2mgh$$
$$\implies 2M+\sqrt{4M^2+m^2g^2h^2} < mgh \implies \sqrt{4M^2+m^2g^2h^2} < mgh-2M$$
$$\implies 4M^2+m^2g^2h^2 < 4M^2+m^2g^2h^2-4mghM \boxed{\therefore 0 < -4mghM}. \tag{5.1.3}$$

This solution is a contradiction, so the positive root in the equation (5.1.2) is not in the x domain. Then, the equilibrium distance is:

$$\boxed{x_{eq} = \frac{2M+mgh-\sqrt{4M^2+m^2g^2h^2}}{2mg}} \tag{5.1.4}$$

as we can see, the equilibrium distance compares the magnetic energy with the gravitational force; then the increase or decrease in some of these magnitudes affects the equilibrium distance of the system, this lets us modulate the equilibrium distance.

5.1.1 Lagrangian analysis

The relevance of the lagrangian analysis is that we can study the dynamic of the system in terms of its coordinates and degrees of freedom. The lagrangian is described as the sum of the kinetic energies minus the potential energies present in the system.

$$L = T - V \implies L = \frac{1}{2}\sum_i m_i \dot{q_i}^2 - V(q_1, q_2, \ldots, q_i), \tag{5.1.5}$$

where m_i is the ith mass of the system, $\dot{q_i}$ is the ith speed of the system, V is the potential energy function and $q_1, q_2, \ldots, q_i$ are the coordinates of the system.

Our system only have a degree of freedom, so the dynamic of the system is described by one coordinate (x), then our lagrangian has the following expression.

$$L = \frac{1}{2}m\dot{x}^2 - V(x), \tag{5.1.6}$$

where $V(x)$ is the potential energy in our system. To find the potential energy we going to use the relation between the work and the potential energy when the system is conservative.

$$-\frac{d}{dx}V(x) = F \implies -V(x) = \int F dx \implies -V(x) = -\int mg dx + \int \frac{M}{x}dx - \int \frac{M}{h-x}dx$$
$$\implies -V(x) = -mg\int dx + M\int \frac{dx}{x} - M\int \frac{dx}{h-x} \implies -V(x) = -mgx + M\ln(x) + M\ln(h-x)$$
$$\implies -V(x) = -\left[mgx - M\ln(x) - M\ln(h-x)\right] \therefore \boxed{V(x) = mgx - M\ln(x) - M\ln(h-x)}. \tag{5.1.7}$$

Substituting this result in the our lagrangian function (equation (4.1.4)), we obtain that our lagrangian is:

$$\boxed{L = \frac{1}{2}m\dot{x}^2 - mgx + M\ln(x) + M\ln(h-x)}. \tag{5.1.8}$$

With our lagrangian, we can find the energy of the system and the equation of motion of the system.

First, we going to find the energy of the system. The energy of the syste is define as.

$$E = \sum_i \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L \implies E = \dot{x}\frac{\partial L}{\partial \dot{x}} - L \therefore \boxed{E = \frac{1}{2}m\dot{x}^2 + mgx - M\ln(x) - M\ln(h-x)}. \tag{5.1.9}$$

Now, using the Euler-Lagrange equation we can find the equation of motion of our system. As we know, the external forces are null ($Q = 0$). So the equation of motion for the coordinate x is.

$$\begin{gathered}\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = Q \implies m\ddot{x} + mg + \frac{M}{h-x} - \frac{M}{x} = 0 \\ \implies \ddot{x} + g + \frac{M}{m(h-x)} - \frac{M}{mx} = 0 \quad \therefore \quad \boxed{\frac{d^2x}{dt^2} + g + \frac{M}{m}\left[\frac{1}{h-x} - \frac{1}{x}\right] = 0}.\end{gathered} \tag{5.1.10}$$

This equation of motion is relevant because it lets us know the accelerations that generate the movement in our system. It's important to remark that the differential equation does not have an analytical solution.

5.1.2 Hamiltonian analysis

Now that we know the equation of motion, it is important to see if the system really oscillates. To do this we going to use the hamiltonian analysis. To use this analysis we need to construct the hamiltonian of our system.

$$\begin{gathered}H = E = \frac{1}{2}m\dot{x}^2 + mgx - M\ln(x) - M\ln(h-x), \\ \boxed{\therefore H = \frac{p^2}{2m} + mgx - M\ln(x) - M\ln(h-x)},\end{gathered} \tag{5.1.11}$$

where H is the hamiltonian of the system and p is the momentum.

Now, using the Hamilton's equations we can find the dynamic of the system. The Hamilton's equations are:

$$\dot{H} = \begin{pmatrix}\dot{x} \\ \dot{p}\end{pmatrix} = \begin{pmatrix}\dfrac{p}{m} \\ \dfrac{M}{x} - mg - \dfrac{M}{h-x}\end{pmatrix} \tag{5.1.12}$$

In the Hamilton's equations we can see that the change of momentum depends of position and vice versa. If we plot the solution of the Hamilton's equations we obtain a phase portrait of our system to see the behavior of it.

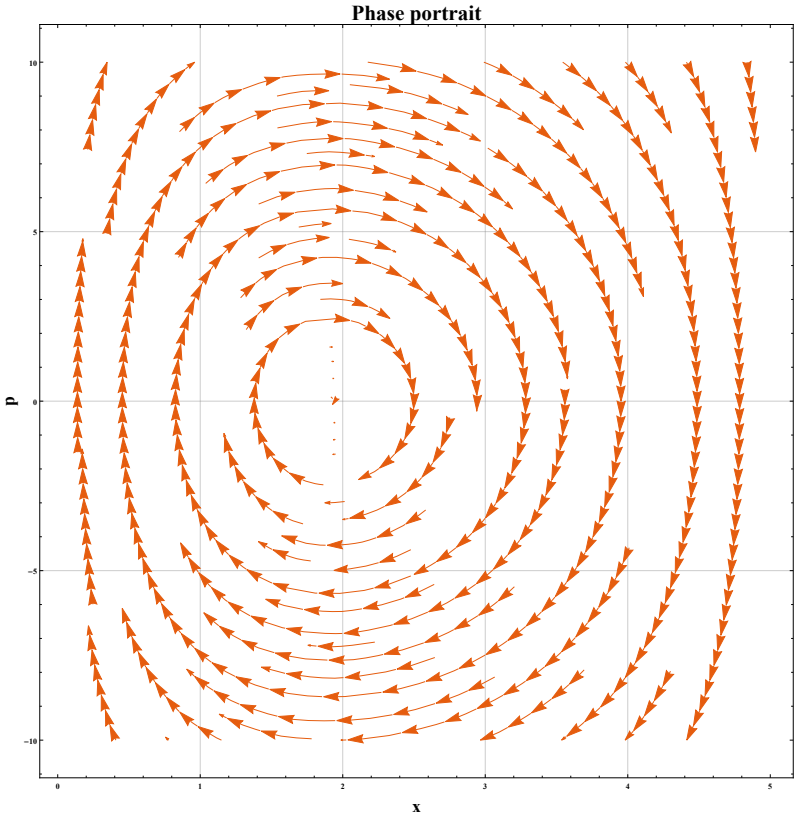


Figure 5.2: Phase portrait of the system with $M = 50$, $h = 5$, $g = 10$ and $m = 1$

As we can see in the figure (5.2), the system follows a closed cyclical trajectory, so we can confirm that the system oscillates.

5.1.3 Dimensional analysis

In our system we have seven variables ($n = 7$), this variables are the mass (m), the time (t), the position (x), the gravity (g), the oscillation amplitude (A), the magnetic energy (M) and the distance between the fixed magnets (h). For the nature of our problem, we only need to use three dimension ($j = 3$) to caracterize all the problem, the dimensions that we need to use is the matter, the longitude and the time. Using the Buckingham's theorem, we know that there are four dimensionless groups ($k = n - j = 4$) that describe our system.

Using a dimensional analysis over every variable:

$$[x] = L, [h] = L, [A] = L, [t] = T, [m] = M, [g] = LT^{-2}, [M] = ML^2T^{-2} \tag{5.1.13}$$

For this system we going to use h, g and M as dimensional variables, this because h is the geometric parameter of the system, g is the cinematic parameter and M is the dynamic parameter. As we saw in the previous chapter, every dimensinal group is the product of the dimensional variables, so we have that the ith group is expressed as:

$$\Pi_i = \delta h^{\alpha} g^{\beta} M^{\lambda}, \tag{5.1.14}$$

where δ is the variable that we want to dimensionless, in our system we have the variables x, t, m and A; α, β and λ are the powers of the distance h, the gravity and the magnetic energy that dimensionless the ith group.

To find the powers of our dimensional variables that dimensionless every group we can construct an equations system, this system is.

$$\begin{pmatrix} L \\ M \\ T \end{pmatrix} + \begin{pmatrix} 1 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & -2 & -2 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \lambda \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \implies \begin{pmatrix} \alpha \\ \beta \\ \lambda \end{pmatrix} = \begin{pmatrix} M - L - \frac{T}{2} \\ M + \frac{T}{2} \\ -M. \end{pmatrix} \tag{5.1.15}$$

It is important to remember that every dimensionless group can be expressed as a fraction. So the dimensionless groups can be rewrite as:

$$\Pi_i = \frac{\delta}{\delta_s} = \tilde{\delta}, \tag{5.1.16}$$

where δ_s is the scale factor of the variable δ y $\tilde{\delta}$ is the dimensionless form of the variable δ. Now, we can obtain the dimensionless groups for the variables x, t, m and A.

Dimensionless group of the position

The x's group is:

$$\begin{pmatrix} \alpha \\ \beta \\ \lambda \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} \quad \therefore \quad \boxed{\Pi_1 = \frac{x}{x_s} = \frac{x}{h} \implies x_s = h} \quad . \tag{5.1.17}$$

With the previous result we can se that the scale factor of the position (x_s) is the equilibrium distance, and the distance can be expressed as the product of the dimensionless distance and the scale factor. So, we obtain the following relation.

$$x = x_s\tilde{x} \implies dx = x_s d\tilde{x} \implies d^2x = x_s d^2\tilde{x} \tag{5.1.18}$$

Dimensionless group of the time

The group of t is.

$$\begin{pmatrix} \alpha \\ \beta \\ \lambda \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix} \quad \therefore \quad \boxed{\Pi_2 = \frac{t}{t_s} = t\sqrt{\frac{g}{h}} \implies t_s = \sqrt{\frac{h}{g}}} \quad . \tag{5.1.19}$$

From this result, we can find the expressions of the scale factors of the frequency and the period of oscillation. These expressions are:

$$\omega_s = \frac{1}{t_s} = \sqrt{\frac{g}{h}} \quad \wedge \quad T_s = \frac{2\pi}{\omega_s} = 2\pi\sqrt{\frac{h}{g}}. \tag{5.1.20}$$

Something important to see in the previous relations is that they are equivalent to the period and frequency of a simple pendulum. This lets us conclude that our system is analogous to a pendulum.

Now, using the relation between the variable and its scale factor, we can obtain.

$$t = t_s\tau \implies dt = t_s d\tau \implies dt^2 = t_s^2 d\tau^2 \tag{5.1.21}$$

Dimensionless group of the mass and the Energie's number

Now, the group of m:

$$\begin{pmatrix}\alpha\\ \beta\\ \lambda\end{pmatrix} = \begin{pmatrix}1\\ 1\\ -1\end{pmatrix} \quad \therefore \quad \boxed{\Pi_3 = \frac{m}{m_s} = \frac{mgh}{M} \implies m_s = \frac{M}{gh}} \quad . \tag{5.1.22}$$

This dimensionless group is very important because give us a relation between the magnetic energy and the mechanic potential energy. Using this dimensionless group we going to define Energie's number as:

$$\Lambda \equiv \frac{1}{\Pi_3} = \frac{M}{mgh}. \tag{5.1.23}$$

The Energie's number going to be so useful because it related all the variables that are not affected by time. An example of some quantity that is not affected by time is the equilibrium distance, so we can rewrite this number interm of the Energie's number as:

$$\begin{aligned} x_{eq} &= \frac{2M + mgh - \sqrt{4M^2 + m^2g^2h^2}}{2mg} = \frac{mgh\left(\frac{2M}{mgh} + 1\right) - \sqrt{m^2g^2h^2\left(\frac{4M^2}{m^2g^2h^2} + 1\right)}}{2mg} \\ &= \frac{mgh\left(\frac{2M}{mgh} + 1 - \sqrt{\frac{4M^2}{m^2g^2h^2} + 1}\right)}{2mg} = \frac{h}{2}\left(2\Lambda + 1 - \sqrt{4\Lambda^2 + 1}\right) \\ \therefore \boxed{x_{eq} = h\left(\frac{2\Lambda + 1 - \sqrt{4\Lambda^2 + 1}}{2}\right) \implies \tilde{x}_{eq} = \frac{x_{eq}}{h} = \frac{2\Lambda + 1 - \sqrt{4\Lambda^2 + 1}}{2}}. \end{aligned} \tag{5.1.24}$$

Dimensionless group of the amplitude

Obtaining the group of A.

$$\begin{pmatrix}\alpha\\ \beta\\ \lambda\end{pmatrix} = \begin{pmatrix}-1\\ 0\\ 0\end{pmatrix} \quad \therefore \quad \boxed{\Pi_4 = \frac{A}{A_s} = \frac{A}{h} \implies A_s = h} \quad . \tag{5.1.25}$$

This result lets us see that the amplitude of oscillation is a fraction of the distance between the fixed magnets. The amplitude does not have a role in the motion equation (equation 5.1.10) so we going to use this dimensionless group as the control number (ϵ) for our approximation.

$$\epsilon = \frac{A}{h} = \tilde{A}. \tag{5.1.26}$$

5.1.4 Dimensionless system

Now that we have our dimensionless groups, we going to use them to find the dimensioless equations for our system. First we going to find the dimensionless form of the equation of motion. So using the groups Π_1, Π_2 and Π_3 in the equation (5.1.10) we obtain.

$$\begin{aligned}
&\frac{d^2x}{dt^2} + g + \frac{M}{m}\left[\frac{1}{h-x} - \frac{1}{x}\right] = 0 \implies \frac{d^2(x_s\tilde{x})}{d(t_s\tau)^2} + g + \frac{M}{m}\left[\frac{1}{h - x_s\tilde{x}} - \frac{1}{x_s\tilde{x}}\right] = 0 \\
\implies &\frac{h}{\frac{h}{g}}\frac{d^2\tilde{x}}{d\tau^2} + g + \frac{M}{m}\left[\frac{1}{h-h\tilde{x}} - \frac{1}{h\tilde{x}}\right] = 0 \implies g\frac{d^2\tilde{x}}{d\tau^2} + g + \frac{M}{mh}\left[\frac{1}{1-\tilde{x}} - \frac{1}{\tilde{x}}\right] = 0 \\
\implies &\frac{d^2\tilde{x}}{d\tau^2} + 1 + \frac{M}{mgh}\left[\frac{1}{1-\tilde{x}} - \frac{1}{\tilde{x}}\right] = 0 \implies \boxed{\frac{d^2\tilde{x}}{d\tau^2} + 1 + \Lambda\left[\frac{1}{1-\tilde{x}} - \frac{1}{\tilde{x}}\right] = 0},
\end{aligned} \tag{5.1.27}$$

this differential equation is so relevant, because lets us see that the position of the free magnte only depends to the time and the Energie's number, then x is function of τ and Λ.

Now, doing some algebra we can reformulate the equation of motion as:

$$\begin{aligned}
&\frac{d^2\tilde{x}}{d\tau^2} + 1 + \Lambda\left[\frac{1}{1-\tilde{x}} - \frac{1}{\tilde{x}}\right] = 0 \implies \frac{d^2\tilde{x}}{d\tau^2} + 1 + \Lambda\left[\frac{2\tilde{x}-1}{\tilde{x}(1-\tilde{x})}\right] = 0 \\
\implies &\tilde{x}(1-\tilde{x})\left(\frac{d^2\tilde{x}}{d\tau^2} + 1 + \Lambda\left[\frac{2\tilde{x}-1}{\tilde{x}(1-\tilde{x})}\right] = 0\right) \implies \tilde{x}(1-\tilde{x})\frac{d^2\tilde{x}}{d\tau^2} + \tilde{x}(1-\tilde{x}) + \Lambda\left(2\tilde{x}-1\right) = 0 \\
&\therefore \boxed{\tilde{x}\frac{d^2\tilde{x}}{d\tau^2} - \tilde{x}^2\frac{d^2\tilde{x}}{d\tau^2} - \tilde{x}^2 + (2\Lambda+1)\,\tilde{x} - \Lambda = 0}.
\end{aligned} \tag{5.1.28}$$

As we can see in this result, the differential equation is non lineal and is impossible to solve it in an anallytical way, so we going to use a numerical solution and an approximation solution.

The next step is to dimensionless the initial conditions of the system, so using the dimensionless groups we obtain the following initial conditions.

$$\begin{aligned}
&x(0) = x_{eq} + A \implies h\tilde{x}(0) = x_{eq} + A \implies \tilde{x}(0) = \frac{x_{eq} + A}{h} \\
&\implies \boxed{\tilde{x}(0) = \frac{x_{eq}}{h} + \epsilon \therefore \tilde{x}(0) = \frac{2\Lambda + 1 - \sqrt{4\Lambda^2+1}}{2} + \epsilon.} \\
&\dot{x}(0) = 0 \implies h\dot{\tilde{x}}(0) = 0 \implies \boxed{\dot{\tilde{x}}(0) = 0}
\end{aligned} \tag{5.1.29}$$

Now, rescaling the domain of the dimensionless amplitude (ϵ) system.

$$0 = x_{eq} + A \implies A = -x_{eq} \implies \epsilon = -\frac{x_{eq}}{h} = -\frac{2\Lambda + 1 - \sqrt{4\Lambda^2 + 1}}{2}$$

$$h = x_{eq} + A \implies A = h - x_{eq} \implies \epsilon = 1 - \frac{x_{eq}}{h} = 1 - \frac{2\Lambda + 1 - \sqrt{4\Lambda^2 + 1}}{2} \tag{5.1.30}$$

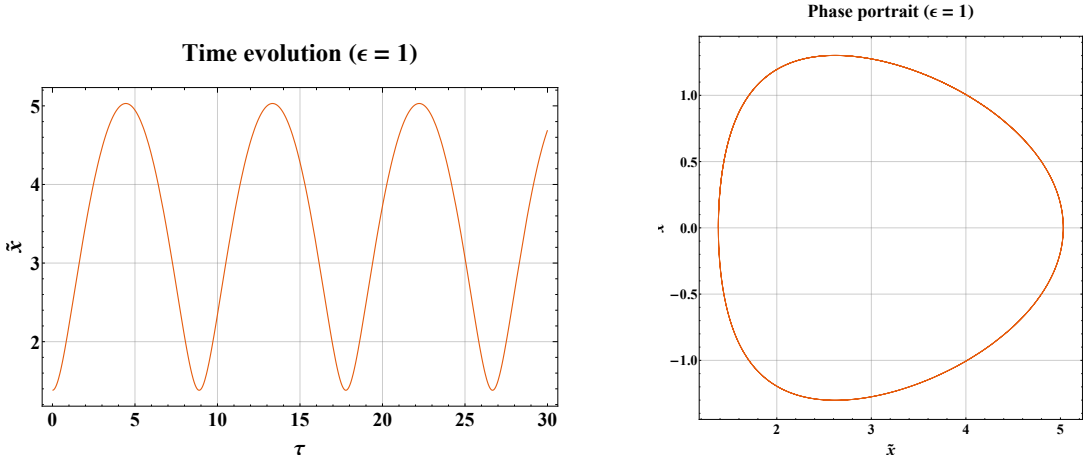
$$\therefore \quad \boxed{-\frac{x_{eq}}{h} < \epsilon < 1 - \frac{x_{eq}}{h} \implies -\frac{2\Lambda + 1 - \sqrt{4\Lambda^2 + 1}}{2} < \epsilon < \frac{1 - 2\Lambda + \sqrt{4\Lambda^2 + 1}}{2}}$$

5.1.5 Numerical solution

As in the previous problem, we need to find the numerical solution to contrast our series approximation with it. And to find the numerical solution, we need to use a numerical method. The numerical method that we are going to use is the Runge-Kutta method.

We are going to use the command ***NDSolve*** of the ***Wolfram Mathematica*** software. For this numerical simulation, we are going to use $\Lambda = 1$.

Suppose we use $\epsilon = 1$ in our approximated solution. In that case, we get the plot that expresses the evolution of the system by time, and the parametric plot communicates the behavior of the system (this can be seen in figure 5.3).



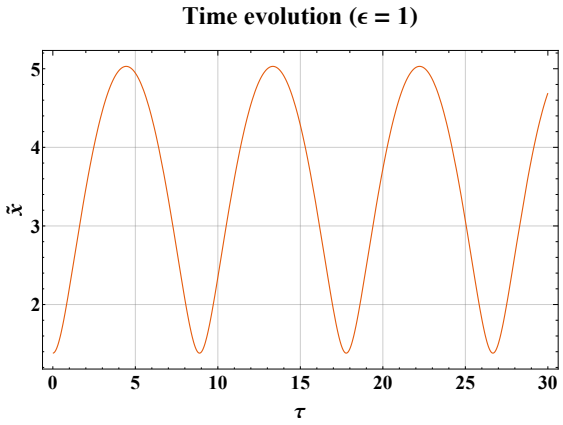
((a)) Time evolution ((b)) Phase portrait

Figure 5.3: Plot and parametric plot of the numerical solution with $\epsilon = 1$

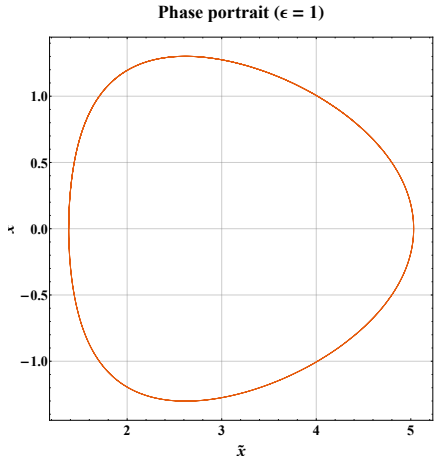
For the previous figure, we can see that the movement of our oscillator is similar to an elastic ball that is bouncing on the ground. So we can imagine that if the displacement from the equilibrium point is minimum, the behavior of the movement is more circular or cyclical. To discover if this is true, we need to simulate the system with different epsilon values.

Now, we simulated three different systems with the intention of contrasting different behaviors of our system. We are going to put these different systems into a figure. In our figure (5.4), we can see how the system acts under the epsilon values of 1, 0.5, and 0.01.

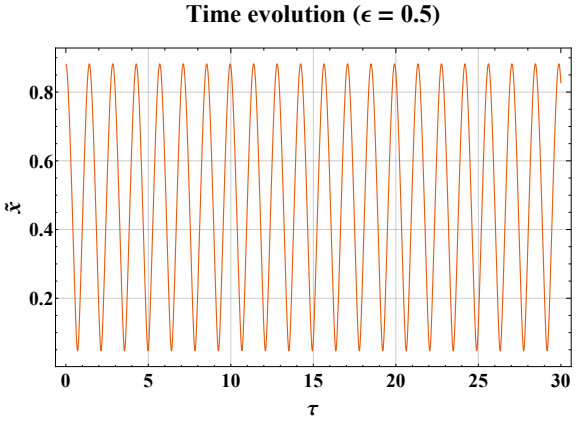
As we can see in the figure (5.4), the system's behavior is clearly similar to a circle when the epsilon value decreases. Then, the complexity of solving the system increases when epsilon increases.



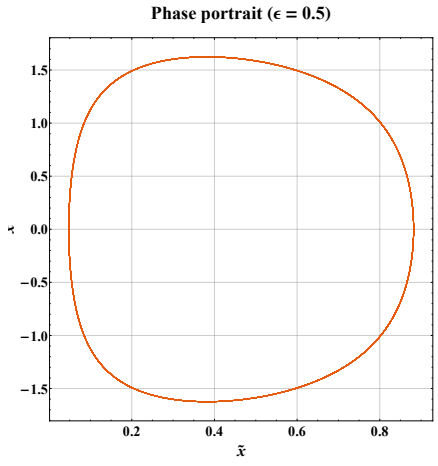
((a)) Time evolution ($\epsilon = 1$)



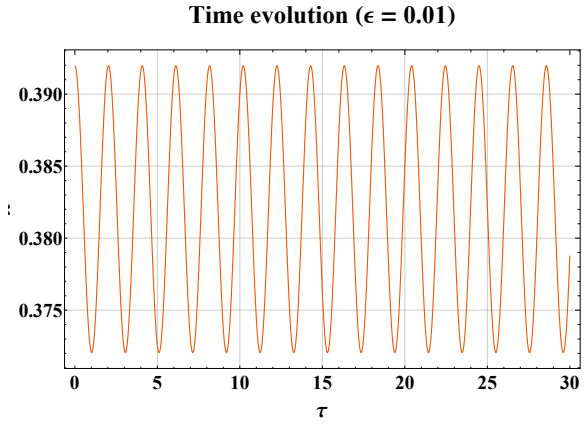
((b)) Phase portrait ($\epsilon = 1$)



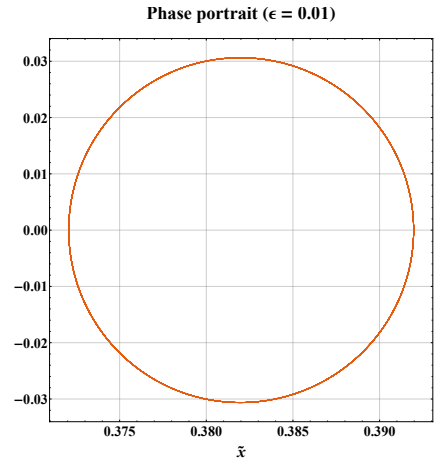
((c)) Time evolution ($\epsilon = 0.5$)



((d)) Phase portrait ($\epsilon = 0.5$)



((e)) Time evolution ($\epsilon = 0.01$)



((f)) Phase portrait ($\epsilon = 0.01$)

Figure 5.4: Time evolution and phase portraits of the system for $\epsilon = 1$ and $\epsilon = 0.5$.

In the figure previously named (figure (5.4)), we can see that when $\epsilon \approx 0$ is easier to fit a cyclical function, a cosine function. So, we can infer that the first-order approximation is a cosine function.

5.1.6 Asymptotic approximation

As we saw in the previous chapter, the asymptotic approximation consists of regulating the apport of every term in a polynomial with the intention to approximate that polynomial to the analytical solution. When the polynomial has infinite terms, the polynomial is equal to the solution. So our solution has the form:

$$x(t) \sim \sum_{i=0}^{N} \epsilon^i x_i(t) \implies x(t) = \lim_{N\to\infty} \sum_{i=0}^{N} \epsilon^i x_i(t). \tag{5.1.31}$$

Applying this to the dimensionless form, we obtain.

$$\tilde{x}(\tau) \sim \sum_{i=0}^{N} \epsilon^i \tilde{x}_i(\tau) \implies \tilde{x}(\tau) = \lim_{N\to\infty} \sum_{i=0}^{N} \epsilon^i \tilde{x}_i(\tau). \tag{5.1.32}$$

Applying to our dimensionless differential equation.

$$\begin{aligned}
\tilde{x}\frac{d^2\tilde{x}}{d\tau^2} - \tilde{x}^2\frac{d^2\tilde{x}}{d\tau^2} - \tilde{x}^2 + (2\Lambda+1)\,\tilde{x} - \Lambda = 0 \\
\sum_{i=0}^{N} \epsilon^i \tilde{x}_i \frac{d^2}{d\tau^2}\left(\sum_{j=0}^{N} \epsilon^j \tilde{x}_j\right) - \left(\sum_{k=0}^{N} \epsilon^k \tilde{x}_k\right)^2 \frac{d^2}{d\tau^2}\left(\sum_{\ell=0}^{N} \epsilon^\ell \tilde{x}_\ell\right) - \left(\sum_{m=0}^{N} \epsilon^m \tilde{x}_m\right)^2 \\
+ (2\Lambda+1)\left(\sum_{n=0}^{N} \epsilon^n \tilde{x}_n\right) - \Lambda = 0 \\
\sum_{i=0}^{N}\sum_{j=0}^{N} \epsilon^{i+j} \tilde{x}_i \frac{d^2\tilde{x}_j}{d\tau^2} - \left(\sum_{k=0}^{N} \epsilon^k \tilde{x}_k\right)^2 \left(\sum_{\ell=0}^{N} \epsilon^\ell \frac{d^2\tilde{x}_\ell}{d\tau^2}\right) - \left(\sum_{m=0}^{N} \epsilon^m \tilde{x}_m\right)^2 \\
+ (2\Lambda+1)\left(\sum_{n=0}^{N} \epsilon^n \tilde{x}_n\right) - \Lambda = 0.
\end{aligned} \tag{5.1.33}$$

5.1.7 First order approximation solution

Before to find the first order solution, we need to find the zero order solution. For that we need to balance all the terms that have ϵ^0 in our differential equation (equation 5.1.33) with the initial conditions. The equation tha we need to solve is:

$$\epsilon^{0+0}\tilde{x}_0\frac{d^2\tilde{x}_0}{d\tau^2} - \left(\epsilon^0\tilde{x}_0\right)^2\left(\epsilon^0\frac{d^2\tilde{x}_0}{d\tau^2}\right) - \left(\epsilon^0\tilde{x}_0\right)^2 + (2\Lambda+1)\left(\epsilon^0\tilde{x}_0\right) - \Lambda = 0. \tag{5.1.34}$$

It is important to highlight that x_0 is the constant term of the polynom, so the derivation of this term is zero and the equation is now:

$$-\tilde{x}_0^2 + (2\Lambda+1)\,\tilde{x}_0 - \Lambda = 0 \implies \boxed{\tilde{x}_0 = \frac{2\Lambda+1-\sqrt{4\Lambda^2+1}}{2} = \frac{x_{eq}}{h}}. \tag{5.1.35}$$

Now, we can continue to find the first order solution. For this order we need all the terms in our polinomic solution with ϵ^1. This terms are:

$$\begin{aligned}\tilde{x}_0\frac{d^2\tilde{x}_1}{d\tau^2}-\tilde{x}_0^2\frac{d^2\tilde{x}_1}{d\tau^2}-\underbrace{2\tilde{x}_0\tilde{x}_1\frac{d^2\tilde{x}_0}{d\tau^2}+\tilde{x}_1\frac{d^2\tilde{x}_0}{d\tau^2}}_{0}-2\tilde{x}_0\tilde{x}_1+2\Lambda\tilde{x}_1+\tilde{x}_1=0\\ (\tilde{x}_0-\tilde{x}_0^2)\frac{d^2\tilde{x}_1}{d\tau^2}+\tilde{x}_1(2\Lambda+1-2\tilde{x}_0)=0\\ \frac{d^2\tilde{x}_1}{d\tau^2}+\left(\frac{2\Lambda+1-2\tilde{x}_0}{\tilde{x}_0-\tilde{x}_0^2}\right)\tilde{x}_1=0.\end{aligned} \tag{5.1.36}$$

The last expression is the differential equation of a harmonic oscillator.

For simplify the calculus we define $\omega=\sqrt{\frac{2\Lambda+1-2\tilde{x}_0}{\tilde{x}_0-\tilde{x}_0^2}}$, which is the is the dimensionless frequency of the system. Using the value of $\tilde{x}_0$ the dimensionless frequency is:

$$\omega_0=\sqrt{4\Lambda+2\sqrt{1+4\Lambda^2}+\frac{1}{\Lambda}}. \tag{5.1.37}$$

Then, the differential equation is:

$$\frac{d^2\tilde{x}_1}{d\tau^2}+\omega^2\tilde{x}_1=0\implies\frac{d^2\tilde{x}_1}{d\tau^2}=-\omega^2\tilde{x}_1\implies\boxed{\tilde{x}_1(\tau)=C_1\cos(\omega\tau)+C_2\sin(\omega\tau)}. \tag{5.1.38}$$

As we can see for the last result is that ω is the dimensionless frequency of the system, so the frequency of oscillation is:

Now, using the solution of $\tilde{x}_1$ in the first order polinmic solution, we obtain that our solution has the next form:

$$\tilde{x}(\tau)\sim\tilde{x}_0+\tilde{x}_1(\tau)\epsilon=\tilde{x}_0+(C_1\cos(\omega\tau)+C_2\sin(\omega\tau))\,\epsilon \tag{5.1.39}$$

Using the initial conditions we can find the values of the constants C_1 and C_2. Starting with the position's initicial condition,

$$\begin{aligned}\tilde{x}(0)\sim\tilde{x}_0+(C_1\cos(0)+C_2\sin(0))\,\epsilon=\tilde{x}_0+C_1\epsilon\\ \frac{x_{eq}}{h}+\epsilon\sim\tilde{x}_0+C_1\epsilon=\frac{x_{eq}}{h}+C_1\epsilon\ \therefore\ \boxed{C_1=1}\end{aligned} \tag{5.1.40}$$

Continuing with the speed's initial condition.

$$\begin{aligned}\dot{\tilde{x}}(0)\sim\frac{d}{d\tau}\left[\tilde{x}_0+(C_1\cos(0)+C_2\sin(0))\,\epsilon\right]=0-(C_1\sin(0)-C_2\cos(0))\epsilon\\ 0\sim C_2\epsilon\ \therefore\ \boxed{C_2=0}\end{aligned} \tag{5.1.41}$$

Therefore, our solution at first order is:

$$\tilde{x}(\tau) \sim \tilde{x}_0 + \cos(\omega\tau)\epsilon \implies \boxed{\tilde{x}(\tau) \sim \frac{2\Lambda + 1 - \sqrt{4\Lambda^2+1}}{2} + \cos\left(\sqrt{4\Lambda + 2\sqrt{1+4\Lambda^2} + \frac{1}{\Lambda}}\tau\right)\epsilon}. \tag{5.1.42}$$

Now, plotting our approximation solution with the numerical solution, we can contrast the precision of our approximation. To contrast, we are going to plot three different epsilon values; they are 0.518034, 0.5, and 0.01.

We selected the number $\epsilon = 0.518034$ because we can not use a value where the distance between the free magnet and the lowest point ($\tilde{x} = 0$) will be bigger than 1, and remember that ϵ is the displacement that we pushed the free magnet out of the equilibrium point. Then, to obtain this specific epsilon value, we use $0.9 - \tilde{x}_0$.

If we see our figure (5.5), we can see that our approximation fits to the numerical solution when epsilon is very small.

5.1.8 Second order approximation solution

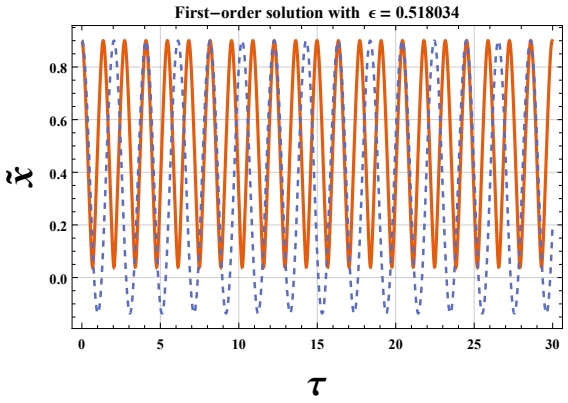
Now, for the second order we need only the terms with ϵ^2. The terms with power of 2 are:

$$\begin{aligned}\tilde{x}_0\frac{d^2\tilde{x}_2}{d\tau^2} - \tilde{x}_0^2\frac{d^2\tilde{x}_2}{d\tau^2} + \tilde{x}_1\frac{d^2\tilde{x}_1}{d\tau^2} - 2\tilde{x}_0\tilde{x}_1\frac{d^2\tilde{x}_1}{d\tau^2} - 2\tilde{x}_0\tilde{x}_2\frac{d^2\tilde{x}_0}{d\tau^2} \\ + \tilde{x}_2\frac{d^2\tilde{x}_0}{d\tau^2} + \tilde{x}_1^2\frac{d^2\tilde{x}_0}{d\tau^2} - 2\tilde{x}_0\tilde{x}_2 + 2\Lambda\tilde{x}_2 + \tilde{x}_2 - \tilde{x}_1^2 = 0\end{aligned} \tag{5.1.43}$$

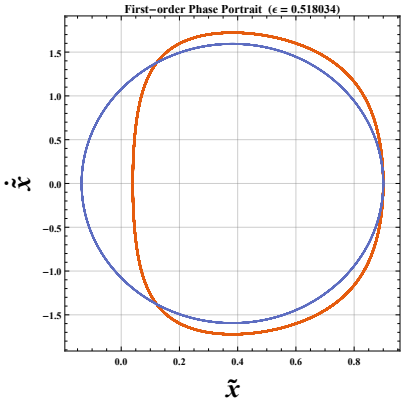
We know that $\tilde{x}_0$ is a constant, that lets us simplify the differential equation. Now, our equation has the next form:

$$\begin{aligned}\tilde{x}_0\frac{d^2\tilde{x}_2}{d\tau^2} - \tilde{x}_0^2\frac{d^2\tilde{x}_2}{d\tau^2} + \tilde{x}_1\frac{d^2\tilde{x}_1}{d\tau^2} - 2\tilde{x}_0\tilde{x}_1\frac{d^2\tilde{x}_1}{d\tau^2} - 2\tilde{x}_0\tilde{x}_2 + 2\Lambda\tilde{x}_2 + \tilde{x}_2 - \tilde{x}_1^2 &= 0 \\ \left(\tilde{x}_0 - \tilde{x}_0^2\right)\frac{d^2\tilde{x}_2}{d\tau^2} + \tilde{x}_1(-\omega^2\tilde{x}_1)\left(1 - 2\tilde{x}_0\right) + \tilde{x}_2\left(2\Lambda + 1 - 2\tilde{x}_0\right) - \tilde{x}_1^2 &= 0 \\ \left(\tilde{x}_0 - \tilde{x}_0^2\right)\frac{d^2\tilde{x}_2}{d\tau^2} - \tilde{x}_1^2\left(\omega_0^2 - 2\omega^2\tilde{x}_0 + 1\right) + \tilde{x}_2\left(2\Lambda + 1 - 2\tilde{x}_0\right) &= 0 \\ \frac{d^2\tilde{x}_2}{d\tau^2} - \tilde{x}_1^2\left(\frac{\omega^2 - 2\omega^2\tilde{x}_0 + 1}{\tilde{x}_0 - \tilde{x}_0^2}\right) + \tilde{x}_2\underbrace{\left(\frac{2\Lambda + 1 - 2\tilde{x}_0}{\tilde{x}_0 - \tilde{x}_0^2}\right)}_{\omega^2} &= 0 \\ \frac{d^2\tilde{x}_2}{d\tau^2} - \tilde{x}_1^2\left(\frac{\omega^2 - 2\omega^2\tilde{x}_0 + 1}{\tilde{x}_0 - \tilde{x}_0^2}\right) + \omega^2\tilde{x}_2 &= 0\end{aligned} \tag{5.1.44}$$

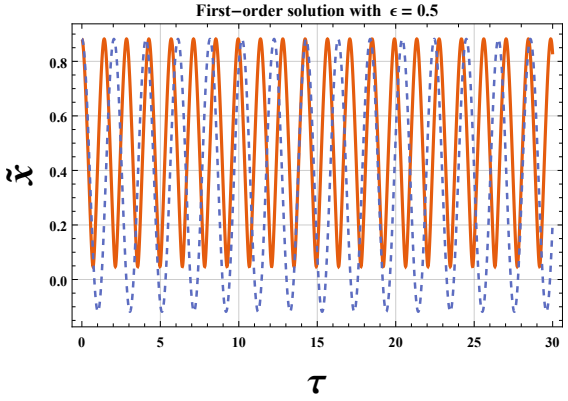
To simplify the differential equation, we define $\alpha^2 = \frac{\omega^2 - 2\omega^2\tilde{x}_0 + 1}{\tilde{x}_0 - \tilde{x}_0^2}$. Then our differential equation, using the definition of $\tilde{x}_1$ is:



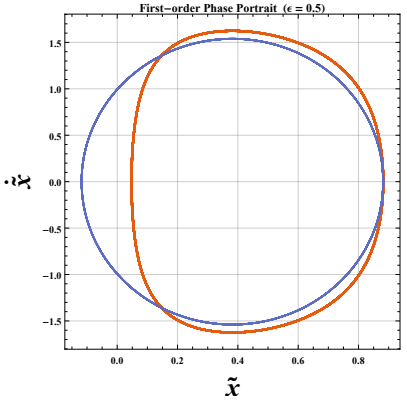
((a)) Time evolution contrast ($\epsilon = 0.518034$)



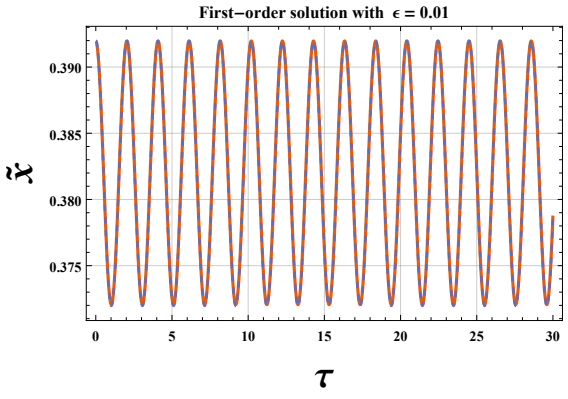
((b)) Phase portrait contrast ($\epsilon = 0.518034$)



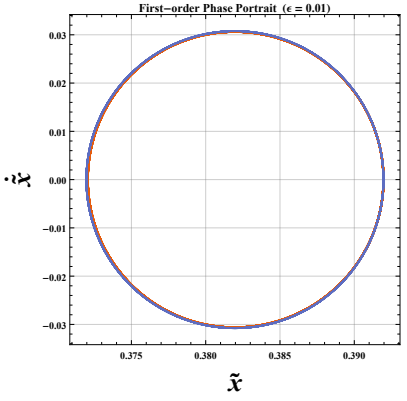
((c)) Time evolution ($\epsilon = 0.5$)



((d)) Phase portrait ($\epsilon = 0.5$)



((e)) Time evolution ($\epsilon = 0.01$)



((f)) Phase portrait ($\epsilon = 0.01$)

Figure 5.5: Approximation of the solution first order solution to many values of ϵ.

$$\frac{d^2\tilde{x}_2}{d\tau^2} - \alpha^2\tilde{x}_1^2 + \omega^2\tilde{x}_2 = 0$$

$$\frac{d^2\tilde{x}_2}{d\tau^2} - \alpha^2\cos^2(\omega\tau) + \omega^2\tilde{x}_2 = 0 \tag{5.1.45}$$

$$\therefore \boxed{\frac{d^2\tilde{x}_2}{d\tau^2} + \omega^2\tilde{x}_2 = \alpha^2\cos^2(\omega\tau)}$$

Now, we proceed to solve the differential equation. First we find the homogeneous solution.

$$\frac{d^2\tilde{x}_2}{d\tau^2} + \omega^2\tilde{x}_2 = 0 \implies \boxed{\tilde{x}_{2h}(\tau) = C_1\cos(\omega\tau) + C_2\sin(\omega\tau)} \tag{5.1.46}$$

The particular solution for our differential equation is.

$$\tilde{x}_{2p} = -\frac{\alpha^2\left[\cos(2\omega\tau) - 3\right]}{6\omega^2} \tag{5.1.47}$$

Therefore, the general solution for our differential equation.

$$\tilde{x}_2(\tau) = \tilde{x}_{2h} + \tilde{x}_{2p} = C_1\cos(\omega\tau) + C_2\sin(\omega\tau) - \frac{\alpha^2\left[\cos(2\omega\tau) - 3\right]}{6\omega^2} \tag{5.1.48}$$

Using the initial conditions to find the constants of the function. Applying the speed's initial condition.

$$\begin{aligned} \dot{\tilde{x}}(0) &= 0 \\ -C_1\omega_0\sin(0) + C_2\omega_0\cos(0) - \frac{\alpha^2\left[-2\omega\sin(0)\right]}{6\omega^2} &= 0 \\ \omega_0 C_2 &= 0 \\ \therefore \boxed{C_2 = 0}&. \end{aligned} \tag{5.1.49}$$

Exploiting the position's initial condition we obtain.

$$\begin{aligned} \tilde{x}(0) = \tilde{x}_{eq} + \epsilon + 0\epsilon^2 &\implies \tilde{x}_2 = 0 \\ \implies 0 = C_1\cos(0) - \frac{\alpha^2\left[\cos(0) - 3\right]}{6\omega^2} & \\ \implies 0 = C_1 + \frac{2\alpha^2}{6\omega^2} & \\ \therefore \boxed{C_1 = -\frac{\alpha^2}{3\omega^2}}&. \end{aligned} \tag{5.1.50}$$

Ergo our solution at second order now has the next form:

$$\tilde{x}_2(\tau) = -\frac{\alpha^2}{3\omega^2}\cos(\omega t) - \frac{\alpha^2\left[\cos(2\omega\tau) - 3\right]}{6\omega^2} = -\frac{\alpha^2}{3\omega_0^2}\left(\cos(\omega\tau) + \frac{1}{2}\left[\cos(2\omega\tau) - 3\right]\right), \tag{5.1.51}$$

to simplify our function we can use some trigonometric identities. Then the function for second-order is:

$$\tilde{x}_2(\tau) = \frac{2\alpha^2}{3\omega_0^2}\sin^2\left(\frac{\omega\tau}{2}\right)(2 + \cos(\omega\tau)) \tag{5.1.52}$$

Therefore, our approximated solution is:

$$\tilde{x}(\tau) \sim \tilde{x}_0 + \tilde{x}_1\epsilon + \tilde{x}_2\epsilon^2 = \tilde{x}_{eq} + \cos(\omega\tau)\epsilon + \frac{2\alpha^2\epsilon^2}{3\omega_0^2}\sin^2\left(\frac{\omega\tau}{2}\right)(2+\cos(\omega\tau)) \tag{5.1.53}$$

As the same as the first order approximation solution, we can plot our second order approximation series with the numerical solution. And it is not a surprise that this approximation is better than the first order. We can see it in the biggest epsilon value, because in the first order (5.5) the approximation passes the width of the numerical solution, but in our second order our approximation does not pass it.

5.2 System with friction

The friction system includes the dissipative forces, in the other hand the external forces does not interact with the physical system. With this, the forces in the system are:

$$F = -F_f - F_g + F_M = -\beta v - mg + \frac{M}{x} - \frac{M}{h-x}, \tag{5.2.1}$$

where F is the force, v is the speed of the free magnet, β is the friction coefficient, m is the mass of the free magnet, g the gravitational acceleration, x the distance between the two magnets and M is the magnetic energy constant. If the system is in equilibrium, then the net force and the speed movement of the free magnet are cero, then we can find the equilibrium distance.

$$\begin{aligned} \frac{M}{x} - mg - \frac{M}{h-x} = 0 \implies M(h-x) &= mgx(h-x) + Mx \\ \therefore \boxed{x_{eq} = \frac{2M + mgh \pm \sqrt{4M^2 + m^2g^2h^2}}{2mg}} &\quad . \end{aligned} \tag{5.2.2}$$

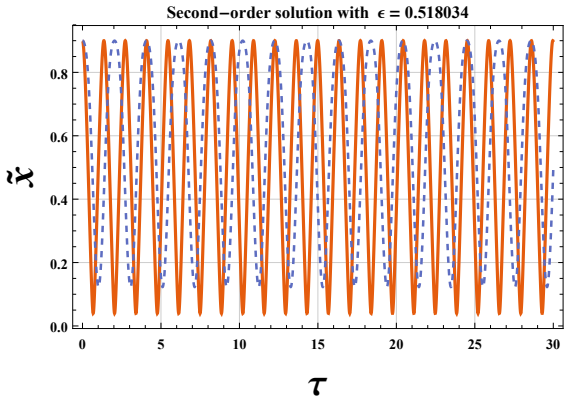
It is important to see that the equilibrium position requieres an static position, this is because our system is unstable for the presence of the friction. The equilibrium position that we obtained is the same that we found to the ideal case (5.1.4).

5.2.1 Lagrangian analysis

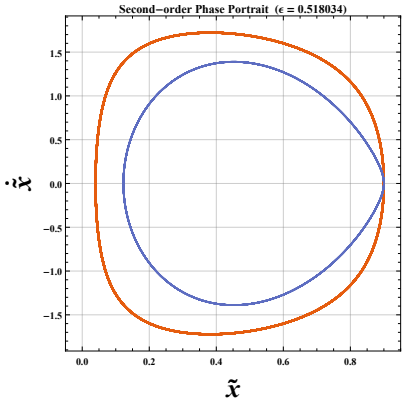
As we have shown, the lagrangian analysis let us know the equations of motion that we need to solve to described all the dynamic of the system. The lagrangian of our system only have a degree of freedom, so the dynamic of the system is described by one coordinate (x), then our lagrangian has the following expression.

$$L = \frac{1}{2}m\dot{x}^2 - V(x), \tag{5.2.3}$$

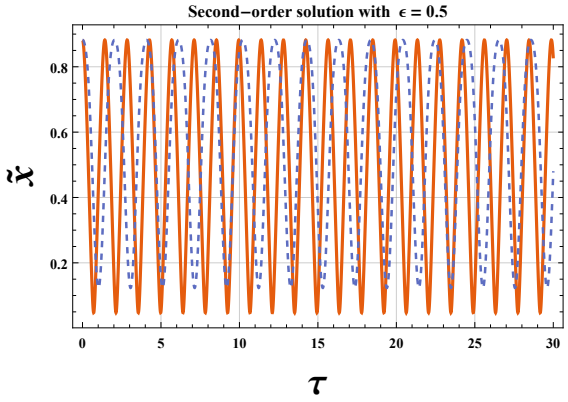
where $V(x)$ is the potential energy in our system. To find the potential energy we going to use the relation between the work and the potential energy when the system is conservative, as we calculated in the equation (5.1.7). Then, the potential energy is.



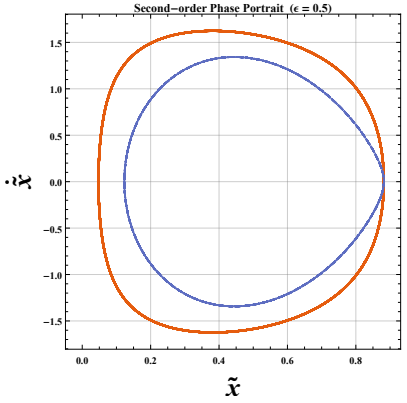
((a)) Time evolution contrast ($\epsilon = 0.518034$)



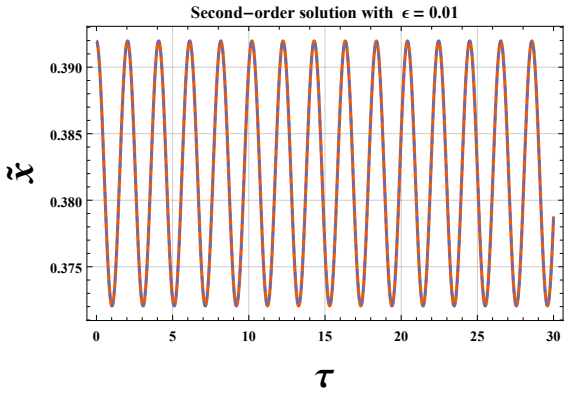
((b)) Phase portrait contrast ($\epsilon = 0.518034$)



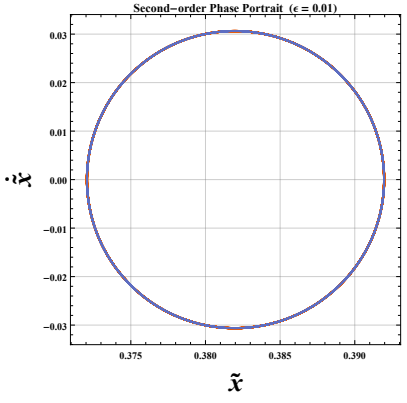
((c)) Time evolution ($\epsilon = 0.5$)



((d)) Phase portrait ($\epsilon = 0.5$)



((e)) Time evolution ($\epsilon = 0.01$)



((f)) Phase portrait ($\epsilon = 0.01$)

Figure 5.6: Approximation of the solution second order solution to many values of ϵ.

$$V(x) = mgx - M\ln(x) - M\ln(h - x). \tag{5.2.4}$$

Introducing our potential energy in the lagrangian, we obtain:

$$\boxed{L = \frac{1}{2}m\dot{x}^2 - mgx + M\ln(x) + M\ln(h-x)}. \tag{5.2.5}$$

With our lagrangian, we can applied the Euler-Lagrange equation to find the equation of motion of our system. As we know, our system present an external force that is the friction force ($Q = F_f$). So the equation of motion for the coordinate x is.

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} &= Q \\ \implies m\ddot{x} + mg + \frac{M}{h-x} - \frac{M}{x} &= -\beta v \\ \implies \ddot{x} + \frac{\beta}{m}\dot{x} + g + \frac{M}{m(h-x)} - \frac{M}{mx} &= 0 \\ \therefore \quad \boxed{\frac{d^2x}{dt^2} + \frac{\beta}{m}\frac{dx}{dt} + g + \frac{M}{m}\left[\frac{1}{h-x} - \frac{1}{x}\right] = 0}. & \end{aligned} \tag{5.2.6}$$

This equation of motion is relevant because it lets us know the accelerations that generate the movement in our system. It's important to remark that the differential equation does not have an analytical solution.

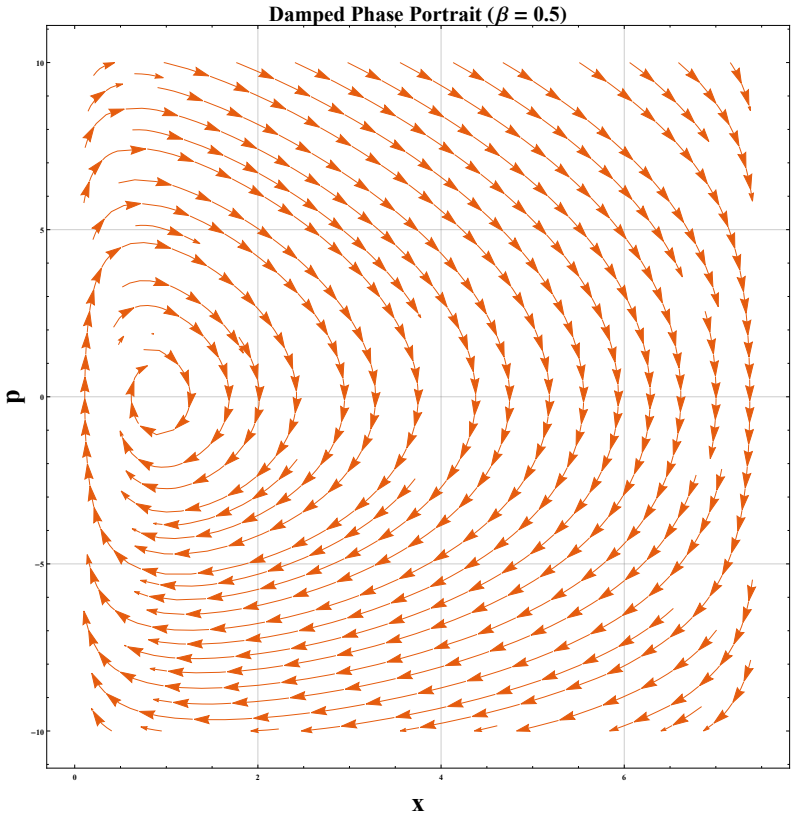


Figure 5.7: Phase portrait of the system with $M = 50$, $\beta = 0.5$, $h = 5$, $g = 10$ and $m = 1$

If we use our equation of motion, we can plot the Phase-portrait of this system to know the behavior of our system. We can see in the figure (5.2), the system follows a closed cyclical trajectory, so we can confirm that the system oscillates.

5.2.2 Dimensional analysis

In our system we have seven variables ($n = 8$), these variables are the mass (m), the time (t), the position (x), the gravity (g), the oscillation amplitude (A), the magnetic energy (M), (β) the friction coefficient and the distance between the fixed magnets (h). For the nature of our problem, we only need to use three dimension ($j = 3$) to caracterize all the problem, the dimensions that we need to use is the matter, the longitude and the time. Using the Buckingham's theorem, we know that there are four dimensionless groups ($k = n - j = 5$) that describe our system.

Using a dimensional analysis over every variable:

$$[x] = L, [h] = L, [A] = L, [t] = T, [m] = M, [g] = LT^{-2}, [M] = ML^2T^{-2}, [\beta] = MT^{-1} \tag{5.2.7}$$

For this system we going to use h, g and M as dimensional variables, this because h is the geometric parameter of the system, g is the cinematic parameter and M is the dynamic parameter. As we saw in the previous chapter, every dimensinal group is the product of the dimensional variables, so we have that the ith group is expressed as:

$$\Pi_i = \delta h^{\alpha} g^{\beta} M^{\lambda}, \tag{5.2.8}$$

where δ is the variable that we want to dimensionless, in our system we have the variables x, t, m and A; α, β and λ are the powers of the distance h, the gravity and the magnetic energy that dimensionless the ith group.

To find the powers of our dimensional variables that dimensionless every group we can construct an equations system, this system is.

$$\begin{pmatrix} L \\ M \\ T \end{pmatrix} + \begin{pmatrix} 1 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & -2 & -2 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \lambda \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \implies \begin{pmatrix} \alpha \\ \beta \\ \lambda \end{pmatrix} = \begin{pmatrix} M - L - \frac{T}{2} \\ M + \frac{T}{2} \\ -M. \end{pmatrix} \tag{5.2.9}$$

It is important to remember that every dimensionless group can be expressed as a fraction. So the dimensionless groups can be rewrite as:

$$\Pi_i = \frac{\delta}{\delta_s} = \tilde{\delta}, \tag{5.2.10}$$

where δ_s is the scale factor of the variable δ y $\tilde{\delta}$ is the dimensionless form of the variable δ. Now, we can obtain the dimensionless groups for the variables x, t, m and A.

Dimensionless group of the position

The x's group is:

$$\begin{pmatrix} \alpha \\ \beta \\ \lambda \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} \quad \therefore \quad \boxed{\Pi_1 = \frac{x}{x_s} = \frac{x}{h} \implies x_s = h} \quad . \tag{5.2.11}$$

Con el resultado anterior se puede observar que el factor de escala de posición es la distancia de equilibrio y la distancia se puede expresar como el productl de la distancia adimensional y el factor de escala, de tal manera que se obtiene.

$$x = x_s\tilde{x} \implies dx = x_s d\tilde{x} \implies d^2x = x_s d^2\tilde{x} \tag{5.2.12}$$

Dimensionless group of the time

The group of t is.

$$\begin{pmatrix}\alpha\\ \beta\\ \lambda\end{pmatrix} = \begin{pmatrix}-\frac{1}{2}\\ \frac{1}{2}\\ 0\end{pmatrix} \quad \therefore \quad \boxed{\Pi_2 = \frac{t}{t_s} = t\sqrt{\frac{g}{h}} \implies t_s = \sqrt{\frac{h}{g}}} \quad . \tag{5.2.13}$$

Del resultado anterior se puede encontrar la expresión de los factores de escala de la frecuencia y el periodo de oscilación, dichas expresiones son las siguientes.

$$\omega_s = \frac{1}{t_s} = \sqrt{\frac{g}{h}} \quad \wedge \quad T_s = \frac{2\pi}{\omega_s} = 2\pi\sqrt{\frac{h}{g}}. \tag{5.2.14}$$

Lo importante a observar de las expresiones anteriores es que son equivalentes a la frecuencia y periodo de un péndulo, por lo que podemos concluir que nuestro sistema es análogo a un movimiento pendular.

Ahora, utilizando la relación entre la variable del sistema y su factor de escala y la variable adimensional, se obtiene.

$$t = t_s\tau \implies dt = t_s d\tau \implies dt^2 = t_s^2 d\tau^2 \tag{5.2.15}$$

Dimensionless group of the mass and the Energie's number

Now, the group of m:

$$\begin{pmatrix}\alpha\\ \beta\\ \lambda\end{pmatrix} = \begin{pmatrix}1\\ 1\\ -1\end{pmatrix} \quad \therefore \quad \boxed{\Pi_3 = \frac{m}{m_s} = \frac{mgh}{M} \implies m_s = \frac{M}{gh}} \quad . \tag{5.2.16}$$

This dimensionless group is very important because give us a relation between the magnetic energy and the mechanic potential energy, Using this dimensionless group we going to define Energie's number as:

$$\Lambda \equiv \frac{1}{\Pi_3} = \frac{M}{mgh}. \tag{5.2.17}$$

The Energie's number going to be so useful because it related all the variables that are not affected by time. An example of some quantity that is not affected by time is the equilibrium distance, so we can rewrite this number interm of the Energie's number as:

$$
\begin{aligned}
x_{eq} &= \frac{2M + mgh - \sqrt{4M^2 + m^2g^2h^2}}{2mg} \\
&= \frac{mgh\left(\frac{2M}{mgh} + 1\right) - \sqrt{m^2g^2h^2\left(\frac{4M^2}{m^2g^2h^2} + 1\right)}}{2mg} \\
&= \frac{mgh\left(\frac{2M}{mgh} + 1 - \sqrt{\frac{4M^2}{m^2g^2h^2} + 1}\right)}{2mg} \\
&= \frac{h}{2}\left(2\Lambda + 1 - \sqrt{4\Lambda^2 + 1}\right) \\
\implies x_{eq} &= h\left(\frac{2\Lambda + 1 - \sqrt{4\Lambda^2 + 1}}{2}\right) \\
\therefore \boxed{\tilde{x}_{eq} = \frac{x_{eq}}{h} = \frac{2\Lambda + 1 - \sqrt{4\Lambda^2 + 1}}{2}}&.
\end{aligned}
\tag{5.2.18}
$$

Dimensionless group of the amplitude

Obtaining the group of A.

$$
\begin{pmatrix} \alpha \\ \beta \\ \lambda \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} \quad \therefore \quad \boxed{\Pi_4 = \frac{A}{A_s} = \frac{A}{h} \implies A_s = h} \quad . \tag{5.2.19}
$$

This result lets us see that the amplitude of oscillation is a fraction of the distance between the fixed magnets. The amplitude does not have a role in the motion equation (equation 5.2.6) so we going to use this dimensionless group as the control number (ϵ) for our approximation.

$$
\epsilon = \frac{A}{h} = \tilde{A}. \tag{5.2.20}
$$

5.2.3 Dimensionless system

Now that we have our dimensionless groups, we will use them to find the dimensionless equations for our system. First, we are going to find the dimensionless form of the equation of motion. So using the groups Π_1, Π_2 and Π_3 in the equation (5.2.6) we obtain.

$$
\begin{aligned}
\frac{d^2x}{dt^2} + \frac{\beta}{m}\frac{dx}{dt} + g + \frac{M}{m}\left[\frac{1}{h-x} - \frac{1}{x}\right] &= 0 \\
\implies \frac{d^2(x_s\tilde{x})}{d(t_s\tau)^2} + \frac{\beta}{m}\frac{d(x_s\tilde{x})}{d(t_s\tau)} + g + \frac{M}{m}\left[\frac{1}{h-x_s\tilde{x}} - \frac{1}{x_s\tilde{x}}\right] &= 0 \\
\implies \frac{h}{\frac{h}{g}}\frac{d^2\tilde{x}}{d\tau^2} + \frac{\beta h}{m\sqrt{\frac{h}{g}}}\frac{d\tilde{x}}{d\tau} + g + \frac{M}{m}\left[\frac{1}{h-h\tilde{x}} - \frac{1}{h\tilde{x}}\right] &= 0 \\
\implies g\frac{d^2\tilde{x}}{d\tau^2} + \frac{\beta}{m}\sqrt{hg}\frac{d\tilde{x}}{d\tau} + g + \frac{M}{mh}\left[\frac{1}{1-\tilde{x}} - \frac{1}{\tilde{x}}\right] &= 0 \\
\implies \frac{d^2\tilde{x}}{d\tau^2} + \frac{\beta}{m}\sqrt{\frac{h}{g}}\frac{d\tilde{x}}{d\tau} + 1 + \frac{M}{mgh}\left[\frac{1}{1-\tilde{x}} - \frac{1}{\tilde{x}}\right] &= 0 \\
\implies \boxed{\frac{d^2\tilde{x}}{d\tau^2} + 2\Phi\frac{d\tilde{x}}{d\tau} + 1 + \Lambda\left[\frac{1}{1-\tilde{x}} - \frac{1}{\tilde{x}}\right] = 0},
\end{aligned}
\tag{5.2.21}
$$

this differential equation is so relevant because it lets us see that the position of the free magnet only depends on the time and the Energie's number, then x is a function of τ and Λ.

Now, doing some algebra, we can reformulate the equation of motion as:

$$
\begin{aligned}
\frac{d^2\tilde{x}}{d\tau^2} + 2\Phi\frac{d\tilde{x}}{d\tau} + 1 + \Lambda\left[\frac{1}{1-\tilde{x}} - \frac{1}{\tilde{x}}\right] &= 0 \\
\implies \frac{d^2\tilde{x}}{d\tau^2} + 2\Phi\frac{d\tilde{x}}{d\tau} + 1 + \Lambda\left[\frac{2\tilde{x}-1}{\tilde{x}(1-\tilde{x})}\right] &= 0 \\
\implies \tilde{x}(1-\tilde{x})\left(\frac{d^2\tilde{x}}{d\tau^2} + 2\Phi\frac{d\tilde{x}}{d\tau} + 1 + \Lambda\left[\frac{2\tilde{x}-1}{\tilde{x}(1-\tilde{x})}\right]\right) &= 0 \\
\implies \tilde{x}(1-\tilde{x})\frac{d^2\tilde{x}}{d\tau^2} + 2\Phi\tilde{x}(1-\tilde{x})\frac{d\tilde{x}}{d\tau} + \tilde{x}(1-\tilde{x}) + \Lambda\left(2\tilde{x}-1\right) &= 0 \\
\therefore \boxed{\tilde{x}\frac{d^2\tilde{x}}{d\tau^2} - \tilde{x}^2\frac{d^2\tilde{x}}{d\tau^2} + 2\Phi\tilde{x}\frac{d\tilde{x}}{d\tau} - 2\Phi\tilde{x}^2\frac{d\tilde{x}}{d\tau} - \tilde{x}^2 + (2\Lambda+1)\,\tilde{x} - \Lambda = 0}&.
\end{aligned}
\tag{5.2.22}
$$

As we can see in this result, the differential equation is non lineal and is impossible to solve it in an anallytical way, so we going to use a numerical solution and an approximation solution.

The next step is to dimensionless the initial conditions of the system, so using the dimensionless groups we obtain the following initial conditions.

$$
\begin{aligned}
x(0) = x_{eq} + A \implies h\tilde{x}(0) = x_{eq} + A \implies \tilde{x}(0) &= \frac{x_{eq}+A}{h} \\
\implies \boxed{\tilde{x}(0) = \frac{x_{eq}}{h} + \epsilon \therefore \tilde{x}(0) = \frac{2\Lambda + 1 - \sqrt{4\Lambda^2+1}}{2} + \epsilon.} & \\
\dot{x}(0) = 0 \implies h\dot{\tilde{x}}(0) = 0 \implies \boxed{\dot{\tilde{x}}(0) = 0} &
\end{aligned}
\tag{5.2.23}
$$

Now, rescaling the domain of the dimensionless amplitude (ϵ) system.

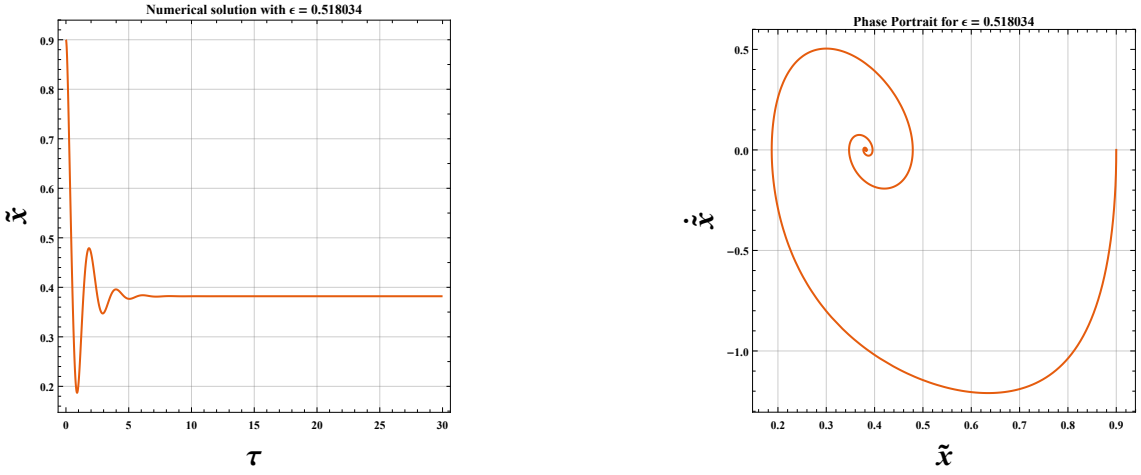
$$
\begin{aligned}
0 = x_{eq} + A \implies A = -x_{eq} \implies \epsilon = -\frac{x_{eq}}{h} = -\frac{2\Lambda + 1 - \sqrt{4\Lambda^2 + 1}}{2} \\
h = x_{eq} + A \implies A = h - x_{eq} \implies \epsilon = 1 - \frac{x_{eq}}{h} = 1 - \frac{2\Lambda + 1 - \sqrt{4\Lambda^2 + 1}}{2}
\end{aligned} \tag{5.2.24}
$$

$$
\therefore \quad \boxed{-\frac{x_{eq}}{h} < \epsilon < 1 - \frac{x_{eq}}{h} \implies -\frac{2\Lambda + 1 - \sqrt{4\Lambda^2 + 1}}{2} < \epsilon < \frac{1 - 2\Lambda + \sqrt{4\Lambda^2 + 1}}{2}}
$$

5.2.4 Numerical solution

Once again, we use the Runge-Kutta method to analyse the evolution of our system. In this numerical simulation, we use the following values: $\Lambda = 1$ and $\Phi = \frac{1}{2}$.

The presence of the friction number indicates to us that the amplitude and speed of our system will decrease over time. And this can be shown with the spiral movement that appears in the phase portrait.



((a)) Time evolution

((b)) Phase portrait

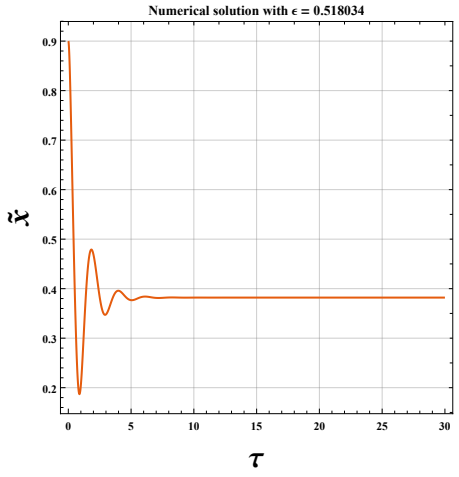
Figure 5.8: Plot and parametric plot of the numerical solution with $\epsilon = 0.518034$

We can see the characteristic spiral movement that a damped oscillator has. To see how the system changes when epsilon changes its values, we plot our system with three different values, those values are: 0.518034, 0.5, and 0.01.

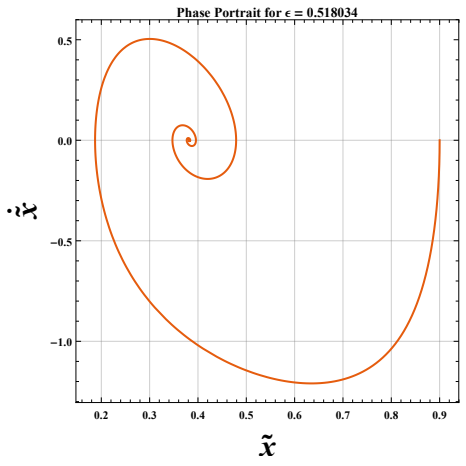
Observing three behaviors, we can see that all have the same behavior but with different amplitudes. This tells us that the friction number generates a homogeneous behavior without relevance to the epsilon value, with the only difference in the amplitude. This can be shown in the figure (5.9).

5.2.5 Asymptotic approximation

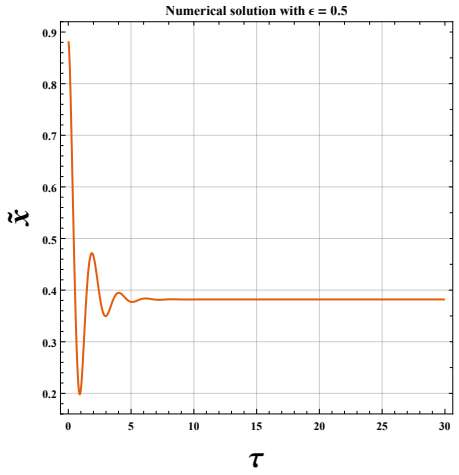
As we saw in the previous chapter, the asymptotic approximation consists in regulate the apport of every term in a polynomial with the intetion to approximate that polynomial to the anallytical solution. When the polynomial has infinity terms, the polynomial is equal to the solution.So our solution has the form:



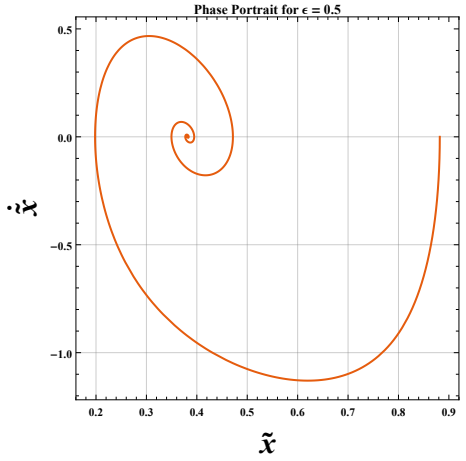
((a)) Time evolution ($\epsilon = 0.518034$)



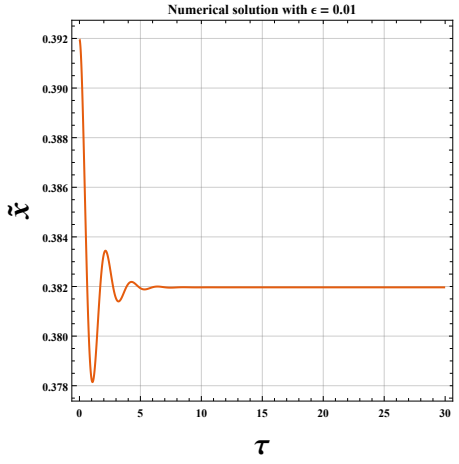
((b)) Phase portrait ($\epsilon = 0.518034$)



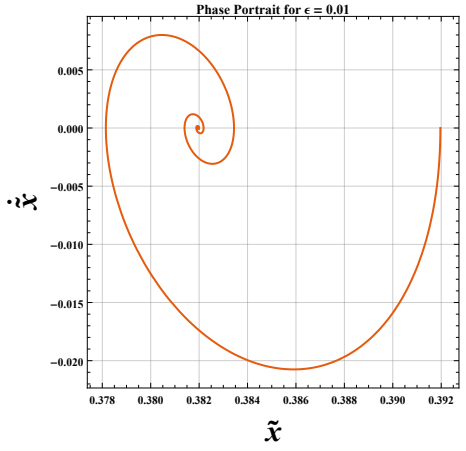
((c)) Time evolution ($\epsilon = 0.5$)



((d)) Phase portrait ($\epsilon = 0.5$)



((e)) Time evolution ($\epsilon = 0.01$)



((f)) Phase portrait ($\epsilon = 0.01$)

Figure 5.9: Time evolution and phase portraits of the system for $\epsilon = 0.518034$, $\epsilon = 0.5$, and $\epsilon = 0.01$.

$$x(t) \sim \sum_{i=0}^{N} \epsilon^i x_i(t) \implies x(t) = \lim_{N\to\infty} \sum_{i=0}^{N} \epsilon^i x_i(t). \tag{5.2.25}$$

Applying this to the dimensionless form we obtain.

$$\tilde{x}(\tau) \sim \sum_{i=0}^{N} \epsilon^i \tilde{x}_i(\tau) \implies \tilde{x}(\tau) = \lim_{N\to\infty} \sum_{i=0}^{N} \epsilon^i \tilde{x}_i(\tau). \tag{5.2.26}$$

Applying to our dimensionless differential equation.

$$\begin{aligned}
&\tilde{x}\frac{d^2\tilde{x}}{d\tau^2} - \tilde{x}^2\frac{d^2\tilde{x}}{d\tau^2} + 2\Phi\tilde{x}\frac{d\tilde{x}}{d\tau} - 2\Phi\tilde{x}^2\frac{d\tilde{x}}{d\tau} - \tilde{x}^2 + (2\Lambda+1)\,\tilde{x} - \Lambda = 0 \\
&\qquad \sum_{i=0}^{N}\sum_{j=0}^{N} \epsilon^{i+j}\tilde{x}_i\frac{d^2\tilde{x}_j}{d\tau^2} - \left(\sum_{k=0}^{N}\epsilon^k\tilde{x}_k\right)^2\left(\sum_{\ell=0}^{N}\epsilon^\ell\frac{d^2\tilde{x}_\ell}{d\tau^2}\right) \\
&+2\Phi\left[\sum_{m=0}^{N}\sum_{n=0}^{N}\epsilon^{m+n}\tilde{x}_m\frac{d\tilde{x}_n}{d\tau} - \left(\sum_{k=0}^{N}\epsilon^p\tilde{x}_p\right)^2\left(\sum_{q=0}^{N}\epsilon^q\frac{d\tilde{x}_q}{d\tau}\right)\right] \\
&\qquad - \left(\sum_{r=0}^{N}\epsilon^r\tilde{x}_r\right)^2 + (2\Lambda+1)\left(\sum_{s=0}^{N}\epsilon^s\tilde{x}_s\right) - \Lambda = 0
\end{aligned} \tag{5.2.27}$$

As we can see, our approximation series is long and complex. But we can work with it. Now, we are going to find the first and second order solutions.

To simplify, we do some algebra.

$$\begin{aligned}
\left(\tilde{x}-\tilde{x}^2\right)\left(\frac{d^2\tilde{x}}{d\tau^2} + 2\Phi\frac{d\tilde{x}}{d\tau} + 1\right) + \Lambda\left(2\tilde{x}-1\right) = 0 \\
\left(\sum_{i=0}^{N}\epsilon^i\tilde{x}_i - \left(\sum_{j=0}^{N}\epsilon^j\tilde{x}_j\right)^2\right)\left(\sum_{k=0}^{N}\epsilon^k\frac{d^2\tilde{x}_k}{d\tau^2} + 2\Phi\sum_{\ell=0}^{N}\epsilon^\ell\frac{d\tilde{x}_\ell}{d\tau} + 1\right) + \Lambda\left(2\sum_{m=0}^{N}\epsilon^m\tilde{x}_m - 1\right) = 0
\end{aligned} \tag{5.2.28}$$

5.2.6 First order approximation solution

Before to find the first order solution, we need to find the zero order solution. For that we need to balance all the terms that have ϵ^0 in our differential equation (equation 5.2.28) with the initial conditions. The equation tha we need to solve is:

$$\left(\epsilon^0\tilde{x}_0 - \left(\epsilon^0\tilde{x}_0\right)^2\right)\left(\epsilon^0\frac{d^2\tilde{x}_0}{d\tau^2} + 2\Phi\epsilon^0\frac{d\tilde{x}_0}{d\tau} + 1\right) + \Lambda\left(2\epsilon^0\tilde{x}_0 - 1\right) = 0 \tag{5.2.29}$$

It is important to highlight that x_0 is the constant term of the polynom, so the derivation of this term is zero and the equation is now:

$$-\tilde{x}_0^2 + (2\Lambda + 1)\,\tilde{x}_0 - \Lambda = 0 \implies \boxed{\tilde{x}_0 = \frac{2\Lambda + 1 - \sqrt{4\Lambda^2 + 1}}{2} = \frac{x_{eq}}{h}}. \tag{5.2.30}$$

Now, with the constant term, we can continue to find the first-order solution. For this order, we need all the terms in our polynomial solution with ϵ^1. These terms are:

$$\begin{aligned}
\left(\epsilon^0\tilde{x}_0 + \epsilon^1\tilde{x}_1 - \left(\epsilon^0\tilde{x}_0 + \epsilon^1\tilde{x}_1\right)^2\right)\left(\epsilon^0\frac{d^2\tilde{x}_0}{d\tau^2} + \epsilon^1\frac{d^2\tilde{x}_1}{d\tau^2} + 2\Phi\left(\epsilon^0\frac{d\tilde{x}_0}{d\tau} + \epsilon^1\frac{d\tilde{x}_1}{d\tau}\right) + 1\right)& \\
+\Lambda\left(2\epsilon^0\tilde{x}_0 + 2\epsilon^1\tilde{x}_1 - 1\right) = 0& \\
\left(\epsilon^0\tilde{x}_0 + \epsilon^1\tilde{x}_1 - \epsilon^0\tilde{x}_0^2 - 2\epsilon^1\tilde{x}_0\tilde{x}_1\right)\left(\underbrace{\epsilon^0\frac{d^2\tilde{x}_0}{d\tau^2}}_{0} + \epsilon^1\frac{d^2\tilde{x}_1}{d\tau^2} + 2\Phi\left(\underbrace{\epsilon^0\frac{d\tilde{x}_0}{d\tau}}_{0} + \epsilon^1\frac{d\tilde{x}_1}{d\tau}\right) + 1\right)& \\
+\Lambda\left(2\epsilon^1\tilde{x}_1\right) = 0& \\
\left(\epsilon^0\tilde{x}_0 + \epsilon^1\tilde{x}_1 - \epsilon^0\tilde{x}_0^2 - 2\epsilon^1\tilde{x}_0\tilde{x}_1\right)\left(\epsilon^1\frac{d^2\tilde{x}_1}{d\tau^2} + 2\Phi\epsilon^1\frac{d\tilde{x}_1}{d\tau} + 1\right) + \Lambda\left(2\epsilon^1\tilde{x}_1\right) = 0& \\
\left(\tilde{x}_0 - \tilde{x}_0^2\right)\left(\frac{d^2\tilde{x}_1}{d\tau^2} + 2\Phi\frac{d\tilde{x}_1}{d\tau}\right) + (2\Lambda + 1 - \tilde{x}_0)\,\tilde{x}_1 = 0& \\
\frac{d^2\tilde{x}_1}{d\tau^2} + 2\Phi\frac{d\tilde{x}_1}{d\tau} + \left(\frac{2\Lambda + 1 - \tilde{x}_0}{\tilde{x}_0 - \tilde{x}_0^2}\right)\tilde{x}_1 = 0&
\end{aligned} \tag{5.2.31}$$

The last expression is the differential equation of a damped oscillator, where the natural frequency is $\omega_0 = \sqrt{\frac{2\Lambda+1-2\tilde{x}_0}{\tilde{x}_0-\tilde{x}_0^2}}$, which is the dimensionless natural frequency of the system. Using the value of $\tilde{x}_0$ the dimensionless frequency is:

$$\omega_0 = \sqrt{4\Lambda + 2\sqrt{1 + 4\Lambda^2} + \frac{1}{\Lambda}}. \tag{5.2.32}$$

Then, the differential equation is:

$$\frac{d^2\tilde{x}_1}{d\tau^2} + 2\Phi\frac{d\tilde{x}_1}{d\tau} + \omega_0^2\tilde{x}_1 = 0 \implies \boxed{\tilde{x}_1(\tau) = e^{-\Phi\tau}\left(C_1 e^{\tau\sqrt{\Phi-\omega_0}} + C_2 e^{-\tau\sqrt{\Phi-\omega_0}}\right)}. \tag{5.2.33}$$

Something important to see is that the behavior depends of Φ and ω_0

If we use Euler's equation, we can transform our solution with exponentials into a trigonometric solution. Then, we obtain

$$\boxed{\tilde{x}_1 = e^{-\Phi t}\left[C_1\cos(\Delta t) + C_2\sin(\Delta t)\right]}. \tag{5.2.34}$$

, where Δ is $\Delta = \sqrt{\omega_0^2 - \Phi^2}$. Now, using the solution of $\tilde{x}_1$ in the first order polynomial solution, we obtain that our solution has the following form:

$$\tilde{x}(\tau) \sim \tilde{x}_0 + \tilde{x}_1(\tau)\epsilon = \tilde{x}_0 + e^{-\Phi t}\left[C_1\cos(\Delta t) + C_2\sin(\Delta t)\right]\epsilon. \tag{5.2.35}$$

Using the initial conditions, we can find the values of the constants C_1 and C_2. Starting with the position's initial condition,

$$\begin{aligned} \tilde{x}(0) &\sim \tilde{x}_0 + (C_1 \cos(0) + C_2 \sin(0))\,\epsilon = \tilde{x}_0 + C_1\epsilon \\ \frac{x_{eq}}{h} + \epsilon &\sim \tilde{x}_0 + C_1\epsilon = \frac{x_{eq}}{h} + C_1\epsilon \;\therefore\; \boxed{C_1 = 1} \end{aligned} \tag{5.2.36}$$

Continuing with the speed's initial condition.

$$\begin{aligned} \dot{\tilde{x}}(0) &\sim \frac{d}{d\tau}\left[\tilde{x}_0 + (C_1 \cos(0) + C_2 \sin(0))\,\epsilon\right] \\ 0 &\sim -e^{-\Phi\tau}\left(\Phi - \Delta C_2\right)\epsilon \;\therefore\; \boxed{C_2 = \frac{\Phi}{\Delta}} \end{aligned} \tag{5.2.37}$$

Therefore, our solution at first order is:

$$\begin{aligned} &\tilde{x}(\tau) \sim \tilde{x}_0 + \cos(\omega\tau)\epsilon \\ \Longrightarrow &\boxed{\tilde{x}(\tau) \sim \frac{2\Lambda + 1 - \sqrt{4\Lambda^2 + 1}}{2} + e^{-\Phi t}\left[\cos(\Delta t) + \frac{\Phi}{\Delta}\sin(\Delta t)\right]\epsilon}. \end{aligned} \tag{5.2.38}$$

Now, we plot our numerical solution with our first-order approximation solution to contrast them. We decide to use three different values to epsilon, they are $\epsilon = 0.518034$, $\epsilon = 0.5$ and $\epsilon = 0.01$.

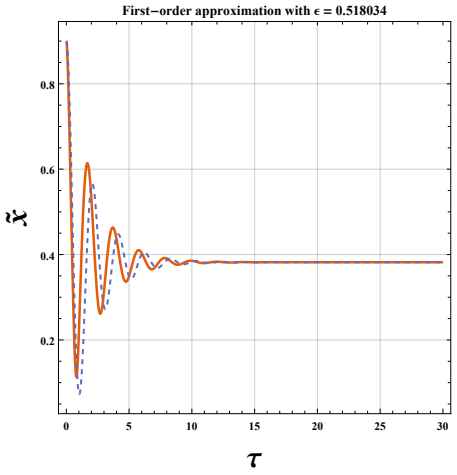
Observing the figure (5.10) we can see that our approximated solution increases its precision when epsilon is small, but is a good first approximation and has the same behavior than the numerical solution.

5.2.7 Second order approximation solution

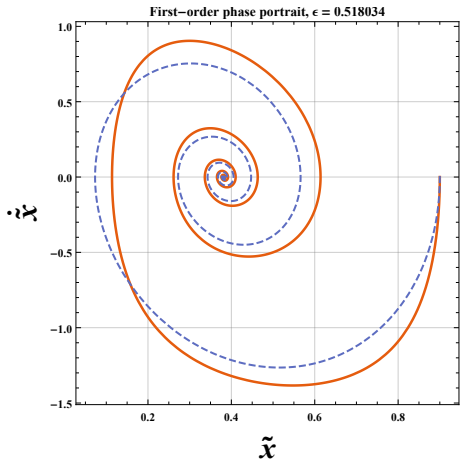
Now, for the second order we need only the terms with ϵ^2. The terms with the power of 2 are:

$$\begin{aligned} &\left(\tilde{x}_0 - \tilde{x}_0^2\right)\left(\frac{d^2\tilde{x}_2}{d\tau^2} + 2\Phi\frac{d\tilde{x}_2}{d\tau}\right) + \left(\tilde{x}_1 - 2\tilde{x}_0\tilde{x}_1\right)\left(\frac{d^2\tilde{x}_1}{d\tau^2} + 2\Phi\frac{d\tilde{x}_1}{d\tau}\right) \\ &+ \left(\tilde{x}_2 - 2\tilde{x}_0\tilde{x}_2 - \tilde{x}_1^2\right)\left(\frac{d^2\tilde{x}_0}{d\tau^2} + 2\Phi\frac{d\tilde{x}_0}{d\tau} + 1\right) + 2\Lambda\,\tilde{x}_2 \;=\; 0. \end{aligned} \tag{5.2.39}$$

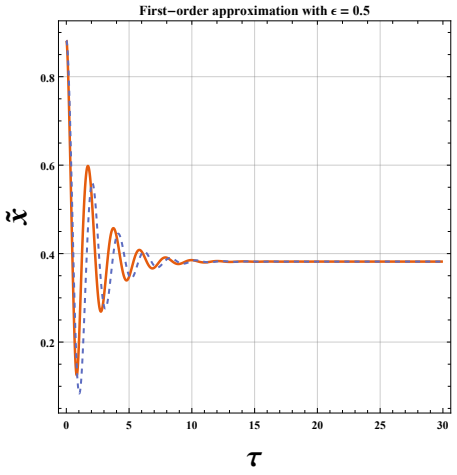
Doing some algebra, we can simplify our expression, the new form of our equation is:



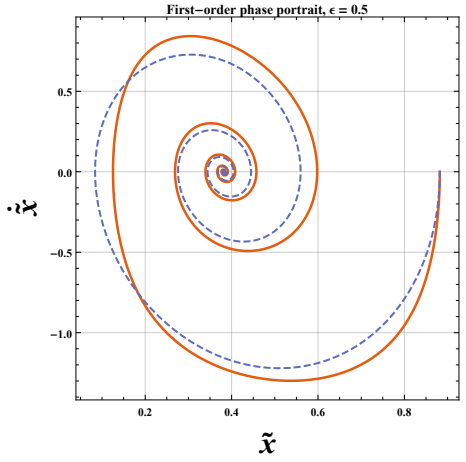
((a)) Time evolution contrast ($\epsilon = 0.518034$)



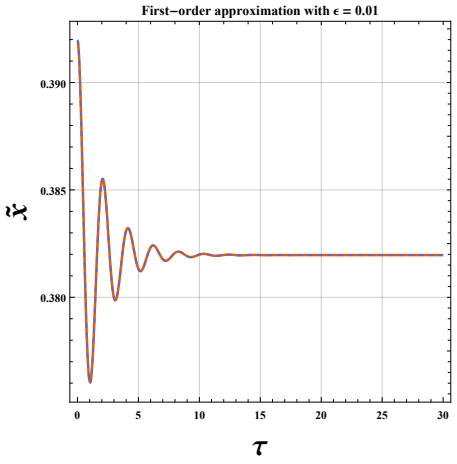
((b)) Phase portrait contrast ($\epsilon = 0.518034$)



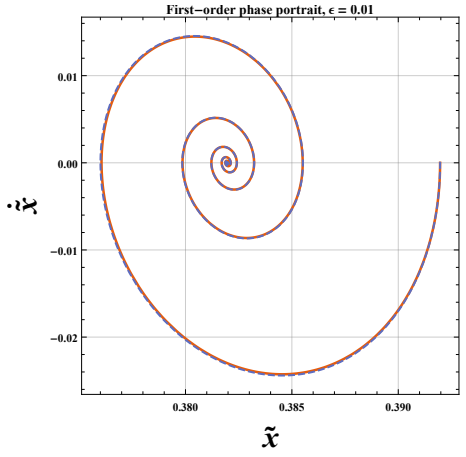
((c)) Time evolution ($\epsilon = 0.5$)



((d)) Phase portrait ($\epsilon = 0.5$)



((e)) Time evolution ($\epsilon = 0.01$)



((f)) Phase portrait ($\epsilon = 0.01$)

Figure 5.10: Approximation of the first-order solution to many values of ϵ with damping.

$$\frac{d^2\tilde{x}_2}{d\tau^2} + 2\Phi\,\frac{d\tilde{x}_2}{d\tau} + \frac{\tilde{x}_1 - 2\tilde{x}_0\tilde{x}_1}{\tilde{x}_0 - \tilde{x}_0^2}\left(\underbrace{\frac{d^2\tilde{x}_1}{d\tau^2} + 2\Phi\,\frac{d\tilde{x}_1}{d\tau}}_{-\omega_0^2\tilde{x}_1}\right)$$

$$+\frac{\tilde{x}_2 - 2\tilde{x}_0\tilde{x}_2 - \tilde{x}_1^2}{\tilde{x}_0 - \tilde{x}_0^2}\left(\underbrace{\frac{d^2\tilde{x}_0}{d\tau^2}}_{0} + 2\Phi\,\underbrace{\frac{d\tilde{x}_0}{d\tau}}_{0} + 1\right) + \frac{2\Lambda\,\tilde{x}_2}{\tilde{x}_0 - \tilde{x}_0^2} = 0$$

$$\frac{d^2\tilde{x}_2}{d\tau^2} + 2\Phi\,\frac{d\tilde{x}_2}{d\tau} + \frac{\tilde{x}_1 - 2\tilde{x}_0\tilde{x}_1}{\tilde{x}_0 - \tilde{x}_0^2}\left(-\omega_0^2\tilde{x}_1\right) + \frac{\tilde{x}_2 - 2\tilde{x}_0\tilde{x}_2 - \tilde{x}_1^2}{\tilde{x}_0 - \tilde{x}_0^2} + \frac{2\Lambda\,\tilde{x}_2}{\tilde{x}_0 - \tilde{x}_0^2} = 0$$

$$\frac{d^2\tilde{x}_2}{d\tau^2} + 2\Phi\,\frac{d\tilde{x}_2}{d\tau} + \left(\underbrace{\frac{2\Lambda + 1 - 2\tilde{x}_0}{\tilde{x}_0 - \tilde{x}_0^2}}_{\omega_0^2}\right)\tilde{x}_2 + \frac{\tilde{x}_1 - 2\tilde{x}_0\tilde{x}_1}{\tilde{x}_0 - \tilde{x}_0^2}\left(-\omega_0^2\tilde{x}_1\right) - \frac{\tilde{x}_1^2}{\tilde{x}_0 - \tilde{x}_0^2} = 0 \tag{5.2.40}$$

$$\frac{d^2\tilde{x}_2}{d\tau^2} + 2\Phi\,\frac{d\tilde{x}_2}{d\tau} + \omega_0^2\tilde{x}_2 + \frac{\tilde{x}_1 - 2\tilde{x}_0\tilde{x}_1}{\tilde{x}_0 - \tilde{x}_0^2}\left(-\omega_0^2\tilde{x}_1\right) - \frac{\tilde{x}_1^2}{\tilde{x}_0 - \tilde{x}_0^2} = 0$$

$$\frac{d^2\tilde{x}_2}{d\tau^2} + 2\Phi\,\frac{d\tilde{x}_2}{d\tau} + \omega_0^2\tilde{x}_2 + \frac{\left(-\omega_0^2\tilde{x}_1\right)\left(\tilde{x}_1 - 2\tilde{x}_0\tilde{x}_1\right) - \tilde{x}_1^2}{\tilde{x}_0 - \tilde{x}_0^2} = 0$$

$$\frac{d^2\tilde{x}_2}{d\tau^2} + 2\Phi\,\frac{d\tilde{x}_2}{d\tau} + \omega_0^2\tilde{x}_2 + \frac{\left(-\omega_0^2\right)\left(1 - 2\tilde{x}_0\right) - 1}{\tilde{x}_0 - \tilde{x}_0^2}\tilde{x}_1^2 = 0$$

$$\frac{d^2\tilde{x}_2}{d\tau^2} + 2\Phi\,\frac{d\tilde{x}_2}{d\tau} + \omega_0^2\tilde{x}_2 - \frac{\omega_0^2 - 2\omega_0^2\tilde{x}_0 + 1}{\tilde{x}_0 - \tilde{x}_0^2}\tilde{x}_1^2 = 0$$

To simplify the differential equation, we define $\alpha^2 = \frac{\omega_0^2 - 2\omega_0^2\tilde{x}_0 + 1}{\tilde{x}_0 - \tilde{x}_0^2}$. Then our differential equation, using the definition of $\tilde{x}_1$ is:

$$\frac{d^2\tilde{x}_2}{d\tau^2} + 2\Phi\frac{d\tilde{x}_2}{d\tau} - \alpha^2\tilde{x}_1^2 + \omega^2\tilde{x}_2 = 0$$

$$\frac{d^2\tilde{x}_2}{d\tau^2} + 2\Phi\frac{d\tilde{x}_2}{d\tau} - \alpha^2 e^{-2\Phi t}\left[\cos\left(\Delta t\right) + \frac{\Phi}{\Delta}\sin\left(\Delta t\right)\right]^2 + \omega^2\tilde{x}_2 = 0 \tag{5.2.41}$$

$$\therefore\ \boxed{\frac{d^2\tilde{x}_2}{d\tau^2} + 2\Phi\frac{d\tilde{x}_2}{d\tau} + \omega^2\tilde{x}_2 = \alpha^2 e^{-2\Phi t}\left[\cos\left(\Delta t\right) + \frac{\Phi}{\Delta}\sin\left(\Delta t\right)\right]^2}$$

Now, we proceed to solve the differential equation. We can see from the previous equation that our system is a forced damped oscillator. So, using the form to solve this kind of differential equation, and using the initial conditions, we find the following solution.

$$\begin{aligned}\tilde{x}_2(\tau) = &- \frac{e^{-2\Phi\tau}\alpha^2\Delta\left(8\Phi^2\omega^2 - 9\omega^4 + (16\Phi^2 - 18\omega^2\Phi^2 + 3\omega^4)\cos(2\Delta t)\right)}{2\omega^2\Delta(9\omega^4 - 17\omega^2\Phi^2 + 8\Phi^4)} \\ &- \frac{2e^{-2\Phi\tau}\alpha^2\Delta^2\left(\Phi(8\Phi^2 - 5\omega^2)\sin(2\Delta t) - (8\Phi^2 + 3\omega^2)\Delta e^{\Phi\tau}\cos(\Delta\tau)\right)}{2\omega^2\Delta(9\omega^4 - 17\omega^2\Phi^2 + 8\Phi^4)} \\ &+ \frac{2e^{-\Phi\tau}\alpha^2\Delta^2\Phi\left(8\Phi^2 - 13\omega^2\right)\sin(\Delta\tau)}{2\omega^2\Delta(9\omega^4 - 17\omega^2\Phi^2 + 8\Phi^4)}\end{aligned} \tag{5.2.42}$$

Therefore, our approximated solution is:

$$\begin{aligned}\tilde{x}(\tau) \sim &\tilde{x}_0 + \tilde{x}_1\epsilon + \tilde{x}_2\epsilon^2 = \tilde{x}_{eq} + e^{-\Phi t}\left[\cos(\Delta t) + \frac{\Phi}{\Delta}\sin(\Delta t)\right]\epsilon \\ &- \frac{e^{-2\Phi\tau}\alpha^2\Delta\left(8\Phi^2\omega^2 - 9\omega^4 + (16\Phi^2 - 18\omega^2\Phi^2 + 3\omega^4)\cos(2\Delta t)\right)}{2\omega^2\Delta(9\omega^4 - 17\omega^2\Phi^2 + 8\Phi^4)}\epsilon^2 \\ &- \frac{2e^{-2\Phi\tau}\alpha^2\Delta^2\left(\Phi(8\Phi^2 - 5\omega^2)\sin(2\Delta t) - (8\Phi^2 + 3\omega^2)\Delta e^{\Phi\tau}\cos(\Delta\tau)\right)}{2\omega^2\Delta(9\omega^4 - 17\omega^2\Phi^2 + 8\Phi^4)}\epsilon^2 \\ &+ \frac{2e^{-\Phi\tau}\alpha^2\Delta^2\Phi\left(8\Phi^2 - 13\omega^2\right)\sin(\Delta\tau)}{2\omega^2\Delta(9\omega^4 - 17\omega^2\Phi^2 + 8\Phi^4)}\epsilon^2\end{aligned} \tag{5.2.43}$$

If we plot the second-order solution of our system and compare it with the numerical solution, we obtain the figure (5.11).

5.3 Forced system

As we mentioned previously, an oscillator forced system is a system where an external force acts over an oscillator system changing the state of motion of the oscillator.

The forces that play a relevant role in the dynamics of the system are the dissipative force presented by the environment and which is proportional to the speed of movement ($F_f = \beta v$) and the external force.

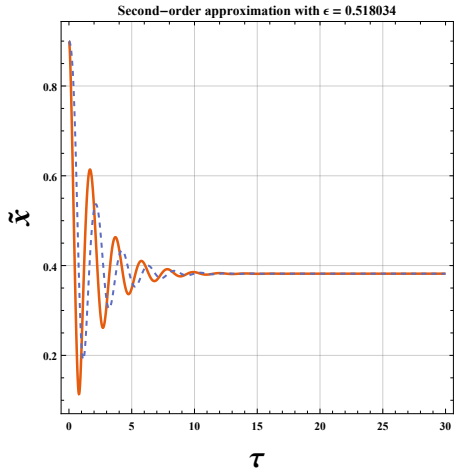
Using the equation of forced oscillations, we can write our system as:

$$\begin{aligned}F + F_f + F_R =& F\cos(\omega t) \\ m\frac{d^2x}{dt^2} + \beta\frac{dx}{dt} + mg - \frac{M}{x} + \frac{M}{h-x} =& F\cos(\omega t).\end{aligned} \tag{5.3.1}$$

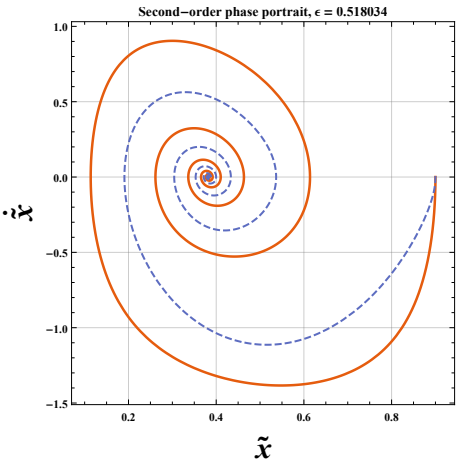
5.3.1 Lagrangian analysis

The Lagrangian that we use is the same Lagrangian of the damped system, then the Lagragian is.

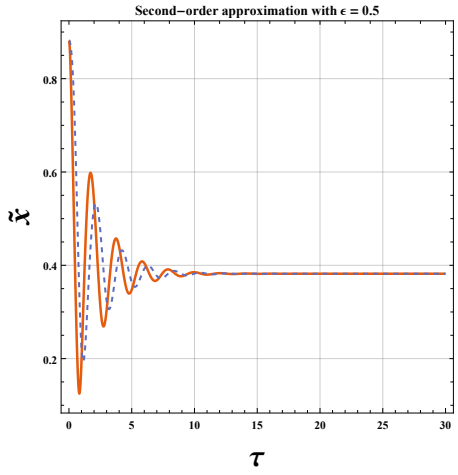
$$\boxed{L = \frac{1}{2}m\dot{x}^2 - mgx + M\ln(x) + M\ln(h-x)}. \tag{5.3.2}$$



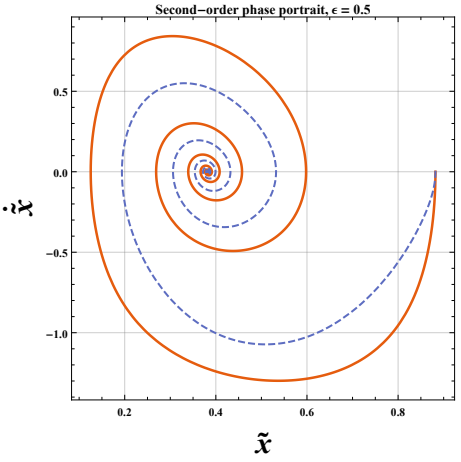
((a)) Time evolution contrast ($\epsilon = 0.518034$)



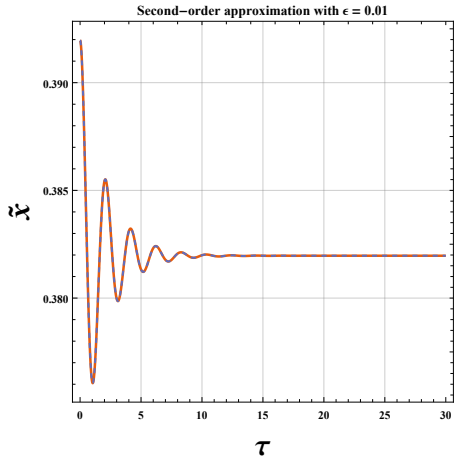
((b)) Phase portrait contrast ($\epsilon = 0.518034$)



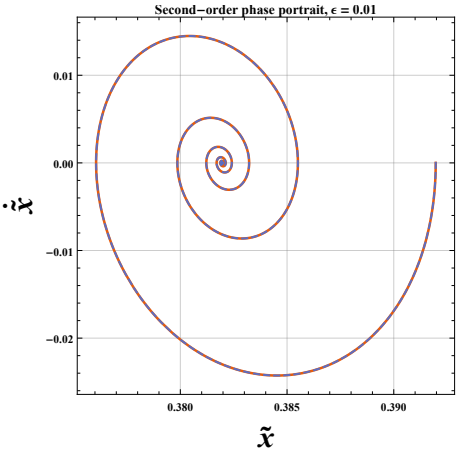
((c)) Time evolution ($\epsilon = 0.5$)



((d)) Phase portrait ($\epsilon = 0.5$)



((e)) Time evolution ($\epsilon = 0.01$)



((f)) Phase portrait ($\epsilon = 0.01$)

Figure 5.11: Approximation of the second-order solution to many values of ϵ with damping.

Applying the Euler-Lagrange equation, we find the equation of motion of our system. As we know, our system presents two external forces that are the friction force and the periodic external force ($Q = F_f + F_E$). So the equation of motion for the coordinate x is.

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} &= Q \\ \implies m\ddot{x} + mg + \frac{M}{h-x} - \frac{M}{x} &= -\beta v + F\cos(\omega t) \\ \implies \ddot{x} + \frac{\beta}{m}\dot{x} + g + \frac{M}{m(h-x)} - \frac{M}{mx} &= \frac{F}{m}\cos(\omega t) \\ \therefore \quad \boxed{\frac{d^2x}{dt^2} + \frac{\beta}{m}\frac{dx}{dt} + g + \frac{M}{m}\left[\frac{1}{h-x} - \frac{1}{x}\right] = \frac{F}{m}\cos(\omega t)}. & \end{aligned} \tag{5.3.3}$$

The previous equation lets us know the different accelerations that act on the system and how they govern the motion of the oscillator.

We can not plot the phase portrait diagram for this system, because the time plays an explicit role in the system, but we can plot it only for a specific time.

5.3.2 Dimensional analysis

For the forced system, we have ten variables ($n = 10$), these variables are the mass (m), the time (t), the position (x), the gravity (g), the oscillation amplitude (A), the magnetic energy (M), (β) the friction coefficient and the distance between the fixed magnets (h), the frequency of the external force (ω) and the external force (F). For the nature of our problem, we only need to use three dimensions ($j = 3$) to characterize the problem; the dimensions that we need to use are the matter, the longitude, and the time. Using Buckingham's theorem, we know that there are four dimensionless groups ($k = n - j = 7$) that describe our system.

Using a dimensional analysis over every variable:

$$\begin{aligned} &[x] = L, [h] = L, [A] = L, [t] = T, [m] = M, [g] = LT^{-2}, \\ &[M] = ML^2T^{-2}, [\beta] = MT^{-1}, [\omega] = T^{-1}, [F] = MLT^{-2}, \end{aligned} \tag{5.3.4}$$

For this system, we are going to use h, g, and M as dimensional variables. This is because h is the geometric parameter of the system, g is the cinematic parameter, and M is the dynamic parameter. As we saw in the previous chapter, every dimensional group is the product of the dimensional variables, so we have that the ith group is expressed as:

$$\Pi_i = \delta h^{\alpha} g^{\beta} M^{\lambda}, \tag{5.3.5}$$

where δ is the variable that we want to dimensionless, in our system we have the variables x, t, m and A; α, β and λ are the powers of the distance h, the gravity and the magnetic energy that dimensionless the ith group.

To find the powers of our dimensional variables that dimensionless every group we can construct an equations system, this system is.

$$\begin{pmatrix} L \\ M \\ T \end{pmatrix} + \begin{pmatrix} 1 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & -2 & -2 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \lambda \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \implies \begin{pmatrix} \alpha \\ \beta \\ \lambda \end{pmatrix} = \begin{pmatrix} M - L - \frac{T}{2} \\ M + \frac{T}{2} \\ -M. \end{pmatrix} \tag{5.3.6}$$

It is important to remember that every dimensionless group can be expressed as a fraction. So the dimensionless groups can be rewrite as:

$$\Pi_i = \frac{\delta}{\delta_s} = \tilde{\delta}, \tag{5.3.7}$$

where δ_s is the scale factor of the variable δ y $\tilde{\delta}$ is the dimensionless form of the variable δ. Now, we can obtain the dimensionless groups for the variables x, t, m and A.

Dimensionless group of the position

The x's group is:

$$\begin{pmatrix} \alpha \\ \beta \\ \lambda \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} \quad \therefore \quad \boxed{\Pi_1 = \frac{x}{x_s} = \frac{x}{h} \implies x_s = h} \quad . \tag{5.3.8}$$

By the previous result, the scale factor of the position is the distance between the two fixed magnets. The position can be expressed as the product of the scale factor of the position and the dimensionless position.

$$x = x_s \tilde{x} \implies dx = x_s d\tilde{x} \implies d^2x = x_s d^2\tilde{x} \tag{5.3.9}$$

Dimensionless group of the time

The group of t is.

$$\begin{pmatrix} \alpha \\ \beta \\ \lambda \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix} \quad \therefore \quad \boxed{\Pi_2 = \frac{t}{t_s} = t\sqrt{\frac{g}{h}} \implies t_s = \sqrt{\frac{h}{g}}} \quad . \tag{5.3.10}$$

Using the previous result and the definitions of the frequency and the oscillation period, we obtain.

$$\omega_s = \frac{1}{t_s} = \sqrt{\frac{g}{h}} \quad \wedge \quad T_s = \frac{2\pi}{\omega_s} = 2\pi\sqrt{\frac{h}{g}}. \tag{5.3.11}$$

If we analyze these expression, we can see that they are the frequency and the period of a pendulum. Then, we can conclude that our system is analogous to a pendulum.

Now, using the relation between the scale factor and the dimensionless variable, we obtain.

$$t = t_s \tau \implies dt = t_s d\tau \implies dt^2 = t_s^2 d\tau^2 \tag{5.3.12}$$

Dimensionless group of the mass and the Energie's number

Now, the group of m:

$$\begin{pmatrix}\alpha\\ \beta\\ \lambda\end{pmatrix} = \begin{pmatrix}1\\ 1\\ -1\end{pmatrix} \quad \therefore \quad \boxed{\Pi_3 = \frac{m}{m_s} = \frac{mgh}{M} \implies m_s = \frac{M}{gh}} \quad . \tag{5.3.13}$$

This dimensionless group is very important because give us a relation between the magnetic energy and the mechanic potential energy, Using this dimensionless group we going to define Energie's number as:

$$\Lambda \equiv \frac{1}{\Pi_3} = \frac{M}{mgh}. \tag{5.3.14}$$

The Energie's number going to be so useful because it related all the variables that are not affected by time. An example of some quantity that is not affected by time is the equilibrium distance, so we can rewrite this number interm of the Energie's number as:

$$\begin{aligned} x_{eq} &= \frac{2M + mgh - \sqrt{4M^2 + m^2g^2h^2}}{2mg} = \frac{mgh\left(\frac{2M}{mgh} + 1\right) - \sqrt{m^2g^2h^2\left(\frac{4M^2}{m^2g^2h^2} + 1\right)}}{2mg} \\ &= \frac{mgh\left(\frac{2M}{mgh} + 1 - \sqrt{\frac{4M^2}{m^2g^2h^2} + 1}\right)}{2mg} = \frac{h}{2}\left(2\Lambda + 1 - \sqrt{4\Lambda^2 + 1}\right) \\ \therefore \boxed{x_{eq} = h\left(\frac{2\Lambda + 1 - \sqrt{4\Lambda^2 + 1}}{2}\right) \implies \tilde{x}_{eq} = \frac{x_{eq}}{h} = \frac{2\Lambda + 1 - \sqrt{4\Lambda^2 + 1}}{2}}&. \end{aligned} \tag{5.3.15}$$

Dimensionless group of the amplitude

Obtaining the group of A.

$$\begin{pmatrix}\alpha\\ \beta\\ \lambda\end{pmatrix} = \begin{pmatrix}-1\\ 0\\ 0\end{pmatrix} \quad \therefore \quad \boxed{\Pi_4 = \frac{A}{A_s} = \frac{A}{h} \implies A_s = h} \quad . \tag{5.3.16}$$

This result lets us see that the amplitude of oscillation is a fraction of the distance between the fixed magnets. The amplitude does not have a role in the motion equation (equation 5.3.3) so we going to use this dimensionless group as the control number (ϵ) for our approximation.

$$\epsilon = \frac{A}{h} = \tilde{A}. \tag{5.3.17}$$

Dimensionless group of the external force

The dimensionless group of F is.

$$\begin{pmatrix}\alpha\\ \beta\\ \lambda\end{pmatrix} = \begin{pmatrix}1\\ 0\\ -1\end{pmatrix} \quad \therefore \quad \boxed{\Pi_5 = \frac{F}{F_s} = \frac{Fh}{M} \implies F_s = \frac{M}{h}} \quad . \tag{5.3.18}$$

This result is so important because the force scale is expressed by the energy provided by the magnets and the distance to the magnets.

5.3.3 Dimensionless system

Applying the dimensionless groups in our differential equation, we can find the equation of motion

$$\begin{aligned}
&\frac{d^2x}{dt^2} + \frac{\beta}{m}\frac{dx}{dt} + g + \frac{M}{m}\left[\frac{1}{h-x} - \frac{1}{x}\right] = \frac{F}{m}\cos(\omega t)\\
\implies &\frac{d^2(x_s\tilde{x})}{d(t_s\tau)^2} + \frac{\beta}{m}\frac{d(x_s\tilde{x})}{d(t_s\tau)} + g + \frac{M}{m}\left[\frac{1}{h-(x_s\tilde{x})} - \frac{1}{(x_s\tilde{x})}\right] = \frac{F}{m}\cos(\omega_s\Omega(t_s\tau))\\
\implies &\frac{x_s}{t_s^2}\frac{d^2\tilde{x}}{d\tau^2} + \frac{\beta}{m}\frac{x_s}{t_s}\frac{d\tilde{x}}{d\tau} + g + \frac{M}{m}\left[\frac{1}{h-x_s\tilde{x}} - \frac{1}{x_s\tilde{x}}\right] = \frac{F}{m}\cos\left(\frac{1}{t_s}\Omega(t_s\tau)\right)\\
\implies &\frac{d^2\tilde{x}}{d\tau^2} + \frac{\beta t_s}{m}\frac{d\tilde{x}}{d\tau} + \frac{gt_s^2}{x_s} + \frac{Mt_s^2}{mx_s}\left[\frac{1}{h-x_s\tilde{x}} - \frac{1}{x_s\tilde{x}}\right] = \frac{Ft_s^2}{mx_s}\cos(\Omega\tau).
\end{aligned} \tag{5.3.19}$$

Now, using the definitions of the scale factors of position and time, we can dimensionless our equation of motion.

$$\begin{aligned}
&\frac{d^2\tilde{x}}{d\tau^2} + \frac{\beta}{m}\sqrt{\frac{h}{g}}\frac{d\tilde{x}}{d\tau} + \frac{g\left(\frac{h}{g}\right)}{h} + \frac{M\left(\frac{h}{g}\right)}{mh}\left[\frac{1}{h-h\tilde{x}} - \frac{1}{h\tilde{x}}\right] = \frac{\left(\frac{h}{g}\right)F}{mh}\cos(\Omega\tau)\\
\implies &\frac{d^2\tilde{x}}{d\tau^2} + \frac{\beta}{m}\sqrt{\frac{h}{g}}\frac{d\tilde{x}}{d\tau} + 1 + \frac{M}{mgh}\left[\frac{1}{1-\tilde{x}} - \frac{1}{\tilde{x}}\right] = \frac{F}{mg}\cos(\Omega\tau)
\end{aligned} \tag{5.3.20}$$

This last equation has three different dimensionless magnitudes that can be defined as dimensionless parameters that characterize the dynamics of the system. Now, using the dimensionless numbers, we obtain the following equation.

$$\frac{d^2\tilde{x}}{d\tau^2} + 2\Phi\frac{d\tilde{x}}{d\tau} + 1 + \Lambda\left[\frac{1}{1-\tilde{x}} - \frac{1}{\tilde{x}}\right] = \tilde{F}\cos(\Omega\tau), \tag{5.3.21}$$

the previous equation is a dimensionless differential equation. The three dimensionless numbers are the ***Energie's number*** (Λ), the ***Friction number*** (Φ) and the dimensionless external force ($\tilde{F}$). Where Φ and $\tilde{F}$ are defined as :

$$\Phi = \frac{\beta}{2m}\sqrt{\frac{h}{g}}, \quad \tilde{F} = \frac{F}{mg} \tag{5.3.22}$$

Now, doing some algebra, we can simplify our differential equation as follows.

$$
\begin{aligned}
\frac{d^2\tilde{x}}{d\tau^2} + 2\Phi\frac{d\tilde{x}}{d\tau} + 1 + \Lambda\left[\frac{\tilde{x}-(1-\tilde{x})}{\tilde{x}(1-\tilde{x})}\right] &= \tilde{F}\cos(\Omega\tau) \\
\implies \frac{d^2\tilde{x}}{d\tau^2} + 2\Phi\frac{d\tilde{x}}{d\tau} + 1 + \Lambda\left[\frac{2\tilde{x}-1}{\tilde{x}(1-\tilde{x})}\right] &= \tilde{F}\cos(\Omega\tau) \\
\implies \tilde{x}(1-\tilde{x})\frac{d^2\tilde{x}}{d\tau^2} + 2\Phi\tilde{x}(1-\tilde{x})\frac{d\tilde{x}}{d\tau} + \tilde{x}(1-\tilde{x}) + \Lambda(2\tilde{x}-1) &= \tilde{x}(1-\tilde{x})\tilde{F}\cos(\Omega\tau) \\
\implies (\tilde{x}-\tilde{x}^2)\left(\frac{d^2\tilde{x}}{d\tau^2} + 2\Phi\frac{d\tilde{x}}{d\tau} + 1 - \tilde{F}\cos(\Omega\tau)\right) + \Lambda(2\tilde{x}-1) &= 0,
\end{aligned}
\tag{5.3.23}
$$

this last equation is an unsolvable differential equation, so we need to use another method to analyze the evolution of the system.

5.3.4 Numerical solution

As we have seen previously, the numerical solution is fundamental to knowing the behavior of our differential equation, which it will follow.

So, we require a numerical method to plot the evolution of the system. There are many numerical methods in the literature, but the complexity and properties of our differential equation forced us to use the more powerful methods, such as the Runge-Kutta method.

Nowadays, there are different software that have the ability to apply the most efficient numerical method to some differential equations, one of them is the Wolfram Mathematica software. So we are going to use the ***NDSolve*** function to find the numerical solution of our system.

Dynamical system stability

To analyze the behavior of our system, we can use our techniques for differential equations. We can not use the phase portrait technique because our dynamical system depends directly on time. Then we are going to use the Poincaré map to find the stability of our system. It is important to highlight that the Poincaré map is very useful in the stroboscopy space, where $\tau = T = \frac{2\pi}{\Omega}$

Firstly, we need to reduce our second-order differential equation to a set of first-order differential equations. The forcing phase relation is $\tilde{\theta} = \Omega\tau$. These equations are:

$$\tilde{v} = \frac{d\tilde{x}}{d\tau} \tag{5.3.24}$$

$$\dot{\tilde{v}} = \tilde{F}\cos(\tilde{\theta}) - 2\Phi\tilde{v} - 1 - \Lambda\left(\frac{1}{1-\tilde{x}} - \frac{1}{\tilde{x}}\right) \tag{5.3.25}$$

$$\Omega = \frac{d\tilde{\theta}}{d\tau} \tag{5.3.26}$$

These equations can be expressed as a vectorial system as:

$$\tilde{r} = \frac{d}{d\tau}\begin{pmatrix}\tilde{x}\\ \tilde{v}\\ \tilde{\theta}\end{pmatrix} = \begin{pmatrix}\tilde{v}\\ \tilde{F}\cos(\tilde{\theta}) - 2\Phi\tilde{v} - 1 - \Lambda\left(\frac{1}{1-\tilde{x}} - \frac{1}{\tilde{x}}\right)\\ \Omega\end{pmatrix}, \tag{5.3.27}$$

where $\tilde{r}$ is the vector that models our system. If we want to know the stability of our system, we need the non-autonomous system, which implies that we do not need to attend to the time. Then our vector $\tilde{r}$ is really formed only by the first and second terms. To find the stability we need the Jacobian matrix of our non-autonomous system. Which is:

$$\mathbf{J}(\tilde{x},\tilde{v}) = \begin{pmatrix}\frac{\partial\tilde{r}_1}{\partial\tilde{x}} & \frac{\partial\tilde{r}_1}{\partial\tilde{v}}\\ \frac{\partial\tilde{r}_2}{\partial\tilde{x}} & \frac{\partial\tilde{r}_2}{\partial\tilde{v}}\end{pmatrix} \implies \mathbf{J}(\tilde{x},\tilde{v}) = \begin{pmatrix}0 & 1\\ -\Lambda\left(\frac{1}{(1-\tilde{x}(\tau))^2} + \frac{1}{\tilde{x}^2(\tau)}\right) & -2\Phi\end{pmatrix} = \mathbf{J}(\tau). \tag{5.3.28}$$

With the Jacobian of the system, we can find the Monodromy matrix (that is, the transformation of our Jacobian matrix to the Poincaré discrete time space), and this matrix follows the next:

$$\dot{\mathbf{M}}(\tau) = \mathbf{J}(\tau)\mathbf{M}(\tau), \quad \mathbf{M}(0) = \mathbf{I}. \tag{5.3.29}$$

This system is so complicated to solve analytically. Then we are going to use Liouville's formula:

$$\det(\mathbf{M}(\tau)) = \det\mathbf{M}(t_0)e^{\int_{t_0}^{\tau}\operatorname{tr}(\mathbf{J}(\tau))d\tau}. \tag{5.3.30}$$

Now, integrating from 0 to T, we obtain:

$$\begin{aligned}\det(\mathbf{M}(T)) &= \det(\mathbf{M}(0))e^{\int_0^T \operatorname{tr}(\mathbf{J}(\tau))d\tau}\\ \det(\mathbf{M}(T)) &= \det(\mathbf{I})e^{\int_0^T -2\Phi d\tau}\\ \therefore \boxed{\det(\mathbf{M}(T)) = e^{-2\Phi T}}&.\end{aligned} \tag{5.3.31}$$

With this result, we can find the Floquet multipliers, which are the eigenvalues of the Monodromy matrix ($\mathbf{M}(T)$) and they are fundamental to determine the stability of the system. Using the definition of Floquet multipliers, we obtain.

$$\rho 1 \cdot \rho 2 = \det(\mathbf{M}(T)) = e^{-2\Phi T}, \tag{5.3.32}$$

where ρ_1 and ρ_2 are the Floquet multipliers. The Floque multipliers let us know if the trajectory that the system follows is stable or unstable. To know this, we have the following table:

Condition	**How many?**	**System behavior**
$\rho < 1$	All	Stable limit cycle (perturbations decay)
$\rho = 1$	At least one (others < 1)	Marginal stability (bifurcation point)
$\rho > 1$	At least one	Unstable limit cycle (perturbations grow)

Table 5.1: Different behaviors of the system in terms of the Floquet multipliers

If $\Phi > 0$, that is the case of a damped system, the product of the Floquet multipliers is always equal or less than one. That implies that the system has a limit cycle. To show that a limit cycle exists, we now make a Poincaré map.

To make our Poincaré map, we are going to use the Runge-Kutta method in our dynamical system, and we are going to save the values of $(\tilde{x}, \tilde{v})$ when we pass for a multiple of T. Doing this, we obtain the following Poincaré map.

Lyapunov exponents

While Floquet multipliers characterize the stability of a periodic orbit, many systems, in particular the forced nonlinear oscillators, can exhibit chaotic behavior where no periodic orbit exists. In such cases, a more general tool is needed: the ***Lyapunov exponents***. The behavior of the system in terms of the Lyapunov exponents follows the next table:

Condition	**How many?**	**System behavior**
$\lambda < 0$	All	Stable dynamics
$\lambda = 0$	At least one (others < 0)	Marginal stability
$\lambda > 0$	At least one	Chaotic dynamics

Table 5.2: Different behaviors of the system in terms of the Lyapunov exponents

When a system is stable, there exists a relation between the Floquet multipliers and the Lyapunov exponents. That relation is:

$$\lambda_i = \frac{1}{T} \ln(\rho_i) \tag{5.3.33}$$

Now, using the Lyapunov exponents equation:

$$\lambda_1 + \lambda_2 = \frac{1}{T} \ln(\rho_1) + \frac{1}{T} \ln(\rho_2) = \frac{1}{T} \ln(\rho_1 \cdot \rho_2) \tag{5.3.34}$$

Using the value of our Floquet product (5.3.32)

$$\lambda_1 + \lambda_2 = \frac{1}{T} \ln\left(e^{-2\Phi T}\right) \implies \lambda_1 + \lambda_2 = -2\Phi \tag{5.3.35}$$

Using the condition of stability:

$$\lambda_1 + \lambda_2 = -2\Phi < 0 \implies \Phi > 0, \tag{5.3.36}$$

this result implies that if the friction number (Φ) increases, the stability increases too. But the amplitude of the oscillation decreases. Another important thing that we can find is the values of Lyapunov exponents that give us stable system

$$\lambda_1 + \lambda_2 = -2\Phi \implies \lambda_1 = -(2\Phi + \lambda_2), \tag{5.3.37}$$

we know that a stable system needs $\lambda_i < 0$, then:

$$\lambda_1 < 0 \implies -(2\Phi + \lambda_2) < 0 \implies 2\Phi + \lambda_2 > 0 \implies \lambda_2 > -2\Phi, \tag{5.3.38}$$

but both Lyapunov exponents need to be less than one; therefore:

$$-2\Phi < \lambda_2 < 0. \tag{5.3.39}$$

Now, we can find the limit for the first Lyapunov exponent domain for a stable system:

$$\begin{aligned} \lambda_1 &= -(2\Phi + 0) = -2\Phi \\ \lambda_1 &= -(2\Phi - 2\Phi) = 0 \\ \therefore &-2\Phi < \lambda_1 < 0. \end{aligned} \tag{5.3.40}$$

We can see that both Lyapunov exponents have the same domain if the system is stable, but it is important to highlight that they can have different values, the only thing that they need to follow is that the sum of both is -2Φ as is presented in the stability condition equation 5.3.36

The previous result indicates that chaos in the system is completely related to the force ($\tilde{F}$) and frequency ($\tilde{\Omega}$) parameters.

Now, that we have our domain for the Lyapunov exponentes, we can start to plot them an see under what kind of parameters the system is stable. It is important to remember that we use a numerical method to find the Lyapunov exponents, and they change as time increases.

For a particular value of $\tilde{F}$, Φ and Ω, we can find the value of time where the system begins to be stable or not. An example of this is the figure 5.12

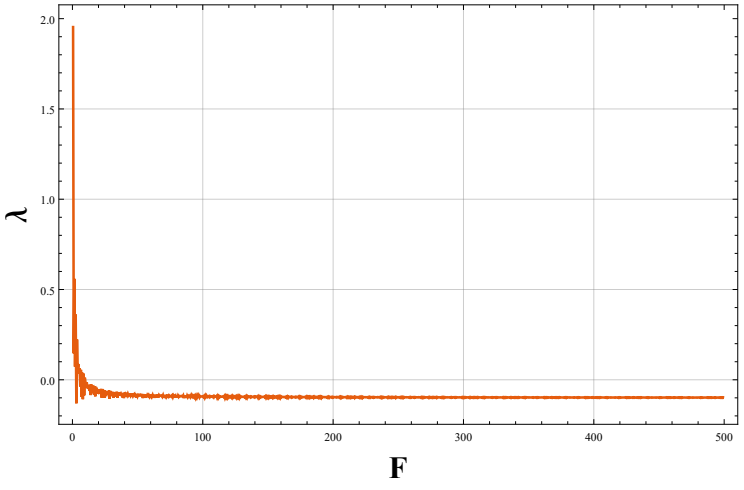


Figure 5.12: Change of the Lyupanov exponents in terms of time for: $\tilde{F} = 1$, $\Phi = 0.1$, $\Lambda = 1$, and $\Omega = 1$

In the previous figure we can see how the system starts acting like a chaotic system, but with the pass of the time the system begins to be stable. This tell us that our system with a that specific parameters is closed to the parameters that generate chaos in the system. So we need to see what kind of values of the force number can generate chaos.

So we decide to run our numerical method to see how Lyapunov exponents change when the value of the force increases. This can be shown in a plot as follows:

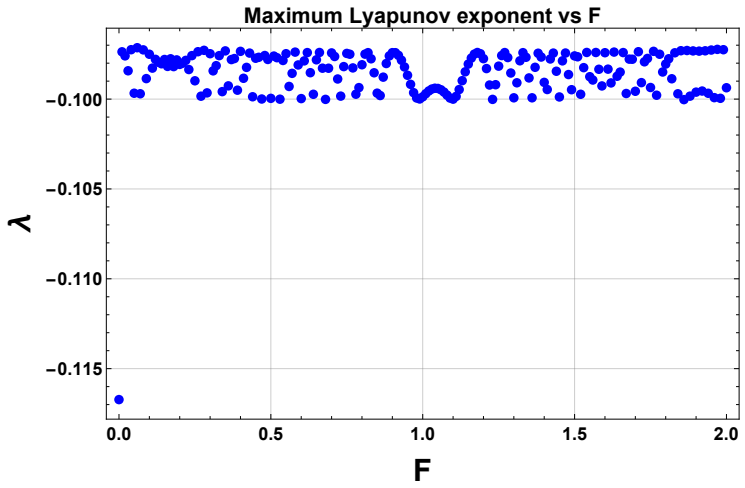


Figure 5.13: Value of Lyupanov exponents when $\tilde{F}$ change it value: $\Phi = 0.1$, and $\Omega = 1$

As we can see in the previous figure, for our system with $\Phi = 0.1$ and $\Omega = 1$, the force does not cross the horizontal axis, so the system in that range of forces will never be chaotic and is always stable. This system is very stable because of the presence of the two magnets that return the free magnet to the equilibrium position, then the presence of the Energies number generates more stability for the system. To see how the value of the Energies number impacts in the dynamics of the system, we plot a bifurcation plot in terms of the force. Firstly, we are going to use the system with $\Lambda = 1$, $\Phi = 0.1$ and $\Omega = 1$.

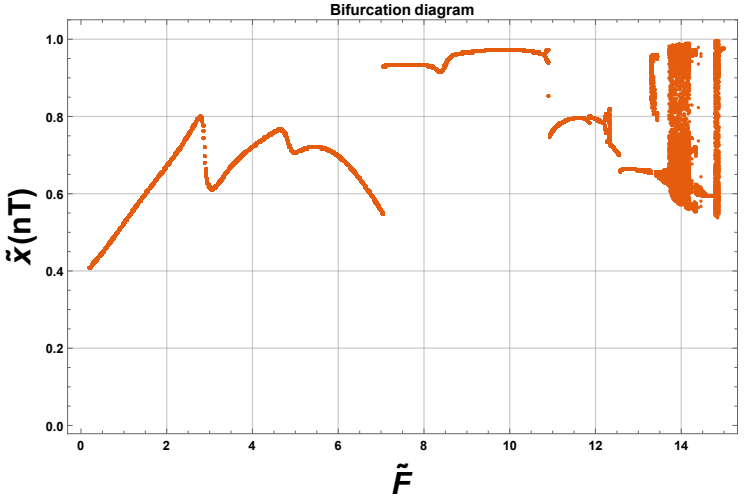


Figure 5.14: Bifurcation diagram for different values of the force with $\Lambda = 1$, $\Phi = 0.1$, and $\Omega = 1$

As we can see in the figure, the force needed to generate chaos in the system is $\tilde{F} \approx 12.5$. Now, we are going to generate another bifurcation diagram with a different Λ value and see how the system reacts to that change. For the following bifurcation diagram, we are going to use $\Lambda = 10$.

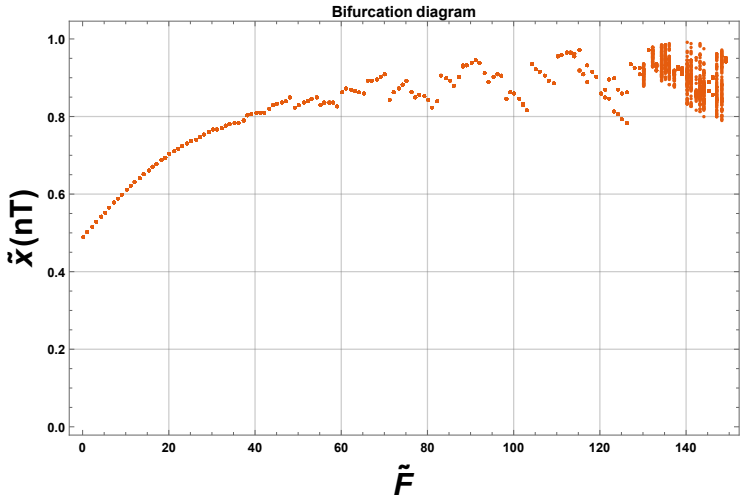


Figure 5.15: Bifurcation diagram for different values of the force with $\Lambda = 1$, $\Phi = 0.1$, and $\Omega = 1$

With the previous figure, we can see that when Λ increases, the force needed to generate chaos increases too. The next bifurcation diagram is with a low value of Λ. For the following plot our Energies number is $\Lambda = 0.1$

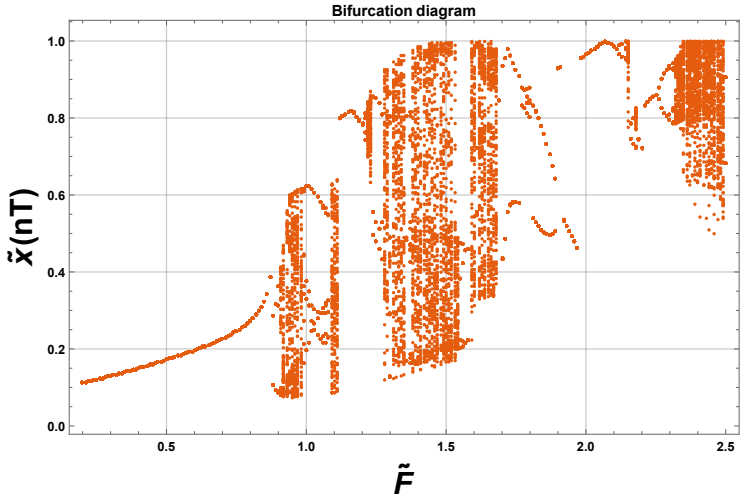
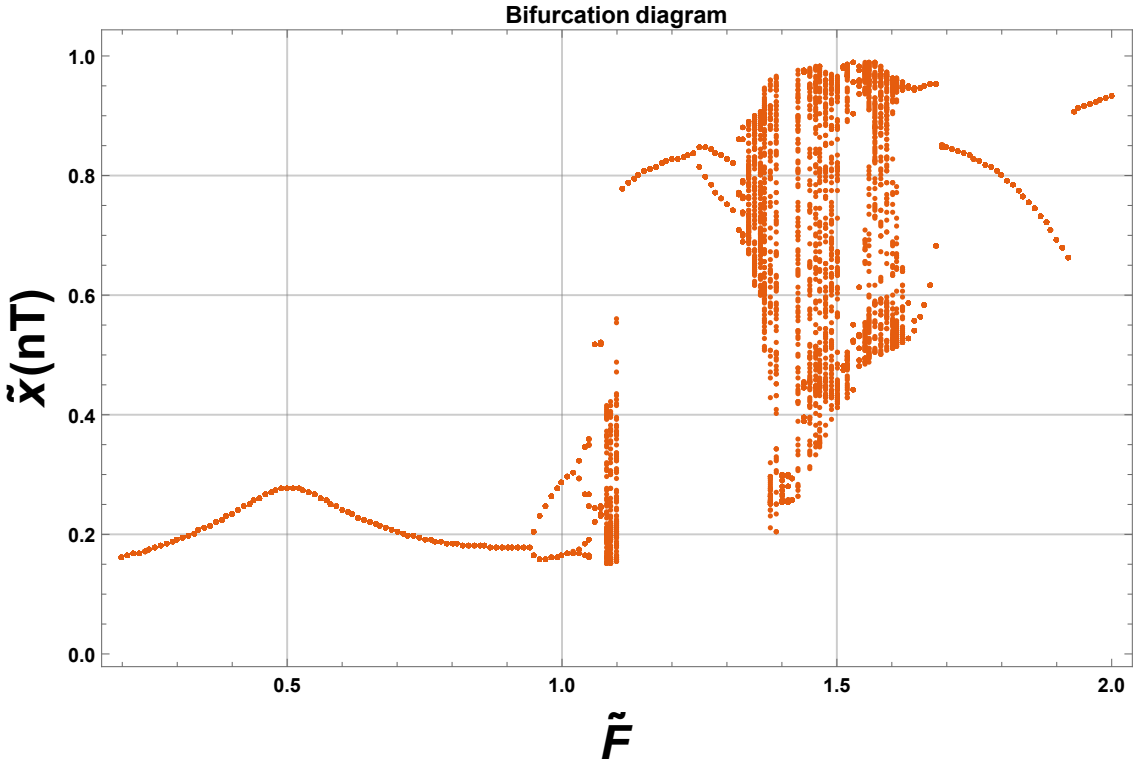
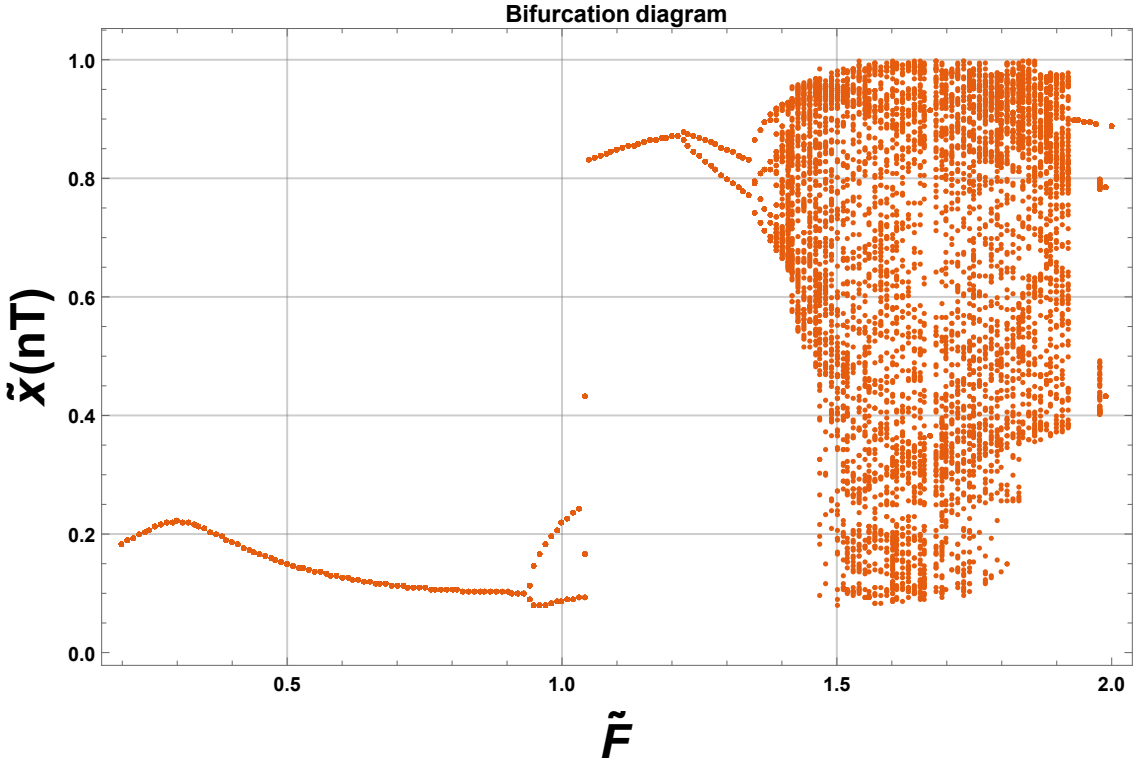


Figure 5.16: Bifurcation diagram for different values of the force with $\Lambda = 0.1$, $\Phi = 0.1$, and $\Omega = 1$

Now, we can see that for a low Energies number, the Force number is low too. So, there is a direct relation between the Energies number and the force number in terms of chaos. Then, to generate chaos in this system, we need to have a very low Energies number value, or a very large value of the force number. For the next step, we are going to see how the frequency number affects the force needed to generate chaos in the system. This can be shown in the figures 5.17 and 5.18.

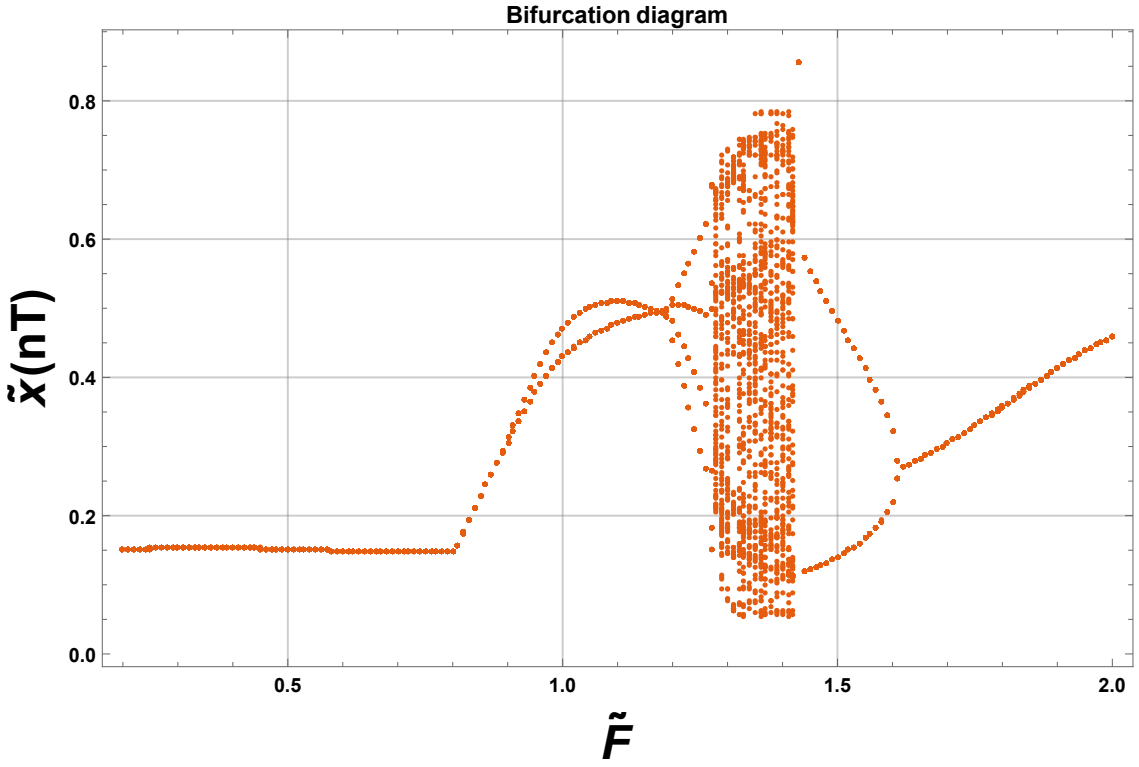


((a)) Bifurcation diagram of the position to different values of $\tilde{F}$ with $\Lambda = 0.15$, $\Phi = 0.1$ and $\Omega = 1$

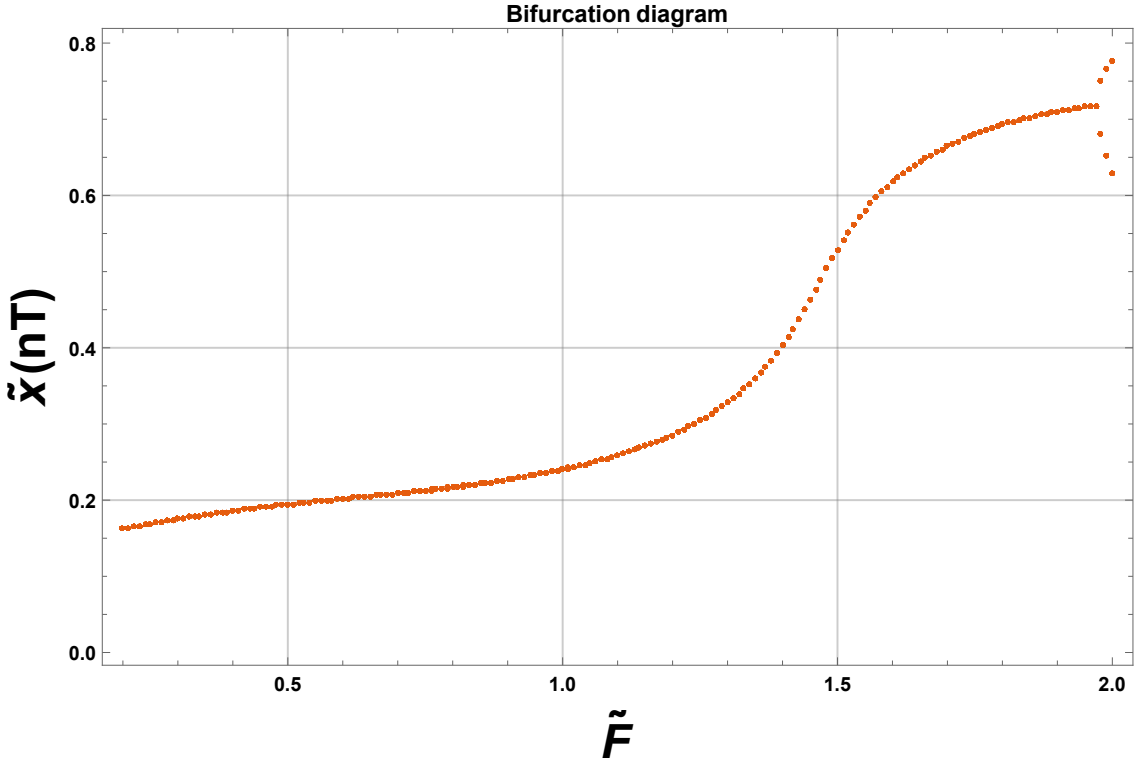


((b)) Bifurcation diagram of the position to different values of $\tilde{F}$ with $\Lambda = 0.15$, $\Phi = 0.1$ and $\Omega = 1.5$

Figure 5.17: Numerical demonstration that the angle (Ω) plays a relevant role in the chaos of the system, and it defines the force value (F) that generates chaos in the system



((a)) Bifurcation diagram of the position to different values of $\tilde{F}$ with $\Lambda = 0.15$, $\Phi = 0.1$ and $\Omega = 1.75$



((b)) Bifurcation diagram of the position to different values of $\tilde{F}$ with $\Lambda = 0.15$, $\Phi = 0.1$ and $\Omega = 2$

Figure 5.18: Numerical demonstration that the angle (Ω) plays a relevant role in the chaos of the system, and it defines the force value (F) that generates chaos in the system

As we can see in the previous figures, the frequency number plays an important role in chaos, because in every case that we check the force needed to generate chaos changes. For this reason, it is important to know the frequency of our system.

Numerical analysis

Something we can appreciate from our differential equation is that it is analytically unsolvable, so the only way to explore the dynamics of the system is to check how the system changes when our dimensionless parameters (Λ, Φ, Ω, and $\tilde{F}$) change. With that, we can make a table to understand the system behavior.

Condition	**System behavior**
$\Lambda < 1$	Linear oscillations
$\Lambda \geq 1$	Nonlinear regime, possible limit cycles
$\Phi = 0$, $F = 0$	Conservative (harmonic) motion
$\Phi > 0$, $F = 0$	Damped oscillation to equilibrium
$\Phi > 0$, $F \neq 0$	Forced nonlinear response
$\Omega \approx \omega_0$	Resonance (maximum amplitude)

Table 5.3: Different behaviors of the system with different values of our dimensionless parameters

As we know, the energy of our system is proportional to its amplitude, so we need to find the set of numbers that maximizes the amplitude. To see the amplitude of a numerical solution, we need to obtain the maximum and minimum values of the steady-state part and compute the mid-point difference between them.

$$A = \frac{max(x_{st}) - min(x_{st})}{2}, \tag{5.3.41}$$

where x_{st} is the steady state position. With this, we can create a code that allows us to compare how the amplitude of the steady-state changes when another dimensionless parameter is altered.

The proportionality of the amplitude to the external force is indicative that the other parameters play a more important role in the dynamics of the system. The second parameter with a less relevant role is the friction number, because the increase in its values reduces the amplitude, so we know that we want a system with the least possible value. The previous can be shown in the following figure (figure 5.19).

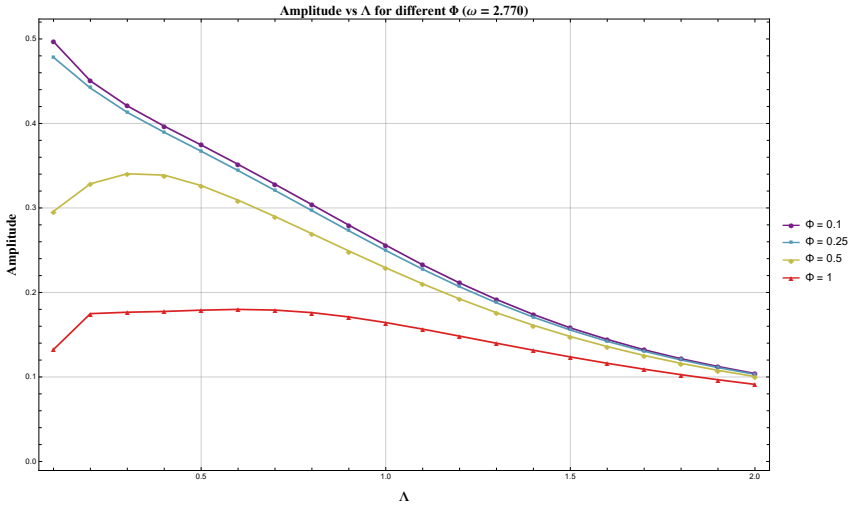


Figure 5.19: Change of the amplitude in terms of Λ with different values of Φ with Ω.

The two principal parameters are the Energie's number (Λ) and the frequency of the external force (Ω). This is because the Energie's number defines the natural frequency of the system, and the external force brings us the resonance, when its value is approximately the same as the natural frequency, increasing the amplitude.

Firstly, we are going to show the behavior of the amplitude with different Λ values at a specific frequency, in a medium with a fixed friction number (Φ), this can be shown in the following figure (5.20).

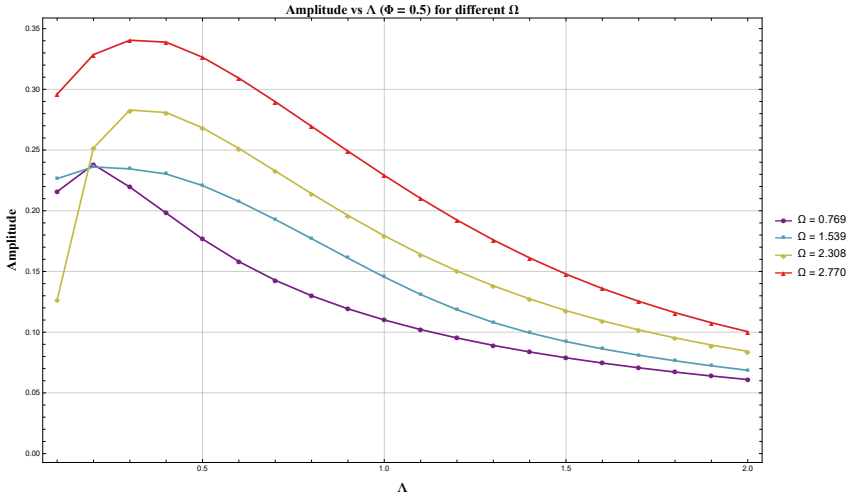


Figure 5.20: Change of the amplitude in terms of Λ with different values of Φ with Ω.

We know that the frequency (Ω) plays an important role, because its values figure (5.21)

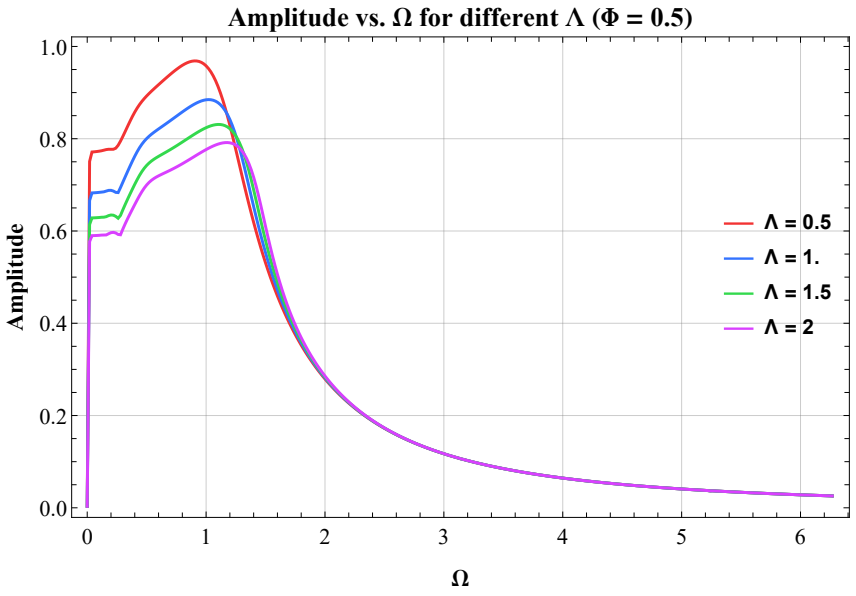


Figure 5.21: Change of the amplitude in terms of Λ with different values of Φ with Ω.

5.3.5 Fluid force

As we mentioned in the fluid chapter, there are a lot of dimensionless numbers that help us to describe the behavior of the fluid, but for the VIV nature, the most important number is the Strouhal number. This is because it describes the vortex shedding behind a body.

Another important dimensionless number is the inertial–gravitational interaction number Γ ($\Gamma = \frac{U}{\sqrt{gD}}$, where U is the speed of the fluid and D is the characteristic length), because it compares the structure acceleration due to flow forcing relative to gravity acceleration, and we know that gravity plays an important role in this system, in particular in the definition of the Energie's number.

Now, we are going to rewrite our differential equation using the fluid expressions. Using fluid dimensionless numbers, like in the previous problem, our external force has the following expression.

$$F_f = \frac{1}{2}\rho U^2 dLC_L, \tag{5.3.42}$$

where L is the axial longitude of the free magnet, C_L is the lift coefficient, d is the width of our free magnet (diameter), and ρ is the density of our fluid. If the external force is the fluid force. Then, we can rewrite the force terms of the Gravity-inertial number as:

$$\begin{aligned} F_0 = \frac{1}{2}\rho U^2 dLC_L \implies \tilde{F}F_s = \frac{1}{2}\rho U^2 dLC_L \\ \tilde{F} = \frac{\rho U^2 dLC_L}{2F_s} \implies \tilde{F} = \frac{\rho U^2 dLC_L}{2\Lambda mg} \\ \tilde{F} = \frac{\rho \Gamma^2 DdLC_L}{2m\Lambda} \therefore \tilde{F} = \frac{1}{2m^*}\frac{C_L\Gamma^2}{\Lambda}. \end{aligned} \tag{5.3.43}$$

We define m^* as the mass ratio between the mass of the structure and the mass of the fluid, that is, $m^* = \left(\frac{m}{\rho DS}\right)$. Something that is relevant to highlight is that the lift coefficient

(C_L) could be a function of time, which would depend on the geometry, but in this specific example, we are going to work with a constant lift coefficient.

$$\frac{d^2\tilde{x}}{d\tau^2} + 2\Phi\frac{d\tilde{x}}{d\tau} + 1 + \Lambda\left[\frac{1}{1-\tilde{x}} - \frac{1}{\tilde{x}}\right] = \frac{1}{2m^*}C_L\,\Gamma^2\cos(\Omega\tau), \tag{5.3.44}$$

as in the general case of a forced system, the amplitude of our oscillation will depend on the energy's number (Λ) and the frequency of the vortices, which in this case is the Strouhal number (St).

Another thing that we need to consider when we are working with a fluid force is the geometry of our structure, cause its shape will define the lift coefficient and the Strouhal number (St). The Stouhal number definition is: $St = \frac{f_s D}{U}$. Then we can find a relation between the Strouhal and the frequency number. This relation is:

$$\begin{aligned} \omega = 2\pi f_s \implies \frac{\Omega}{t_s} &= 2\pi\left(\frac{StU}{D}\right) \\ \implies \Omega = \frac{2\pi StUt_s}{D} \implies \Omega &= \frac{2\pi StU}{D}\sqrt{\frac{h}{g}} \\ \implies \Omega = \frac{2\pi StU}{\sqrt{gD}}\sqrt{\frac{h}{D}} \implies &\boxed{\Omega = 2\pi St\Gamma\sqrt{\frac{h}{D}} = \frac{2\pi StUt_s}{D}}. \end{aligned} \tag{5.3.45}$$

Then, the frequency of the system is related to the Strouhal and Gravity-inertial numbers. If the characteristic length is equal to the equilibrium distance ($D = h$), the frequency number can be simplified as:

$$\Omega = 2\pi St\Gamma. \tag{5.3.46}$$

Then, we can rewrite our dimensionless differential equation as:

$$\frac{d^2\tilde{x}}{d\tau^2} + 2\Phi\frac{d\tilde{x}}{d\tau} + 1 + \Lambda\left[\frac{1}{1-\tilde{x}} - \frac{1}{\tilde{x}}\right] = \frac{1}{2m^*}C_L\,\frac{\Gamma^2}{\Lambda}\cos(2\pi St\Gamma\tau), \tag{5.3.47}$$

It is important to see that our system now depends on the values of the Strouhal (St), Gravity-inertia(Γ) and Energie's (Λ) numbers and the friction number (Φ). In this system, the most important numbers are the Strouhal and Gravity-inertial numbers because they define the frequency of the oscillation.

Something interesting that we can find in our equation is the relation $R = \frac{\Gamma^2}{\Lambda}$, because it relates the amplitude of the system to the properties of the fluid and the energy of the magnets. So, it is important to analyse how the system reacts to this variable, for this reason, we made the following table.

As observed, the optimal operating regime for our system occurs when $R \approx 1$, as this allows the resonance to maintain the oscillations within the optimal operatin range. In contrast, if $R \gg$, the fluid force dominates the magnetic forces, leading to excessive oscillation amplitudes that may cause the free magnet to interact undesirably with the structure. On the other hand, if $R \ll 1$, the fluid forcing is too weak, and the system remains under-excited.

Regime	R	Physical Interpretation
Gravity–dominated	$R \ll 1$	Fluid inertial forces are much weaker than magnetic forces. Motion is primarily governed by magnetic attraction/repulsion and gravity. The system behaves like a nonlinear magnetic oscillator with weak fluid forcing.
Balanced regime	$R \approx 1$	Fluid inertial effects and magnetic forces are comparable in magnitude. Possible resonant responses depending on forcing. System sensitivity to initial conditions increases.
Fluid–dominated	$R \gg 1$	Fluid inertial forces dominate the dynamics over magnetic forces. Motion resembles a fluid–structure interaction problem, where drag/lift control the oscillations.

Table 5.4: Different physical regimes determined by the ratio $R = \Gamma^2/\Lambda$.

New dimensionless equation

If we pay attention to the Strouhal number-frequency relation (equation (5.3.45)), we find that there is another possible dimensionless form that could help us to let the dimensionless frequency be only in terms of the Strouhal numbers. This new dimensionless use fluid and structure properties. The new relation is:

$$\Omega = \frac{2\pi StUt_s}{D} = \Omega = 2\pi St \implies \frac{Ut_s}{D} = 1 \quad \therefore \quad \boxed{t_s = \frac{D}{U} = \frac{h}{U}}, \tag{5.3.48}$$

this new time scale lets us to rewrite our differential equation. To rewrite it, we are going to use the dimensionless groups that we obtained previously. Starting the new dimensionless analysis in the equation of motion (5.3.3).

Applying the dimensionless groups in our differential equation, we can find the equation of motion

$$\begin{aligned}
&\frac{d^2x}{dt^2} + \frac{\beta}{m}\frac{dx}{dt} + g + \frac{M}{m}\left[\frac{1}{h-x} - \frac{1}{x}\right] = \frac{F}{m}\cos(\omega t) \\
\implies &\frac{d^2\tilde{x}}{d\tau^2} + \frac{\beta t_s}{m}\frac{d\tilde{x}}{d\tau} + \frac{gt_s^2}{x_s} + \frac{Mt_s^2}{mx_s}\left[\frac{1}{h - x_s\tilde{x}} - \frac{1}{x_s\tilde{x}}\right] = \frac{Ft_s^2}{mx_s}\cos(2\pi St\tau) \\
\implies &\frac{d^2\tilde{x}}{d\tau^2} + \frac{\beta h}{mU}\frac{d\tilde{x}}{d\tau} + \underbrace{\frac{gh^2}{hU^2}}_{1/\Gamma^2} + \frac{Mh^2}{mhU^2}\left[\frac{1}{h - h\tilde{x}} - \frac{1}{h\tilde{x}}\right] = \frac{Fh^2}{mhU^2}\cos(2\pi St\tau) \\
&\implies \frac{d^2\tilde{x}}{d\tau^2} + \frac{\beta h}{mU}\frac{d\tilde{x}}{d\tau} + \frac{1}{\Gamma^2} + \frac{M}{mU^2}\left[\frac{1}{1-\tilde{x}} - \frac{1}{\tilde{x}}\right] = \frac{Fh}{mU^2}\cos(2\pi St\tau)
\end{aligned} \tag{5.3.49}$$

Now, using the definitions of the Energie's number ($\Lambda = \frac{M}{mgh}$), and the dimensionless form of the force ($\tilde{F} = \frac{Fh}{M}$); we can dimensionless our equation of motion.

$$
\begin{aligned}
&\frac{d^2\tilde{x}}{d\tau^2} + \frac{\beta h}{mU}\frac{d\tilde{x}}{d\tau} + \frac{1}{\Gamma^2} + \frac{\Lambda g h}{U^2}\left[\frac{1}{1-\tilde{x}} - \frac{1}{\tilde{x}}\right] = \frac{F_s \tilde{F} h}{mU^2}\cos(2\pi St\tau) \\
\implies &\frac{d^2\tilde{x}}{d\tau^2} + \frac{\beta h}{mU}\frac{d\tilde{x}}{d\tau} + \frac{1}{\Gamma^2} + \frac{\Lambda}{\Gamma^2}\left[\frac{1}{1-\tilde{x}} - \frac{1}{\tilde{x}}\right] = \frac{\tilde{F} M}{mU^2}\cos(2\pi St\tau) \\
\implies &\frac{d^2\tilde{x}}{d\tau^2} + \frac{\beta h}{mU}\frac{d\tilde{x}}{d\tau} + \frac{1}{\Gamma^2} + \frac{\Lambda}{\Gamma^2}\left[\frac{1}{1-\tilde{x}} - \frac{1}{\tilde{x}}\right] = \frac{\tilde{F}\Lambda m g h}{mU^2}\cos(2\pi St\tau) \\
\implies &\frac{d^2\tilde{x}}{d\tau^2} + \frac{\beta h}{mU}\frac{d\tilde{x}}{d\tau} + \frac{1}{\Gamma^2} + \frac{\Lambda}{\Gamma^2}\left[\frac{1}{1-\tilde{x}} - \frac{1}{\tilde{x}}\right] = \frac{\tilde{F}\Lambda g h}{U^2}\cos(2\pi St\tau) \\
\implies &\frac{d^2\tilde{x}}{d\tau^2} + \frac{\beta h}{mU}\frac{d\tilde{x}}{d\tau} + \frac{1}{\Gamma^2} + \frac{\Lambda}{\Gamma^2}\left[\frac{1}{1-\tilde{x}} - \frac{1}{\tilde{x}}\right] = \frac{\tilde{F}\Lambda}{\Gamma^2}\cos(2\pi St\tau).
\end{aligned}
\tag{5.3.50}
$$

Now, we can define the Friction number in terms of the new time scale, this new friction number is:

$$
\Phi = \frac{\beta h}{2mU}. \tag{5.3.51}
$$

Then using this new definition, the magneto-fluid ratio ($R = \frac{Fr^2}{\Lambda}$) and the fluid force (equation (5.3.43))definitions, our differential equation is:

$$
\begin{aligned}
\implies &\frac{d^2\tilde{x}}{d\tau^2} + \frac{\beta h}{mU}\frac{d\tilde{x}}{d\tau} + \frac{1}{\Gamma^2} + \frac{\Lambda}{\Gamma^2}\left[\frac{1}{1-\tilde{x}} - \frac{1}{\tilde{x}}\right] = \frac{\tilde{F}\Lambda}{\Gamma^2}\cos(2\pi St\tau) \\
\implies &\frac{d^2\tilde{x}}{d\tau^2} + 2\Phi\frac{d\tilde{x}}{d\tau} + \frac{1}{\Gamma^2} + \frac{1}{R}\left[\frac{1}{1-\tilde{x}} - \frac{1}{\tilde{x}}\right] = \frac{\tilde{F}}{R}\cos(2\pi St\tau) \\
\implies &\frac{d^2\tilde{x}}{d\tau^2} + 2\Phi\frac{d\tilde{x}}{d\tau} + \frac{1}{\Gamma^2} + \frac{1}{R}\left[\frac{1}{1-\tilde{x}} - \frac{1}{\tilde{x}}\right] = \frac{\frac{1}{2m^*}C_L R}{R}\cos(2\pi St\tau) \\
\therefore \quad &\frac{d^2\tilde{x}}{d\tau^2} + 2\Phi\frac{d\tilde{x}}{d\tau} + \frac{1}{\Gamma^2} + \frac{1}{R}\left[\frac{1}{1-\tilde{x}} - \frac{1}{\tilde{x}}\right] = \frac{1}{2m^*}C_L\cos(2\pi St\tau).
\end{aligned}
\tag{5.3.52}
$$

This last result is important because it lets us the equation of motion in terms of four dimensionless parameters that are: the Strouhal (St) and Gravity-inertia(Γ) numbers, the friction number (Φ) and the magneto-fluid ratio (R), where two are directly obtained by the fluid, one is by the environment and the last is the relation between the movement of the fluid and the energy of the free magnet. And it is very important because the frequency of the external force only depends in terms of the Strouhal number, or in other words, the vortex–shedding frequency of the von Kármán street.

If we use the optimal regime of R in our system, where is when $R \approx 1$, then our differential equation can be re expressed in terms of only three dimensionless parametes (St, Γ, Φ), like we show in the following equation.

$$
\frac{d^2\tilde{x}}{d\tau^2} + 2\Phi\frac{d\tilde{x}}{d\tau} + \frac{1}{\Gamma^2} + \frac{1}{1-\tilde{x}} - \frac{1}{\tilde{x}} = \frac{1}{2m^*}C_L\cos(2\pi St\tau) \tag{5.3.53}
$$

5.4 Harnessing energy model

With the equation of motion of the system, we know the behavior of our dynamical system, and we are able to predict its motion. That lets us know where and when the system is near or far from a point. So, using this knowledge and Faraday's law, we can induce a voltage in an open circuit. Then, introducing Faraday's law:

$$\varepsilon = -n\frac{d\Phi_B}{dt}, \tag{5.4.1}$$

where n is the number of turns of a coil, Φ_B is the magnetic flux. Now, using the definition of the magnetic flux:

$$\Phi_B = BS\cos(\alpha), \tag{5.4.2}$$

where B is the magnetic field, S the surface area of the coil, and α is the angle between the magnetic field and the normal vector of the surface area. It is important to remark that in our system the magnet and the coil are always aligned with $\alpha = \pi$. Using this information, our magnetic flux is:

$$\Phi_B = -BS. \tag{5.4.3}$$

If we use the magnetic field of a dipole magnet, we obtain that it depends on the distance. The magnetic field is.

$$B = \frac{\mu_0 \mathcal{M}_D}{2x^3}, \tag{5.4.4}$$

where μ_0 is the magnetic premeability of the free space, $\mathcal{M}_D$ is the magnetic dipole moment,and x is the distance between the magnet and the coil. Applying the previous definition to the induction Faraday's law.

$$\varepsilon = n\frac{d}{dt}(BS) \implies \varepsilon = \frac{n\mu_0\mathcal{M}_D S}{2}\frac{d}{dt}\left(\frac{1}{x(t)^3}\right). \tag{5.4.5}$$

For simplicity we define $\zeta = \frac{n\mu_0\mathcal{M}_D S}{2}$. Then, our generated voltage is:

$$\varepsilon = \zeta\frac{-3x(t)^2\cdot\dot{x}(t)}{x(t)^6} = -\frac{3\zeta\dot{x}(t)}{x(t)^4}. \tag{5.4.6}$$

Now, using the dimensionless parameters and solutions.

$$\varepsilon = -3\zeta\frac{\frac{d(x_s\tilde{x})}{d(t_s\tau)}}{(x_s\tilde{x})^4} = -3\zeta\left(\frac{x_s}{t_s x_s^4}\right)\frac{\dot{\tilde{x}}(\tau)}{\tilde{x}(\tau)^4} = -\frac{3\zeta}{t_s x_s^3}\left(\frac{\dot{\tilde{x}}(\tau)}{\tilde{x}(\tau)^4}\right) \quad \therefore \quad \boxed{\varepsilon = -\frac{3\zeta U}{h^4}\left(\frac{\dot{\tilde{x}}(\tau)}{\tilde{x}(\tau)^4}\right)} \tag{5.4.7}$$

Now, using the equation of the induced voltage and the numerical solution of the system, we can find the response of the voltage to the motion of our system. For a specific case, we use in the system the following values: $\Lambda = 1$, $\Phi = 0.5$, $\epsilon = 0.5$, $\Omega = 0.9\Omega_0$, $F = 1$; and

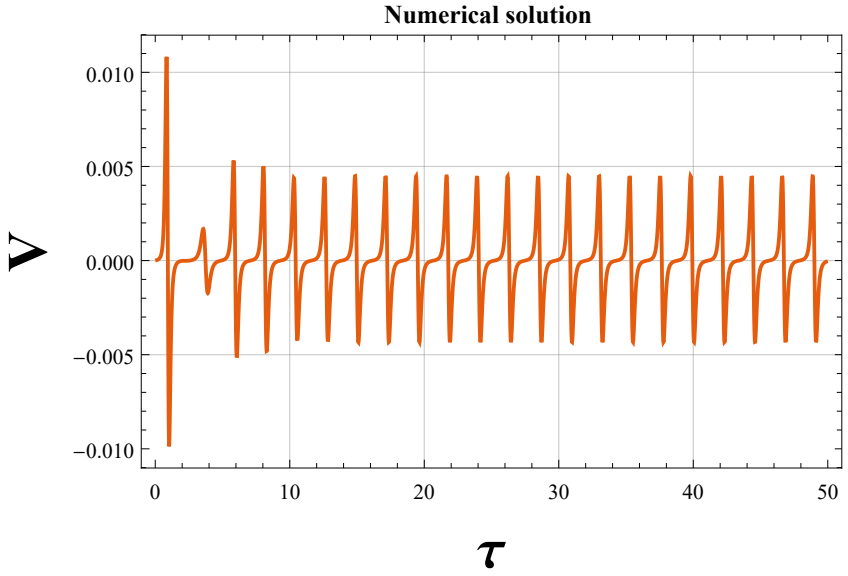


Figure 5.22: Voltage generated by the system

the coil has the following parameters: $n = 10$, $h = 1$, $S = 1$, $\mathcal{M} = 1$, $U = 1$. Then, the voltage generated over time is represented in the following figure.

Now, to see the dependence of the voltage to the friction, we plot the voltage generated by different Φ values

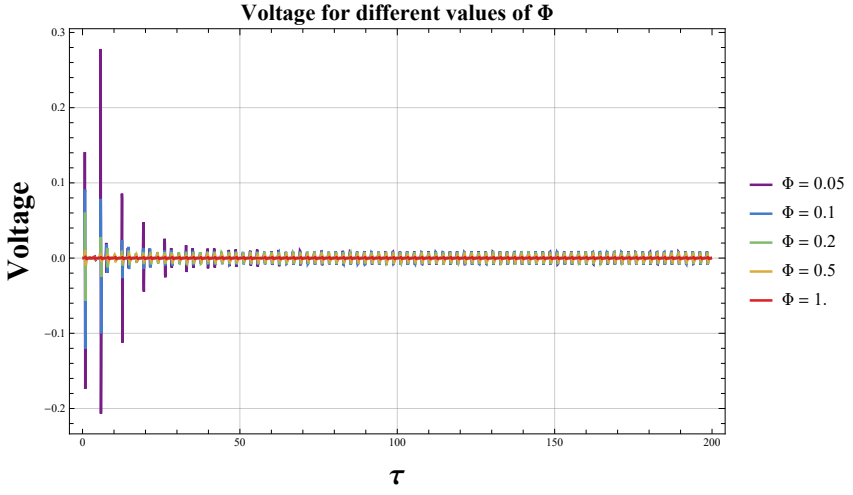


Figure 5.23: Voltage generated by different values of friction.

For the previous figure, we can see that the voltage decreases when the damping increases. Another interesting case is how the voltage reacts to different frequency values. These frequencies are the frequencies of the vortices.

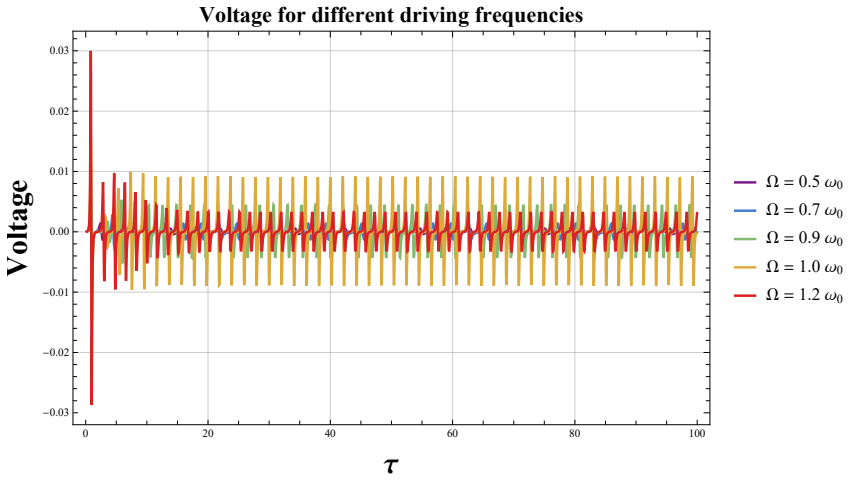


Figure 5.24: Voltage generated by different frequencies, multiples of the natural frequency Ω_0

As we can see in the previous figure, the magnitude of the voltage reduces when the frequency tends to the natural frequency of the system, but the voltage is more recurrent.

5.5 Conclusions

In this chapter, we confirm that our system can oscillate, and the interaction with a fluid in motion could be the source of energy that it needs to maintain the oscillation. In contrast to the first system (magneto-gravitational system), this system has a symmetric oscillation, which is caused by the structural design that establishes a region where the system moves.

The presence of the two fixed magnets indicates that the magnetic energy plays an important role. This can be shown in the expressions of the natural frequency and the equilibrium position, which are written in terms of the Energie's number (Λ).

$$\omega_0 = \sqrt{4\Lambda + 2\sqrt{1+4\Lambda^2} + \frac{1}{\Lambda}}, \tag{5.5.1}$$

$$\tilde{x}_{eq} = \frac{x_{eq}}{h} = \frac{2\Lambda + 1 - \sqrt{4\Lambda^2+1}}{2} \tag{5.5.2}$$

In terms of the external force, we can see that the frequency of the vortices generated by the fluid-structure interaction is responsible for the motion and the quantity of the voltage generated.

For this system, the amplitude of the oscillation depends on some dimensionless numbers, which are the Energie's number (Λ), the inertial-gravitational interaction number (Γ), the friction number (Φ), the lift coefficient (C_L), the mass ratio (m^*), and the Strouhal number (St). One of the most important parameters is the lift coefficient, because it changes with the shape of the body. For this reason, it becomes essential to study different geometries and choose the one that maximizes the lift coefficient. A better shape would increase the force that the fluid transfers to the system, and it can increase the voltage that we generate.

We also observed that the fluid parameters play an important role. The Strouhal number sets the resonance condition that creates the oscillation, while the inertial gravity interaction number affects not only the resonance itself but also the stability and complexity of the dimensionless equation. This shows how sensitive the system is to the properties of the fluid and how these parameters can change the global behavior of the solution.

As in the previous chapter, the mathematical tools used throughout the chapter were very useful to understand how the system reacts and why it behaves the way it does. These methods helped us find expressions that describe the dynamics more clearly and opened the possibility of using this kind of system in future projects related to clean energy generation. The ideas developed here could be the base to design simple and stable oscillators powered only by natural fluid motion.

CHAPTER 6

Rigid pendulum with magnets

> The production of electricity from magnetism is a result of motion... when a magnet passes before a conductor, or a conductor before a magnet, a power is awakened that flows as a current, manifesting the beautiful unity of force and motion.
>
> Michael Faraday, *"Experimental Researches in Electricity"*, 1831

In this chapter, we analyze a system composed of a body that oscillates like a pendulum, with magnets at the extremum points that introduce repulsive forces. To see the system, we present the following figure.

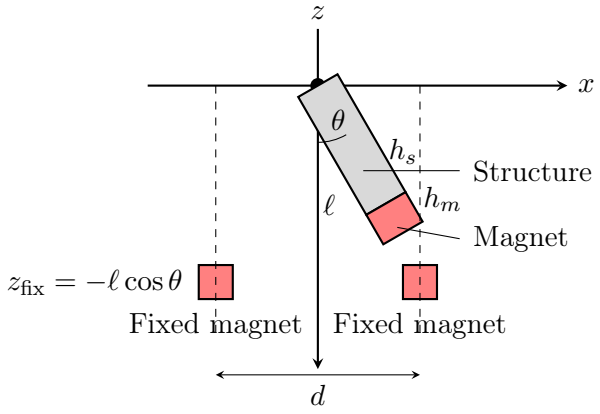


Figure 6.1: Graphic representation of our system

Before starting the analysis of this system, we need to describe it and find its center. The system consists of two coupled bodies and a set of magnets that will be used to create repulsive forces.

The bodies are quadrangular prisms, but for simplicity, we will assume that they are square prisms. The smaller body (a magnet) is positioned beneath the larger one (the oscillating structure). Both bodies are aligned along the vertical symmetry axis, so the centers of both bodies are aligned.

If both bodies are homogeneous, their centers of mass (COM) will coincide with their geometric centers, meaning they are aligned. Now, if the structure has a mass m_s and a height h_s, and the magnet has a mass m_m and a height h_m, we can determine the COM of the entire system. It is important to note that the bodies undergo pendular motion, so it is better to use polar coordinates. Thus, we have:

$$\vec{R}_{COM} = \frac{1}{M}\sum_{i=1}^{N} m_i \vec{r}_i = \frac{m_s \vec{r}_{COMs} + m_m \vec{r}_{COMm}}{m_m + m_s}, \tag{6.0.1}$$

where $\vec{r}_{COMs}$ is the position vector of the COM of the structure and $\vec{r}_{COMm}$ is the position vector of the COM of the magnet. Next, we will calculate the position vector of the COM for the structural body. To do that, we suppose that the structure has a homogeneous, and then the density is:

$$\rho(\vec{r}) = \frac{m_s}{V} = \frac{m_s}{LWH}, \tag{6.0.2}$$

where V is the volume, L is the length, W is the width, and H is the height of the structure body. Now, using the previous result and the formula of the position vector of the COM, we obtain:

$$\begin{aligned}
\vec{r}_{COMs} &= \frac{1}{m_s}\int_V \vec{r}\rho(\vec{r})dV \\
&= \frac{1}{m_s}\int_V \frac{m_s\vec{r}}{LWH}dV \\
&= \frac{1}{LWH}\int_V \vec{r}dV \\
&= \frac{1}{LWH}\int_V x\hat{x} + y\hat{y} + z\hat{z}dV \\
&= \frac{1}{LWH}\int_{z_0}^{z_1}\int_{y_0}^{y_1}\int_{x_0}^{x_1} x\hat{x} + y\hat{y} + z\hat{z}dxdydz,
\end{aligned} \tag{6.0.3}$$

To simplify the calculus, we are going to do the integrals using Einstein's sum convention. Hence, we obtain:

$$\begin{aligned}
\int_{\alpha_0}^{\alpha_1}\int_{\beta_0}^{\beta_1}\int_{\gamma_0}^{\gamma_1} x_i\hat{x}_i dx_i dx_j dx_k &= (x_j)\Big|_{\beta_0}^{\beta_1}(x_k)\Big|_{\alpha_0}^{\alpha_1}\left(\frac{x_i^2}{2}\right)\Big|_{\gamma_0}^{\gamma_1}\hat{x}_i \\
&= \frac{1}{2}(\alpha_1-\alpha_0)(\beta_1-\beta_0)\left(\gamma_1^2-\gamma_0^2\right)\hat{x}_i
\end{aligned} \tag{6.0.4}$$

Substituting this result in the equation 6.0.3 and doing the algebra, we can see:

$$\begin{aligned}
\vec{r}_{COMs} &= \frac{1}{LWH}\int_{z_0}^{z_1}\int_{y_0}^{y_1}\int_{x_0}^{x_1} x\hat{x} + y\hat{y} + z\hat{z}dxdydz \\
&= \frac{1}{LWH}[\frac{1}{2}(z_1-z_0)(y_1-y_0)\left(x_1^2-x_0^2\right)\hat{x} + \frac{1}{2}(z_1-z_0)(x_1-x_0)\left(y_1^2-y_0^2\right)\hat{y} \\
&+ \frac{1}{2}(x_1-x_0)(y_1-y_0)\left(z_1^2-z_0^2\right)\hat{z}]
\end{aligned} \tag{6.0.5}$$

If the integration limits are: $x_0 = -\frac{L}{2}$, $x_1 = \frac{L}{2}$ $y_0 = -\frac{W}{2}$, $x_1 = \frac{W}{2}$ $z_0 = -H$, $z_1 = 0$. Then, the position vector of the COM of the structure is:

$$\begin{aligned}\vec{r}_{COMs} &= \frac{1}{LWH}[\frac{1}{2}(z_1 - z_0)(y_1 - y_0)\left(x_1^2 - x_0^2\right)\hat{x} + \frac{1}{2}(z_1 - z_0)(x_1 - x_0)\left(y_1^2 - y_0^2\right)\hat{y} \\ &+ \frac{1}{2}(x_1 - x_0)(y_1 - y_0)\left(z_1^2 - z_0^2\right)\hat{z}] \\ &= \frac{1}{2LWH}\left[0\hat{x} + 0\hat{y} - LWH^2\hat{z}\right] \\ &= 0\hat{x} + 0\hat{y} - \frac{H}{2}\hat{z}.\end{aligned} \tag{6.0.6}$$

As we know, the height of the structure is h_s, then the position vector is:

$$\vec{r}_{COMs} = 0\hat{x} + 0\hat{y} - \frac{h_s}{2}\hat{z} \tag{6.0.7}$$

.

Now we can reduce the dimensions of the motion because the movement of the structure is constrained to a plane, specifically the x-z plane. Then, the position vector now is:

$$\vec{r}_{COMs} = 0\hat{x} - \frac{h_s}{2}\hat{z} \tag{6.0.8}$$

Using the transform relations to change the cartesian coordinates to polar coordinates:

$$r = \sqrt{x^2 + z^2} = \sqrt{0 + \left(\frac{h_s}{2}\right)^2} = \frac{h_s}{2}, \tag{6.0.9}$$

$$\theta = \frac{3\pi}{2}. \tag{6.0.10}$$

Therefore, the position of the COM in polar coordinates is:

$$\vec{r}_{COMs} = \frac{h_s}{2}\hat{r} \tag{6.0.11}$$

Analogously, we can do the same process to the magnet body, then the position vector of COM of the magnet is:

$$\vec{r}_{COMm} = \frac{h_m}{2}\hat{r} \tag{6.0.12}$$

But, as we know, the magnet is under the structure, then the relative height of the magnet with respect of the origin is $h_s + h_m$. Therefore, the position vector of COM of the magnet is:

$$\vec{r}_{COMm} = \left(h_s + \frac{h_m}{2}\right)\hat{r} \tag{6.0.13}$$

Then, the COM of the system has the following form:

$$\begin{aligned}
\vec{R}_{COM} &= \frac{m_s\vec{r}_{COMs} + m_m\vec{r}_{COMm}}{m_m + m_s} \\
&= \frac{1}{m_s + m_m}\left(\frac{m_s h_s}{2} + m_m\left\{h_s + \frac{h_m}{2}\right\}\right)\hat{r} \\
&= \frac{1}{m_s + m_m}\left[\frac{1}{2}\left(m_s h_s + 2m_m h_s + m_m h_m\right)\hat{r}\right] \\
&= \frac{1}{2}\left(\frac{1}{m_s + m_m}\right)\left[h_s(m_m + m_s) + m_m(h_s + h_m)\right]\hat{r} \\
\boxed{\therefore \vec{R}_{COM} = \frac{1}{2}\left[h_s + \frac{m_m}{m_s + m_m}(h_s + h_m)\right]\hat{r}}&.
\end{aligned} \tag{6.0.14}$$

The previous result is important because it lets us know that the position of the center of mass of the system depends on the masses involved. The center of mass is closer to the larger mass and the distance to the origin is always constant.

Another important thing is to define the space where our pendulum will be acting. We have the frame structure in the form of a rectangle that has a width of d and a height of h, like:

At the top, in the middle of the width of the frame structure, we place a pivot. We decide to place the origin of the coordinate system in the pivot. Then, the coordinates of the pivot are $(0, 0)$.

As we know, we are going to use a pendulum formed by two bodies, then our pendulum is a ***"physical pendulum"***, and the length of that pendulum (ℓ) is the sum of the height of the prism and the magnet ($\ell = h_s + h_m$). Now, we can find some geometric relations between the length of the pendulum and the frame structure, which are:

$$\begin{aligned}
&d \geq 2\ell \implies \ell \leq \frac{d}{2}, \\
&h \geq \ell,
\end{aligned} \tag{6.0.15}$$

These equations define the conditions for a pendulum acting within our frame structure.

We can restrict the movement of the pendulum by placing two magnets in the system. These magnets are static and are located symmetrically about the system's origin. They are at a vertical distance η, with the horizontal distance from the middle of the structure where they are $\frac{\delta}{2}$. Then, the coordinates are $(\frac{\delta}{2}, -\eta)$ for the right magnet and $(-\frac{\delta}{2}, -\eta)$ for the left magnet. With the coordinates of the fixed magnets, we can find the maximum angle (Θ) that our system could have:

$$\delta = 2\ell\sin(\Theta) \implies \sin(\Theta) = \frac{\delta}{2\ell} \implies \Theta = \arcsin\left(\frac{\delta}{2\ell}\right), \tag{6.0.16}$$

$$\eta = -\ell\cos(\Theta) \implies \cos(\Theta) = -\frac{\eta}{\ell} \implies \Theta = \arccos\left(-\frac{\eta}{\ell}\right). \tag{6.0.17}$$

Both expressions give us a relation to find the maximum angle (Θ) of the system, so we can find a relation between the arguments of the trigonometric functions; this relation is the Pythagorean relation of a unitary circle, then:

$$\left(\frac{\delta}{2\ell}\right)^2 + \left(-\frac{\eta}{\ell}\right)^2 = 1 \implies \ell^2 = \frac{\delta^2}{4} + \eta^2. \tag{6.0.18}$$

The previous result gives us a relation between the length of our pendulum and the coordinates of the fixed magnets.

Then, we can find the domains of our operative space and structural parameters. The operative domain in the horizontal direction is:

$$-\frac{d}{2} \leq x \leq \frac{d}{2} \implies Dom(x) = \left[-\frac{d}{2}, \frac{d}{2}\right]. \tag{6.0.19}$$

In the vertical direction, the domain is:

$$0 \leq z \leq h \implies Dom(z) = [-h, 0]. \tag{6.0.20}$$

Then, the domain of the horizontal position of the fixed magnets is:

$$0 \leq \delta \leq \frac{d}{2} \implies Dom(\delta) = \left[0, \frac{d}{2}\right]. \tag{6.0.21}$$

Finally, the domain of angle θ is:

$$-\Theta \leq \theta \leq \Theta \implies Dom(\theta) = [-\Theta, \Theta] \tag{6.0.22}$$

It is important to highlight that the system is going to be analyzed in polar coordinates, because the movement follows a curved trajectory.

6.1 Ideal case

For the ideal case, as we know, the dissipation forces are considered as null. Then, the only forces that play in our system are the gravitational force, the magnetic repulsive forces, and the torque of the structure. Then, the system of forces is described by:

$$\vec{F} = \vec{T} + \vec{g} + \vec{F}_{m1} + \vec{F}_{m2} \tag{6.1.1}$$

We know that this way is too complicated, so, as Landau and Lifshits said, it is simplest to work with the energies that act on the system. For that reason, we are going to use the Lagrangian and Hamiltonian analysis.

6.1.1 Lagrangian analysis

As we have seen previously, analytical mechanics requires knowing the expressions for the energies. For that reason, we are going to find them.

To find the expressions for the energies, we need to remember that we are working with a body, so we can express everything in terms of the center of mass and the moments of inertia of the body. The center of mass of a quadrangular prism is:

$$I_{COM} = \frac{1}{12} m(w^2 + h^2), \tag{6.1.2}$$

where w is the width of our prism. Something important to remember is that our body is a pendulum, so we can use the parallel axis theorem to find the moment of inertia at the pivot point.

$$I_{pivot} = I_{COM} + mR_i^2, \tag{6.1.3}$$

where R_i is the distance from the origin to the center of mass of our body. It is important to remember that we have two bodies, so we need to use the pivot moment of inertia of both bodies.

$$\begin{aligned} I_0 &= I_s + I_m \\ &= I_{COMs} + mR_s^2 + I_{COMm} + mR_m^2 \\ &= \frac{1}{12} m_s(w^2 + h_s^2) + m\left(\frac{h_s}{2}\right)^2 + \frac{1}{12} m_m(w^2 + h_m^2) + m\left(h_s + \frac{h_m}{2}\right)^2. \end{aligned} \tag{6.1.4}$$

It is important to highlight that I_0 is a constant because every term in the moment of inertia is defined by the geometry of our body.

Now, we can find the expression of the kinetic energy of a system. This can be found by the following equation:

$$T = \frac{1}{2} I_0 \dot{\theta}^2. \tag{6.1.5}$$

The potential energy is integrated by the gravitational potential energy and the two magnetic repulsion energies. The gravitational potential energy is the same as that of a simple pendulum.

$$U_g = mgR(1 - \cos(\theta)), \tag{6.1.6}$$

where R is the distance between the pivot and the center of mass of our system ($R = |\vec{R}_{COM}|$), m is the mass of all the system ($m = m_s + m_m$), g is the gravity acceleration and θ is the angle of the position of our system. In the case of the magnetic repulsion energy, we need to consider the angle of the pendulum. Now checking how the distance between magnets changes in terms of the angle is, for the right magnet:

$$
\begin{aligned}
r_R(\theta) &= \sqrt{\left(\frac{\delta}{2} - \ell\sin(\theta)\right)^2 + (\eta + \ell\cos(\theta))^2} \\
&= \sqrt{\frac{\delta^2}{4} - \delta\ell\sin(\theta) + \ell^2\sin^2(\theta) + \eta^2 + 2\eta\ell\cos(\theta) + \ell^2\cos^2(\theta)} \\
&= \sqrt{\frac{\delta^2}{4} + \eta^2 + \ell^2 - \ell(\delta\sin(\theta) - 2\eta\cos(\theta))},
\end{aligned}
\tag{6.1.7}
$$

Now, using the Pythagorean relation (6.0.18)and the relations between position and maximum angle (6.0.16, 6.0.17):

$$
\begin{aligned}
r_R(\theta) &= \sqrt{2\ell^2 - \ell(\delta\sin(\theta) - 2\eta\cos(\theta))} \\
&= \sqrt{2\ell^2 - \ell(2\ell\sin(\Theta)\sin(\theta) + 2\ell\cos(\Theta)\cos(\theta))} \\
&= \sqrt{2\ell^2 - 2\ell^2(\sin(\Theta)\sin(\theta) + \cos(\Theta)\cos(\theta))} \\
\therefore r_R(\theta) &= \ell\sqrt{2(1 - \cos(\Theta - \theta))}
\end{aligned}
\tag{6.1.8}
$$

Now, for the left magnet:

$$
\begin{aligned}
r_L(\theta) &= \sqrt{\left(-\frac{\delta}{2} - \ell\sin(\theta)\right)^2 + (\eta + \ell\cos(\theta))^2} \\
&= \sqrt{\frac{\delta^2}{4} + \delta\ell\sin(\theta) + \ell^2\sin^2(\theta) + \eta^2 + 2\eta\ell\cos(\theta) + \ell^2\cos^2(\theta)} \\
&= \sqrt{2\ell^2 + \ell(\delta\sin(\theta) + 2\eta\cos(\theta))} \\
&= \sqrt{2\ell^2 + \ell(2\ell\sin(\Theta)\sin(\theta) - 2\ell\cos(\Theta)\cos(\theta))} \\
&= \sqrt{2\ell^2 - 2\ell^2(\cos(\Theta)\cos(\theta) - \sin(\Theta)\sin(\theta))} \\
\therefore r_L(\theta) &= \ell\sqrt{2(1 - \cos(\Theta + \theta))}
\end{aligned}
\tag{6.1.9}
$$

With the distance between the fixed magnets and the free movement magnet, we can find the expression of the repulsive magnetic energy in terms of the distance; for this, we applied the expression of the magnetic energy that we used in the previous chapters. Then, the magnetic energies are:

$$
U_{MR} = M\ln(r_R(\theta)) \quad \text{and} \quad U_{ML} = M\ln(r_L(\theta)). \tag{6.1.10}
$$

Then the Lagrangian of our system is:

$$
\begin{aligned}
L &= T - U \\
&= \frac{1}{2}I_0\dot{\theta}^2 + mgR(1 - \cos(\theta)) + M\ln(r_R(\theta)) + M\ln(r_L(\theta))
\end{aligned}
\tag{6.1.11}
$$

Now, applying the Euler-Lagrange equation:

$$
\begin{aligned}
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} &= 0 \\
\frac{d}{dt}\left(I_0\dot{\theta}\right) + mgR\sin(\theta) - \frac{M}{r_R(\theta)}\frac{dr_R(\theta)}{d\theta} - \frac{M}{r_L(\theta)}\frac{dr_L(\theta)}{d\theta} &= 0 \\
I_0\ddot{\theta} + mgR\sin(\theta) - \frac{M}{r_R(\theta)}\frac{dr_R(\theta)}{d\theta} - \frac{M}{r_L(\theta)}\frac{dr_L(\theta)}{d\theta} &= 0 \\
I_0\ddot{\theta} + mgR\sin(\theta) - \frac{M}{r_R(\theta)}\left(\frac{\ell\sin(\Theta+\theta)}{\sqrt{2(1-\cos(\Theta+\theta))}}\right) - \frac{M}{r_L(\theta)}\left(-\frac{\ell\sin(\Theta-\theta)}{\sqrt{2(1-\cos(\Theta-\theta))}}\right) &= 0 \\
I_0\ddot{\theta} + mgR\sin(\theta) - \frac{M\sin(\Theta+\theta)}{2(1-\cos(\Theta+\theta))} + \frac{M\sin(\Theta-\theta)}{2(1-\cos(\Theta-\theta))} &= 0 \\
I_0\ddot{\theta} + mgR\sin(\theta) - \frac{M}{2}\left(\frac{\sin(\Theta+\theta)}{1-\cos(\Theta+\theta)} - \frac{\sin(\Theta-\theta)}{1-\cos(\Theta-\theta)}\right) &= 0.
\end{aligned}
\tag{6.1.12}
$$

Now, using trigonometric identities and doing some algebra to simplify the magnetic term:

$$
\begin{aligned}
\frac{\sin(\Theta+\theta)}{1-\cos(\Theta+\theta)} - \frac{\sin(\Theta-\theta)}{1-\cos(\Theta-\theta)} =& \frac{\sin(\Theta+\theta)(1-\cos(\Theta-\theta)) - \sin(\Theta-\theta)(1-\cos(\Theta+\theta))}{(1-\cos(\Theta+\theta))(1-\cos(\Theta-\theta))} \\
=& \frac{\sin(\Theta+\theta) - \frac{1}{2}(\sin(2\Theta)+\sin(2\theta)) - \sin(\Theta-\theta))}{1-\cos(\Theta-\theta)-\cos(\Theta+\theta)+\cos(\Theta+\theta)\cos(\Theta-\theta)} \\
& - \frac{\frac{1}{4}(\sin(2\Theta)-\sin(2\theta))}{1-\cos(\Theta-\theta)-\cos(\Theta+\theta)+\cos(\Theta+\theta)\cos(\Theta-\theta)} \\
=& \frac{\sin(\Theta+\theta)-\sin(\Theta-\theta)-\sin(2\theta)}{(\cos(\theta)-\cos(\Theta))^2} \\
=& -\frac{2\sin(\theta)(\cos(\theta)-\cos(\Theta))}{(\cos(\theta)-\cos(\Theta))^2} \\
=& -\frac{2\sin(\theta)}{\cos(\theta)-\cos(\Theta)}.
\end{aligned}
\tag{6.1.13}
$$

Then, our differential equation in terms of the maximum angle is:

$$
\boxed{I_0\ddot{\theta} + \left[mgR + \frac{M}{\cos(\theta)-\cos(\Theta)}\right]\sin(\theta) = 0}. \tag{6.1.14}
$$

It is important to highlight that our system does not depend on the length of the pendulum or the coordinates of the fixed magnets. Our system depends on the maximum angle (Θ), the distance between the pivot and the system's center of mass (R), the mass of the structure (m), the magnetic energy (M), and the structure's moment of inertia (I_0).

Then, we can find that our differential equation is:

$$
\ddot{\theta} + \left[\frac{mgR}{I_0} + \frac{M}{I_0}\frac{1}{\cos(\theta)-\cos(\Theta)}\right]\sin(\theta) = 0. \tag{6.1.15}
$$

This last result is very important because it lets us see that if the magnets are eliminated ($M = 0$), the equation of motion is the same as that of a physical pendulum. To simplify our differential equation, we define two constants: the natural frequency of a physical pendulum (ω_{0p}) as $\omega_{0p}^2 = \frac{mgR}{I_o}$ and the frequency of the magnetic repulsion (ϕ_M) as $\phi_M^2 = \frac{M}{I_0}$

$$\ddot{\theta} + \left[\omega_{0p}^2 + \frac{\phi_M^2}{\cos(\theta) - \cos(\Theta)}\right] \sin(\theta) = 0. \tag{6.1.16}$$

The last equation is similar to the equation of motion of a simple pendulum. So, using the analogy, we can conclude that the natural frequency of the system (ω_0) is all the terms that multiply $\sin(\theta)$. Then, our system is as follows:

$$\omega_0^2(\theta) = \omega_{0p}^2 + \frac{\phi_M^2}{\cos(\theta) - \cos(\Theta)} \implies \boxed{\ddot{\theta} + \omega_0^2(\theta)\sin(\theta) = 0}. \tag{6.1.17}$$

It is important to note that the natural frequency of the system is a function of the angle (θ), and it can be a constant for a specific case, that is, the small angle case ($\theta \ll 1$). For the small angle case our equation of motion turns:

$$\omega_0^2 = \omega_{0p}^2 + \frac{\phi_M^2}{\cos(\Theta) + 1} \implies \boxed{\ddot{\theta} + \omega_0^2\theta = 0}. \tag{6.1.18}$$

6.1.2 Hamiltonian analysis

Now, with our Lagrangian, we are going to use the definition of the Hamiltonian to find it for our system. With the Hamiltonian, we can generate a phase portrait to see the behavior of the system. Although we know that the system oscillates like a physical pendulum, because the equation of motion (6.1.15) is the same as that of a physical pendulum when the magnetic energy is null. The principal reason for finding the Hamiltonian of our system is to find the expression of the energy that is in the system.

Remembering, the Hamiltonian is defined through the Legendre transformation of the Lagrangian. For a system described by a generalized coordinate θ and its conjugate momentum p_θ, the Hamiltonian is defined as:

$$H = p_\theta\dot{\theta} - L. \tag{6.1.19}$$

The generalized momentum associated with the coordinate θ is obtained from the Lagrangian as:

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}}. \tag{6.1.20}$$

Using the Lagrangian obtained previously (6.1.11), the Hamiltonian of the system takes the following expression:

$$\boxed{H(\theta, p_\theta) = \frac{p_\theta^2}{2I_0} - mgR(1 - \cos(\theta)) - M\ln(r_R(\theta)) - M\ln(r_L(\theta))} \tag{6.1.21}$$

Since the Hamiltonian does not depend explicitly on time, it represents a conserved quantity of the system. Physically, this corresponds to the total mechanical energy of the pendulum, constituted by the kinetic energy and the potential energies associated with gravity and magnetic interaction.

The equations of motion of the system can now be obtained from Hamilton's equations:

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta}, \tag{6.1.22}$$

$$\dot{p}_\theta = -\frac{\partial H}{\partial \theta}. \tag{6.1.23}$$

Computing the derivatives, we obtain

$$\dot{\theta} = \frac{p_\theta}{I_0}, \tag{6.1.24}$$

and

$$\dot{p}_\theta = -\left[mgR + \frac{M}{\cos(\theta) - \cos(\Theta)}\right]\sin(\theta). \tag{6.1.25}$$

Now, we can generate the phase portrait diagram. It is important to see that the magnetic energy (M) and the maximum angle (Θ) change the phase portrait. To see the impact of M and Θ on the dynamic, we plot a set of diagrams with different values of M and Θ.

For the following diagrams, we want to see how the system reacts to different values of the magnetic energy M, and we use $m = 2\,kg$, $g = 10\,\frac{m}{s^2}$, $R = \frac{5}{8}\,m$, $I_0 = \frac{15}{8}\,kg\,m^2$, and $\Theta = 1.57$.

Now, we are going to see how the system reacts to different values for the maximum angle and a fixed value for the magnetic energy $M = 1$.

As we can see in both sets of diagrams, the system oscillates, but the region of oscillation changes when Θ and M change.

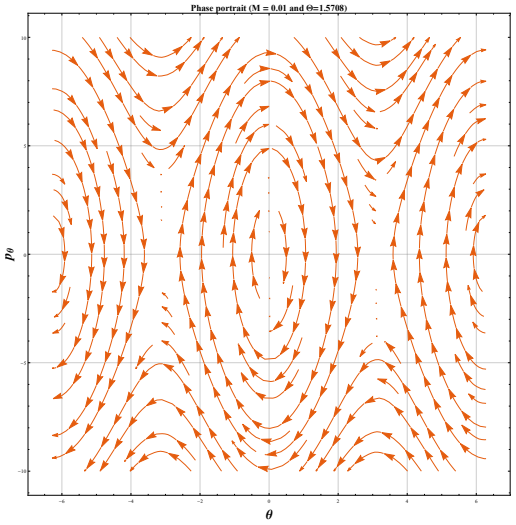
Another thing that we can find with Hamilton's equations is the equilibrium point. To do this, we need $\dot{p}_\theta = 0$, so the equilibrium is:

$$-\left[mgR + \frac{M}{\cos(\theta) - \cos(\Theta)}\right]\sin(\theta) = 0 \implies \begin{cases} \sin(\theta) = 0, & \theta = 0 \\ mgR + \frac{M}{\cos(\theta) - \cos(\Theta)} = 0, & \theta = \pm \arccos\left(\frac{-M - mgR\cos(\Theta)}{mgR}\right) \end{cases} \tag{6.1.26}$$

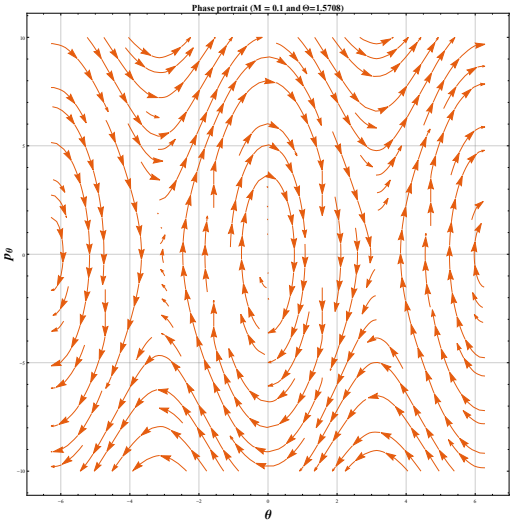
6.1.3 Dimensional analysis

Now, something important to analyze in our differential equation is that the angle does not have units, so the system is partially dimensionless; to make our differential equation dimensionless, we need to work with time.

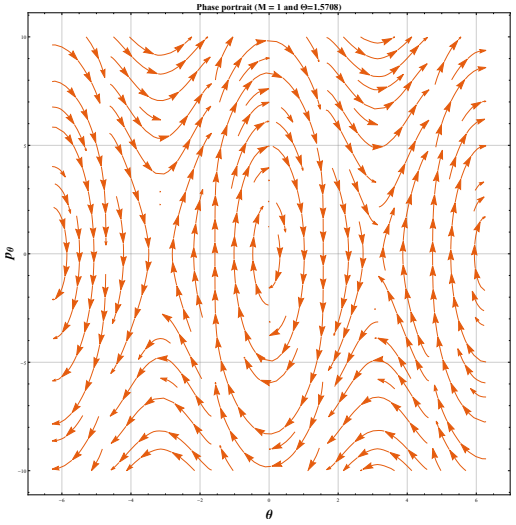
Firstly, we can obtain the dimensions of all the factors that appear in our differential equation (6.1.15). These factors, with their dimensions, are:



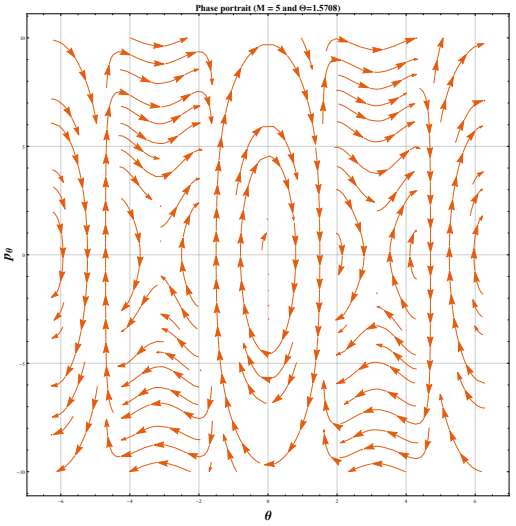
((a)) Phase portrait of the system with $M = 0.01$



((b)) Phase portrait of the system with $M = 0.1$



((c)) Phase portrait of the system with $M = 1$



((d)) Phase portrait of the system with $M = 5$

Figure 6.2: System reaction to different values of M.

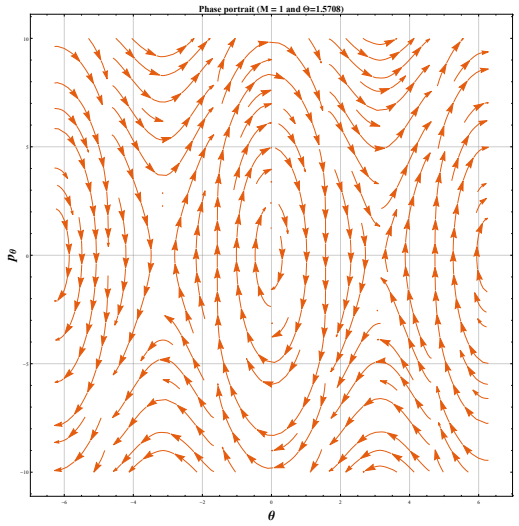
$$[m] = M, [R] = L, [g] = LT^{-2}, [t] = T, [I_0] = ML^2, [M] = ML^2T^{-2}. \tag{6.1.27}$$

It is important that Θ and θ are dimensionless.

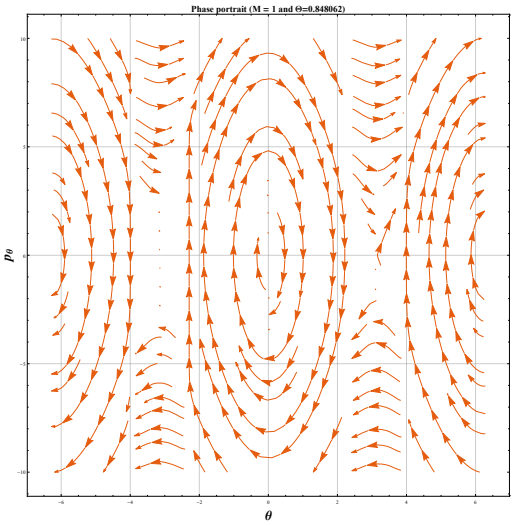
For this system, we are going to use R, g, and I_0 as dimensional variables, because R is the geometric parameter of the system, g is the cinematic parameter, and I_0 is the dynamic parameter. Now, expressing our parameters in terms of the dimensionless groups:

$$\Pi_i = \delta R^\alpha g^\beta I_0^\lambda, \tag{6.1.28}$$

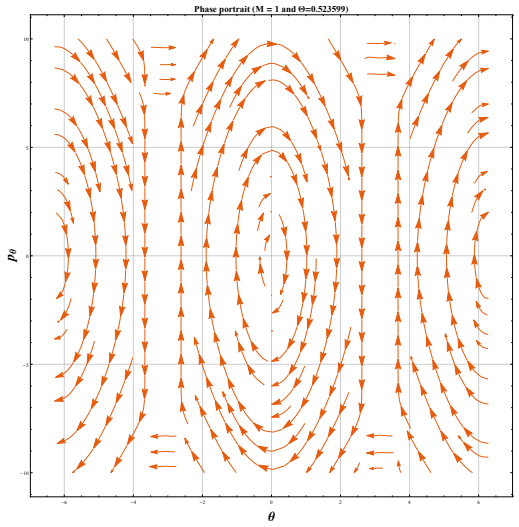
where δ is the variable that we want to be dimensionless, in our system we have the variables m, t, and M. And α, β, and λ are the powers that generate our dimensionless groups.



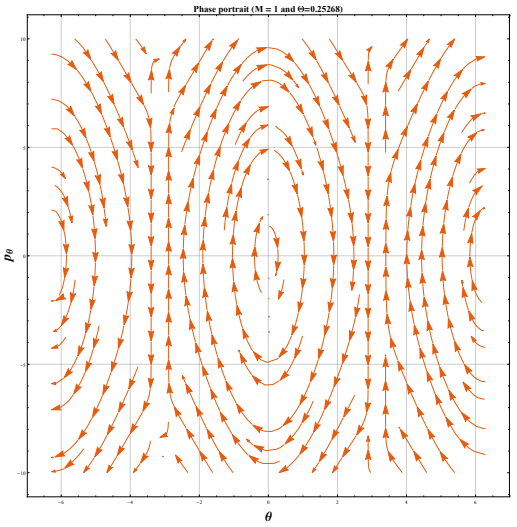
((a)) Phase portrait of the system with $\Theta = 1.57$



((b)) Phase portrait of the system with $\Theta = 0.84$



((c)) Phase portrait of the system with $\Theta = 0.52$



((d)) Phase portrait of the system with $\Theta = 0.25$

Figure 6.3: System reaction to different values of Θ.

To find the powers of our dimensional variables that dimensionless every group, we can construct an equation system. This system is.

$$\begin{pmatrix} L \\ M \\ T \end{pmatrix} + \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & -2 & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \lambda \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \implies \begin{pmatrix} \alpha \\ \beta \\ \lambda \end{pmatrix} = \begin{pmatrix} 2M - L - \frac{T}{2} \\ \frac{T}{2} \\ -M. \end{pmatrix} \tag{6.1.29}$$

It is important to remember that every dimensionless group can be expressed as a fraction. So, the dimensionless groups can be rewritten as:

$$\Pi_i = \frac{\delta}{\delta_s} = \tilde{\delta}, \tag{6.1.30}$$

where δ_s is the scale factor of the variable δ y $\tilde{\delta}$ is the dimensionless form of the variable δ. Now, we can obtain dimensionless groups for the variables m, t, and M.

Dimensionless group of the mass and the Inertia number

The m's group is:

$$\begin{pmatrix}\alpha\\ \beta\\ \lambda\end{pmatrix} = \begin{pmatrix}2\\ 0\\ -1\end{pmatrix} \quad \therefore \quad \boxed{\Pi_1 = \frac{m}{m_s} = \frac{mR^2}{I_0} \implies m_s = \frac{I_0}{R^2}} \quad . \tag{6.1.31}$$

With this we can define a dimensionless number that compares the inertia of the center of mass with the moment of inertia of the structure. This number is:

$$\chi \equiv \Pi_1 = \frac{mR^2}{I_0} \tag{6.1.32}$$

Dimensionless group of the time

The group of t is.

$$\begin{pmatrix}\alpha\\ \beta\\ \lambda\end{pmatrix} = \begin{pmatrix}-\frac{1}{2}\\ \frac{1}{2}\\ 0\end{pmatrix} \quad \therefore \quad \boxed{\Pi_2 = \frac{t}{t_s} = t\sqrt{\frac{g}{R}} \implies t_s = \sqrt{\frac{R}{g}}} \quad . \tag{6.1.33}$$

This result is important; the time scale is in terms of the distance between the pivot and the center of mass of the structure. With this, we can find the frequency scale.

$$\omega_s = \frac{1}{t_s} = \sqrt{\frac{g}{h}} \quad \wedge \quad T_s = \frac{2\pi}{\omega_s} = 2\pi\sqrt{\frac{h}{g}}. \tag{6.1.34}$$

These are the frequency and period of time of a pendulum. Now, we use the time scale to find the derivative of time.

$$t = t_s\tau \implies dt = t_s d\tau \implies dt^2 = t_s^2 d\tau^2 \tag{6.1.35}$$

Dimensionless group of the magnetic energy and the Energie's number

Now, the group of M:

$$\begin{pmatrix}\alpha\\ \beta\\ \lambda\end{pmatrix} = \begin{pmatrix}1\\ -1\\ -1\end{pmatrix} \quad \therefore \quad \boxed{\Pi_3 = \frac{M}{M_s} = \frac{MR}{I_0 g} \implies M_s = \frac{gI_0}{R}} \quad . \tag{6.1.36}$$

This dimensionless group relates the magnetic energy and the mechanical potential energy. Using this dimensionless group, we are going to define Energie's number as:

$$\Lambda \equiv \Pi_3 = \frac{MR}{gI_0}. \tag{6.1.37}$$

The Energie's number is going to be so useful because it relates all the variables that are not affected by time. An example of some quantity that is not affected by time is the equilibrium distance, so we can rewrite this number in terms of the Energie's number as:

6.1.4 Dimensionless system

Once we get the dimensionless groups, we can dimensionless our differential equation (6.1.15).

$$\begin{aligned}
\frac{d^2\theta}{dt^2} + \left[\frac{mgR}{I_0} + \frac{M}{I_0}\frac{1}{\cos(\theta) - \cos(\Theta)}\right]\sin(\theta) = 0 \\
\frac{d^2\theta}{d(t_s\tau)^2} + \left[\frac{mgR}{I_0} + \frac{M}{I_0}\frac{1}{\cos(\theta) - \cos(\Theta)}\right]\sin(\theta) = 0 \\
\frac{1}{t_s^2}\frac{d^2\theta}{d\tau^2} + \left[\frac{mgR}{I_0} + \frac{M}{I_0}\frac{1}{\cos(\theta) - \cos(\Theta)}\right]\sin(\theta) = 0 \\
\frac{d^2\theta}{d\tau^2} + \left[\frac{mgRt_s^2}{I_0} + \frac{Mt_s^2}{I_0}\frac{1}{\cos(\theta) - \cos(\Theta)}\right]\sin(\theta) = 0.
\end{aligned} \tag{6.1.38}$$

Now, using the time scale factor, and the deffinition of our dimensionless numbers:

$$\begin{aligned}
\frac{d^2\theta}{d\tau^2} + \left[\frac{mgRt_s^2}{I_0} + \frac{Mt_s^2}{I_0}\frac{1}{\cos(\theta) - \cos(\Theta)}\right]\sin(\theta) = 0 \\
\frac{d^2\theta}{d\tau^2} + \left[\frac{mgR}{I_0}\left(\frac{R}{g}\right) + \frac{M}{I_0}\left(\frac{R}{g}\right)\frac{1}{\cos(\theta) - \cos(\Theta)}\right]\sin(\theta) = 0 \\
\frac{d^2\theta}{d\tau^2} + \left[\frac{mgR^2}{gI_0} + \frac{MR}{gI_0}\frac{1}{\cos(\theta) - \cos(\Theta)}\right]\sin(\theta) = 0 \\
\therefore \boxed{\frac{d^2\theta}{d\tau^2} + \left[\chi + \frac{\Lambda}{\cos(\theta) - \cos(\Theta)}\right]\sin(\theta) = 0}\,.
\end{aligned} \tag{6.1.39}$$

Something important to note is that the dimensionless numbers χ and Λ are dimensionless frequencies of the system. This can be seen in the following equations:

$$\chi = \frac{mR^2}{I_0} \implies \chi = \frac{mgR}{I_0}\left(\frac{R}{g}\right) \therefore \chi = \omega_{0p}^2 t_s^2\,, \tag{6.1.40}$$

$$\Lambda = \frac{MR}{gI_0} \implies \Lambda = \frac{M}{I_0}\left(\frac{R}{g}\right) \therefore \Lambda = \phi_M^2 t_s^2\,. \tag{6.1.41}$$

Our differential equation is similar to a simple pendulum system. Then, just as that system, this system is analytically unsolvable. The unique case where our differential equation is analytically solvable is the small-angle system, but for a general case, we need to use a numerical method.

6.1.5 Small-angle approximation

In the case of the small-angle system ($\theta \ll 1$), using Taylor's series, we obtain:

$$\sin\theta \approx \theta \quad \text{and} \quad \cos(\theta) \approx 1. \tag{6.1.42}$$

Then our differential equation transfroms into the following system:

$$\ddot{\theta} + \left[\chi + \frac{\Lambda}{1-\cos(\Theta)}\right]\theta = 0. \tag{6.1.43}$$

Now, using the trigonometric identity of sine square, we obtain.

$$\ddot{\theta} + \left[\chi + \frac{\Lambda}{1-\cos(\Theta)}\right]\theta = \ddot{\theta} + \left[\chi + \frac{\Lambda}{2\sin^2\left(\frac{\Theta}{2}\right)}\right]\theta = 0. \tag{6.1.44}$$

As we know, the operative range of θ is defined by Θ, then we can conclude that for our system, the maximum angle is also less than one. This can be shown in the following equation:

$$\ddot{\theta} + \left[\chi + \frac{\Lambda}{2\left(\frac{\Theta}{2}\right)^2}\right]\theta = 0 \implies \ddot{\theta} + \left(\chi + \frac{2\Lambda}{\Theta^2}\right)\theta = 0. \tag{6.1.45}$$

Now we can find the dimensionless natural frequency of our system, which is:

$$\Omega_0^2 = \chi + \frac{2\Lambda}{\Theta^2} \implies \Omega_0 = \sqrt{\chi + \frac{2\Lambda}{\Theta^2}}. \tag{6.1.46}$$

It is important to see that if $\Lambda = 0$, then the system returns to being completely a physical pendulum. Now, we can find the solution for our small-angle approximation differential equation, that is:

$$\theta(\tau) = A\cos(\Omega_0\tau) + B\sin(\Omega_0\tau). \tag{6.1.47}$$

Using the initial conditions, which are: $\theta(0) = \Theta$ and $\dot{\theta}(0) = 0$, we obtain:

$$\theta(\tau) = \Theta\cos(\Omega_0\tau). \tag{6.1.48}$$

6.1.6 Stability

As we can see in the phase portrait diagrams, the system generates closed trajectories. So we can conclude that the system is stable. A more formal demonstration of the idea is that our Hamiltonian system has only one degree of freedom, and for the Poincaré-Bendixson theorem, chaos cannot occur in systems of one degree of freedom.

6.2 Damping system

The next step in the analysis of our system is adding the damping term. It is important because the damping system lets us know how the dissipative forces affect the behavior of the ideal system and adds some stability to the forced system.

If we remember, the damping force is the damping coefficient (β) times the speed of the system (v). Then, we can express it as:

$$F_d = -\beta v. \tag{6.2.1}$$

Now, we rewrite the speed in terms of the angle. Using the relation between the tangential velocity and the angular velocity, it is important to note that tangential velocity is applied over the center of mass of the system, we obtain.

$$F_d = -\beta v \implies F_d = -\beta R\dot{\theta}. \tag{6.2.2}$$

But our system works with torques. Then, dissipative torque of the system (τ_d) is:

$$\tau_d = F_d R \implies \tau_d = -\underbrace{\beta R^2}_{b}\dot{\theta} \therefore \tau_d = -b\dot{\theta}, \tag{6.2.3}$$

where b is a damping coefficient in terms of torques.

6.2.1 Lagrangian analysis

For this analysis, we are going to use the same Lagrangian that we find for the ideal case (6.1.11). It is important to remember that in this system, we are adding the damping term, and it is a non-conservative force. As we have seen previously, when the system has a non-conservative force, we can add it to our Euler-Lagrange equation:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \left(\frac{\partial L}{\partial \theta}\right) = Q_i, \tag{6.2.4}$$

where Q_i is the non-conservative force, that in our system is the damping torque ($\tau_d = -b\dot{\theta}$). Therefore, the equation of motion has the following form:

$$I_0\ddot{\theta} + b\dot{\theta} + \left[mgR + \frac{M}{\cos(\theta) - \cos(\Theta)}\right]\sin(\theta) = 0. \tag{6.2.5}$$

Dividing in terms of the moment of inertia (I_0).

$$\ddot{\theta} + \frac{b}{I_0}\dot{\theta} + \left[\frac{mgR}{I_0} + \frac{M/I_0}{\cos(\theta) - \cos(\Theta)}\right]\sin(\theta) = 0. \tag{6.2.6}$$

Now that we have our differential equation, we can do the Hamiltonian analysis to see the behavior of our system.

6.2.2 Hamiltonian analysis

The Hamiltonian of our system is the same as that of the ideal system (6.1.21). Something important to remember is that the Hamiltonian of the ideal system does not include the damping term, because it is a conservative Hamiltonian; but in Hamilton's equations, the damping term is included. Hamilton's equations are:

$$\begin{pmatrix}\dot{\theta}\\ \dot{p_\theta}\end{pmatrix} = \begin{pmatrix}\frac{\partial H}{\partial p_\theta}\\ -\frac{\partial H}{\partial \theta} - \frac{bp_\theta}{I_0}\end{pmatrix} = \begin{pmatrix}\frac{p_\theta}{I_0}\\ -\left[mgR + \frac{M}{\cos(\theta)-\cos(\Theta)}\right]\sin(\theta) - \frac{bp_\theta}{I_0}\end{pmatrix}. \tag{6.2.7}$$

With our dynamical system, we can plot the phase portrait of our system with the intention of seeing the behavior that the system follows.

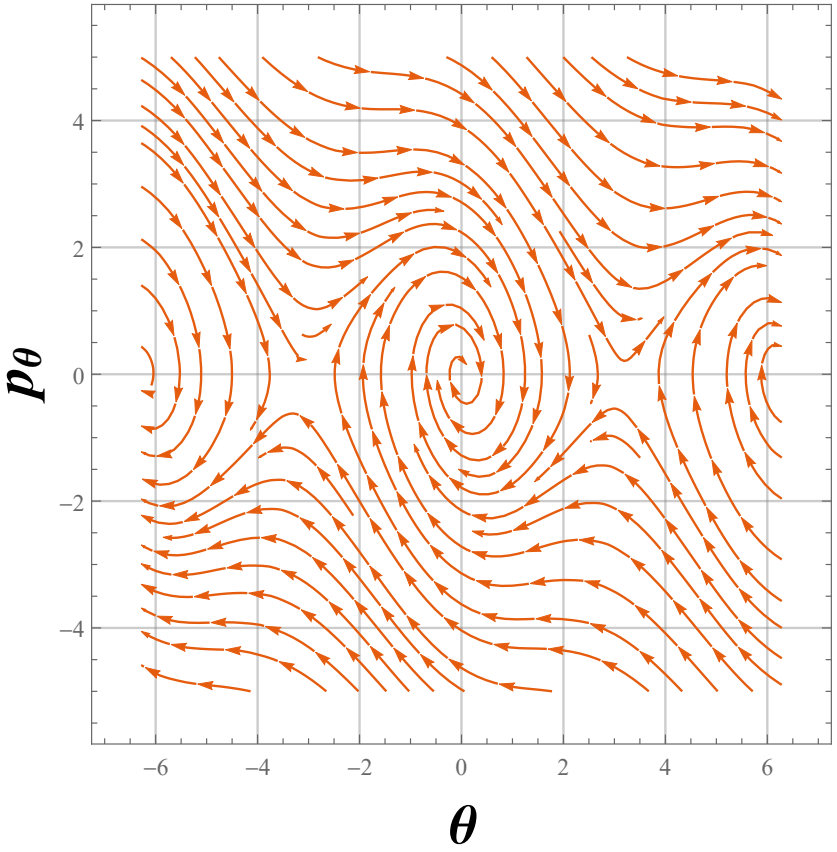


Figure 6.4: Phase portrait of the damiping system.

6.2.3 Dimensional analysis

The next step for us is to dimensionless our differential equation (6.2.6). To do this, we are going to continue using the same dimensionless variable as in the ideal case, but we are going to find the dimensionless group for the damping term

Dimensionless group of the damping coefficient and the Damping number

The b's group is:

$$\begin{pmatrix}\alpha\\ \beta\\ \lambda\end{pmatrix} = \begin{pmatrix}\frac{1}{2}\\ -\frac{1}{2}\\ -1\end{pmatrix} \quad \therefore \quad \boxed{\Pi_4 = \frac{b}{b_s} = \frac{b}{I_0}\sqrt{\frac{R}{g}} \implies b_s = I_0\sqrt{\frac{g}{R}} = \frac{I_0}{t_s}} \quad . \tag{6.2.8}$$

With this, we can define a dimensionless number that is the ratio of damping torque to inertial torque. This number is:

$$2\Phi \equiv \Pi_4 = \frac{b}{I_0}\sqrt{\frac{R}{g}} = \frac{bt_s}{I_0} \tag{6.2.9}$$

6.2.4 Dimensionless system

Now, applying our dimensionless groups, we obtain our dimensionless system.

$$
\begin{aligned}
\frac{d^2\theta}{dt^2} + \frac{b}{I_0}\frac{d\theta}{dt} + \left[\frac{mgR}{I_0} + \frac{M}{I_0}\frac{1}{\cos(\theta) - \cos(\Theta)}\right]\sin(\theta) = 0 \\
\frac{d^2\theta}{d(t_s\tau)^2} + \frac{b}{I_0}\frac{d\theta}{d(t_s\tau)} + \left[\frac{mgR}{I_0} + \frac{M}{I_0}\frac{1}{\cos(\theta) - \cos(\Theta)}\right]\sin(\theta) = 0 \\
\frac{1}{t_s^2}\frac{d^2\theta}{d\tau^2} + \frac{b}{I_0 t_s}\frac{d\theta}{d\tau} + \left[\frac{mgR}{I_0} + \frac{M}{I_0}\frac{1}{\cos(\theta) - \cos(\Theta)}\right]\sin(\theta) = 0 \\
\frac{d^2\theta}{d\tau^2} + \frac{bt_s}{I_0}\frac{d\theta}{d\tau} + \left[\frac{mgRt_s^2}{I_0} + \frac{Mt_s^2}{I_0}\frac{1}{\cos(\theta) - \cos(\Theta)}\right]\sin(\theta) = 0 \\
\boxed{\frac{d^2\theta}{d\tau^2} + 2\Phi\frac{d\theta}{d\tau} + \left[\chi + \frac{\Lambda}{\cos(\theta) - \cos(\Theta)}\right]\sin(\theta) = 0}.
\end{aligned}
\tag{6.2.10}
$$

This last dimensionless differential equation is, as in the ideal case (6.1.39), analytically unsolvable. And as in the ideal case, the unique case that we can solve analytically is the small-angle approximation.

6.2.5 Small-angle approximation

Remembering the case of the small-angle system ($\theta \ll 1$), using Taylor's series, we obtain:

$$
\sin\theta \approx \theta \quad \text{and} \quad \cos(\theta) \approx 1. \tag{6.2.11}
$$

Then our differential equation is now:

$$
\ddot{\theta} + 2\Phi\dot{\theta} + \left[\chi + \frac{\Lambda}{1 - \cos(\Theta)}\right]\theta = 0. \tag{6.2.12}
$$

As we know, the operative range of θ is defined by Θ, then we can conclude that for our system, the maximum angle is also less than one. This can be shown in the following equation:

$$
\ddot{\theta} + 2\Phi\dot{\theta} + \left[\chi + \frac{\Lambda}{2\sin^2\left(\frac{\Theta}{2}\right)}\right]\theta = 0 \implies \ddot{\theta} + 2\Phi\dot{\theta} + \left(\chi + \frac{2\Lambda}{\Theta^2}\right)\theta = 0. \tag{6.2.13}
$$

Using the definition of the natural frequency of the system:

$$
\ddot{\theta} + 2\Phi\dot{\theta} + \Omega_0^2\theta = 0. \tag{6.2.14}
$$

Now, finding the characteristic equation of our differential equation.

$$
s^2 + 2\Phi s + \Omega_0^2 = 0. \tag{6.2.15}
$$

The solutions of our characteristic equations are:

$$
s = -\Phi \pm \sqrt{\Phi^2 - \Omega_0^2} = -\Phi \pm i\sqrt{\Omega_0^2 - \Phi^2} = -\Phi \pm i\eth, \tag{6.2.16}
$$

where $\eth$ is the discriminant of our quadratic solution, and it is defined as $\eth = \sqrt{\Omega_0^2 - \Phi^2}$. With the solutions of our characteristic equation, the solution of our differential equation is:

$$\theta(\tau) = Ae^{s_1\tau} + Be^{s_2\tau} \implies \theta(\tau) = Ae^{(-\Phi+i\eth)\tau} + Be^{(-\Phi-i\eth)\tau}. \tag{6.2.17}$$

Doing some algebra and using Euler's complex identity:

$$\begin{aligned}
\theta(\tau) &= Ae^{(-\Phi+i\eth)\tau} + Be^{(-\Phi-i\eth)\tau} \\
&= e^{-\Phi\tau}\left(Ae^{i\eth\tau} + Be^{-i\eth\tau}\right) \\
&= e^{-\Phi\tau}\left(A\cos(\eth\tau) + iA\sin(\eth\tau) + B\cos(\eth\tau) - iB\sin(\eth\tau)\right) \\
&= e^{-\Phi\tau}\left[(A+B)\cos(\eth\tau) + i(A-B)\sin(\eth\tau)\right] \\
\therefore \boxed{\theta(\tau) = e^{-\Phi\tau}\left(C\cos(\eth\tau) + D\sin(\eth\tau)\right)}&.
\end{aligned} \tag{6.2.18}$$

Using the initial conditions, which are: $\theta(0) = \Theta$ and $\dot{\theta}(0) = 0$. Using the first initial condition, we obtain:

$$\begin{aligned}
\theta(0) = \Theta &= e^{-\Phi(0)}\left(C\cos(\eth(0)) + D\sin(\eth(0))\right) \\
&= 1(C\cos(0) + D\sin(0)) \\
\therefore \boxed{\Theta = C}&.
\end{aligned} \tag{6.2.19}$$

Obtaining the first derivative of θ.

$$\begin{aligned}
\dot{\theta}(\tau) &= -\Phi e^{-\Phi\tau}\left(\Theta\cos(\eth\tau) + D\sin(\eth\tau)\right) + e^{-\Phi\tau}\left(-\eth\Theta\sin(\eth\tau) + \eth D\cos(\eth\tau)\right) \\
&= e^{-\Phi\tau}\left[(\eth D - \Phi\Theta)\cos(\eth\tau) - (\Phi D + \eth\Theta)\sin(\eth\tau)\right].
\end{aligned} \tag{6.2.20}$$

Now, applying the second initial condition:

$$\begin{aligned}
\dot{\theta}(0) = 0 &= e^{-\Phi(0)}\left[(\eth D - \Phi\Theta)\cos(\eth(0)) - (\Phi D + \eth\Theta)\sin(\eth(0))\right] \\
&= 1\left[(\eth D - \Phi\Theta)\cos(0) - (\Phi D + \eth\Theta)\sin(0)\right] \\
0 = \eth D - \Phi\Theta &\implies \eth D = \Phi\Theta \\
\therefore \boxed{D = \frac{\Phi\Theta}{\eth}}&.
\end{aligned} \tag{6.2.21}$$

So, the solution of our differential equation is:

$$\theta(\tau) = \Theta e^{-\Phi\tau}\left[\cos(\eth\tau) + \frac{\Phi}{\eth}\sin(\eth\tau)\right] \tag{6.2.22}$$

6.3 Forced system

We now introduce an external force into our system. This force is applied over the structure and generates a torque in the system. For this reason, in our differential equation, the external force is expressed as a torque (T_0).

For our system, we are going to consider a periodic external force. Then, it generates a periodic external torque $T_0\cos(\omega t)$, where ω is the frequency of the external torque.

6.3.1 Lagrangian analysis

As with the damped system, the forced system has the same Lagrangian (6.1.11). To find the equation of motion, we apply the Euler-Lagrange equation, adding as the non-conservative force, the damping force, and the external torque. The generalize external force is:

$$Q_i = T_0 \cos(\omega t) - b\dot{\theta}. \tag{6.3.1}$$

Then, applying the Euler-Lagrange equation, our differential equation is:

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} &= Q_i \\ I_0\ddot{\theta} + \left[mgR + \frac{M}{\cos(\theta) - \cos(\Theta)}\right]\sin(\theta) &= T_0\cos(\omega t) - b\dot{\theta} \\ I_0\ddot{\theta} + b\dot{\theta} + \left[mgR + \frac{M}{\cos(\theta) - \cos(\Theta)}\right]\sin(\theta) &= T_0\cos(\omega t). \end{aligned} \tag{6.3.2}$$

For our differential equation, we can see that this system is non-autonomous, because there is a term in our differential equation that has a clear time dependence. For that reason, we cannot generate a phase portrait that describes the system at every time because the system changes in every instant of time. To know if our system generates a closed trajectory, we need to analyze the stability of the system, but before doing that, we are going to dimensionless our differential equation.

6.3.2 Dimensional analysis

To dimensionless our differential equation, we are going to continue using the same dimensionless variables and dimensionless groups that we used in the ideal and damped cases. For that reason, we only need to find the dimensionless groups of the external torque and the frequency.

Dimensionless group of the external torque and the torque number

The T_0's group is:

$$\begin{pmatrix}\alpha\\ \beta\\ \lambda\end{pmatrix} = \begin{pmatrix}1\\ -1\\ -1\end{pmatrix} \quad \therefore \quad \boxed{\Pi_5 = \frac{T_0}{T_s} = \frac{T_0 R}{gI_0} \implies T_s = \frac{gI_0}{R} = \frac{I_0}{t_s^2}} \quad . \tag{6.3.3}$$

With this, we can define a dimensionless number as the ratio of external torque to inertial torque. This number is:

$$\sigma \equiv \Pi_5 = \frac{T_0 R}{I_0 g} = \frac{T_0 t_s^2}{I_0} \tag{6.3.4}$$

Dimensionless group of frequency and the frequency number

The ω's group is:

$$\begin{pmatrix}\alpha\\ \beta\\ \lambda\end{pmatrix} = \begin{pmatrix}\frac{1}{2}\\ -\frac{1}{2}\\ 0\end{pmatrix} \quad \therefore \quad \boxed{\Pi_6 = \frac{\omega}{\omega_s} = \omega\sqrt{\frac{R}{g}} \implies \omega_s = \sqrt{\frac{g}{R}} = \frac{1}{t_s}} \quad . \tag{6.3.5}$$

With this, we can define a dimensionless number as the ratio of the external frequency and the frequency of a pendulum. This number is:

$$\Omega \equiv \Pi_6 = \omega\sqrt{\frac{R}{g}} = \omega t_s. \tag{6.3.6}$$

With

6.3.3 Dimensionless system

Now, applying our dimensionless groups, we obtain our dimensionless system.

$$\begin{aligned}
I_0\frac{d^2\theta}{dt^2} + b\frac{d\theta}{dt} + \left[mgR + \frac{M}{\cos(\theta)-\cos(\Theta)}\right]\sin(\theta) &= T_0\cos(\omega t)\\
\frac{d^2\theta}{dt^2} + \frac{b}{I_0}\frac{d\theta}{dt} + \left[\frac{mgR}{I_0} + \frac{M}{I_0}\frac{1}{\cos(\theta)-\cos(\Theta)}\right]\sin(\theta) &= \frac{T_0}{I_0}\cos(\omega t)\\
\frac{d^2\theta}{d(t_s\tau)^2} + \frac{b}{I_0}\frac{d\theta}{d(t_s\tau)} + \left[\frac{mgR}{I_0} + \frac{M}{I_0}\frac{1}{\cos(\theta)-\cos(\Theta)}\right]\sin(\theta) &= \frac{T_0}{I_0}\cos(\omega(t_s\tau))\\
\frac{1}{t_s^2}\frac{d^2\theta}{d\tau^2} + \frac{b}{I_0 t_s}\frac{d\theta}{d\tau} + \left[\frac{mgR}{I_0} + \frac{M}{I_0}\frac{1}{\cos(\theta)-\cos(\Theta)}\right]\sin(\theta) &= \frac{T_0}{I_0}\cos(\omega t_s\tau)\\
\frac{d^2\theta}{d\tau^2} + \frac{bt_s}{I_0}\frac{d\theta}{d\tau} + \left[\frac{mgRt_s^2}{I_0} + \frac{Mt_s^2}{I_0}\frac{1}{\cos(\theta)-\cos(\Theta)}\right]\sin(\theta) &= \frac{T_0t_s^2}{I_0}\cos(\omega t_s\tau)\\
\boxed{\frac{d^2\theta}{d\tau^2} + 2\Phi\frac{d\theta}{d\tau} + \left[\chi + \frac{\Lambda}{\cos(\theta)-\cos(\Theta)}\right]\sin(\theta) = \sigma\cos(\Omega\tau)}&.
\end{aligned} \tag{6.3.7}$$

6.3.4 Stability

Starting from the idea that the general solution is analytically unsolvable, we need to know if our system is stable or chaotic. To know this, we are going to analyze the stability of the system.

Linearizing our differential equation, we obtain the following set of first-order differential equations.

$$v(\theta_1,\theta_2,\theta_3) = \begin{pmatrix}\dot{\theta_1}\\ \dot{\theta_2}\\ \dot{\theta_3}\end{pmatrix} = \begin{pmatrix}\theta_2\\ \sigma\cos(\theta_3) - 2\Phi\theta_2 - \left[\chi + \frac{\Lambda}{\cos(\theta_1)-\cos(\Theta)}\right]\sin(\theta_1)\\ \Omega\end{pmatrix}, \tag{6.3.8}$$

where $\theta_3 = \Omega\tau$ is the angle induced by the forcing. With our dynamical system vector, we can build our Jacobian matrix. The Jacobian matrix is:

$$\mathbf{J}(\theta_1,\theta_2,\theta_3)=\begin{pmatrix}\frac{\partial v_1}{\partial\theta_1} & \frac{\partial v_1}{\partial\theta_2} & \frac{\partial v_1}{\partial\theta_3}\\ \frac{\partial v_2}{\partial\theta_1} & \frac{\partial v_2}{\partial\theta_2} & \frac{\partial v_2}{\partial\theta_3}\\ \frac{\partial v_3}{\partial\theta_1} & \frac{\partial v_3}{\partial\theta_2} & \frac{\partial v_3}{\partial\theta_3}\end{pmatrix}$$

$$\implies \mathbf{J}(\theta_1,\theta_2,\theta_3)=\begin{pmatrix}0 & 1 & 0\\ -\left(\left[\chi+\frac{\Lambda}{\cos(\theta_1)-\cos(\Theta)}\right]\cos(\theta_1)+\frac{\Lambda\sin^2(\theta_1)}{(\cos(\theta_1)-\cos(\Theta))^2}\right) & -2\Phi & -\sigma\sin(\theta_3)\\ 0 & 0 & 0\end{pmatrix}=\mathbf{J}(\tau). \tag{6.3.9}$$

Then, the trace of this Jacobian is:

$$\mathrm{tr}(\mathbf{J}(\tau))=-2\Phi. \tag{6.3.10}$$

The following step is to find the Monodromy matrix (, and it follows the next:

$$\dot{\mathbf{M}}(\tau)=\mathbf{J}(\tau)\mathbf{M}(\tau),\quad \mathbf{M}(0)=\mathbf{I}. \tag{6.3.11}$$

Now, we can use Liouville's formula:

$$\det\left(\mathbf{M}(\tau)\right)=\det\left(\mathbf{M}(t_0)\right)e^{\int_{t_0}^{T}\mathrm{tr}(\mathbf{J}(\tau))d\tau}. \tag{6.3.12}$$

Integrating from 0 to T, we obtain:

$$\det\left(\mathbf{M}(T)\right)=e^{-2\Phi T}. \tag{6.3.13}$$

With this result, we can find the Floquet multipliers, which are the eigenvalues of the Monodromy matrix ($\mathbf{M}(T)$). Using the definition of Floquet multipliers, we obtain.

$$\rho_1\cdot\rho_2\cdot\rho_3=\det\left(\mathbf{M}(T)\right)=e^{-2\Phi T}, \tag{6.3.14}$$

where ρ_1, ρ_2, and ρ_3 are the Floquet multipliers. It is important to remember that in an autonomous system in 3D that has a periodic orbit, one of the multipliers is always 1, then $\rho_3=1$. The Floque multipliers let us know if the trajectory that the system follows is stable or unstable. To know this, we have the following table:

Condition	**How many?**	**System behavior**
$\lvert\rho_i\rvert<1$	All except the one equal to 1	Stable limit cycle (perturbations decay)
$\lvert\rho_i\rvert=1$	At least one other besides the neutral 1	Marginal stability (bifurcation point)
$\lvert\rho_i\rvert>1$	At least one (excluding the neutral 1)	Unstable limit cycle (perturbations grow)

Table 6.1: Stability criteria based on Floquet multipliers ρ_i for an autonomous system with a periodic orbit. One multiplier is always 1 (neutral direction along the orbit). The table applies to the remaining multipliers.

For a damped system ($\Phi>0$), if a periodic orbit exists, the product of its Floquet multipliers is $e^{-2\Phi T}<1$. This is a necessary condition for asymptotic stability (all non-neutral multipliers inside the unit circle). To determine whether such a periodic orbit actually exists, we construct a Poincaré map.

To make our Poincaré map, we use the Runge-Kutta method on our dynamical system, and we record the values of $(\theta, \dot{\theta})$ at times that are multiples of the period T (i.e., a stroboscopic map). Doing this, we obtain the following Poincaré map.

Lyapunov exponents

While Floquet multipliers characterize the stability of a periodic orbit, many systems, in particular the forced nonlinear oscillators, can exhibit chaotic behavior where no periodic orbit exists. In such cases, a more general tool is needed: the ***Lyapunov exponents***. The behavior of the system in terms of the Lyapunov exponents follows the next table:

Condition	**How many?**	**System behavior**
$\lambda < 0$	All (except possibly one zero for a periodic orbit)	Stable dynamics
$\lambda = 0$	At least one (others < 0)	Marginal stability
$\lambda > 0$	At least one	Chaotic dynamics

Table 6.2: Interpretation of Lyapunov exponents λ_i (ordered $\lambda_1 \geq \lambda_2 \geq \cdots$) for an autonomous system. For a stable periodic orbit, one exponent is zero (direction along the orbit) and the remaining are negative. A positive exponent indicates chaos.

When a system is stable, there exists a relation between the Floquet multipliers and the Lyapunov exponents. That relation is:

$$\lambda_i = \frac{1}{T} \ln(\rho_i) \tag{6.3.15}$$

Now, using the Lyapunov exponents equation:

$$\lambda_1+\lambda_2+\lambda_3 = \frac{1}{T}\ln(\rho_1)+\frac{1}{T}\ln(\rho_2)+\frac{1}{T}\ln(\rho_3) = \frac{1}{T}\ln(\rho_1\cdot\rho_2\cdot\rho_3) = \frac{1}{T}\ln(\rho_1\cdot\rho_2) \tag{6.3.16}$$

Using the value of $\rho_3 = 1 \implies \lambda_3 = 0$, and our Floquet product (6.3.14)

$$\lambda_1 + \lambda_2 = \frac{1}{T}\ln\left(e^{-2\Phi T}\right) \implies \lambda_1 + \lambda_2 = -2\Phi \tag{6.3.17}$$

Using the condition of stability:

$$\lambda_1 + \lambda_2 = -2\Phi < 0 \implies \Phi > 0, \tag{6.3.18}$$

this result implies that if the friction number (Φ) increases, the stability increases too. But the amplitude of the oscillation decreases. Another important thing that we can find is the values of Lyapunov exponents that give us stable system

$$\lambda_1 + \lambda_2 = -2\Phi \implies \lambda_1 = -(2\Phi + \lambda_2), \tag{6.3.19}$$

we know that a stable system needs $\lambda_i < 0$, then:

$$\lambda_1 < 0 \implies -(2\Phi + \lambda_2) < 0 \implies 2\Phi + \lambda_2 > 0 \implies \lambda_2 > -2\Phi, \tag{6.3.20}$$

but both Lyapunov exponents need to be less than one; therefore:

$$-2\Phi < \lambda_2 < 0. \tag{6.3.21}$$

Now, we can find the limit for the first Lyapunov exponent domain for a stable system:

$$\begin{aligned} \lambda_1 &= -(2\Phi + 0) = -2\Phi \\ \lambda_1 &= -(2\Phi - 2\Phi) = 0 \\ \therefore &-2\Phi < \lambda_1 < 0. \end{aligned} \tag{6.3.22}$$

We can see that both Lyapunov exponents have the same domain if the system is stable, but it is important to highlight that they can have different values; the only thing that they need to follow is that the sum of both is -2Φ as is presented in the stability condition equation 6.3.18.

To see how the system responds to the external moment, we present a Lyapunov exponent graph. We are going to study the stability of the system with low friction. For this analysis, we are going to use $\Phi = 1$. Then, the Lyapunov diagram is.

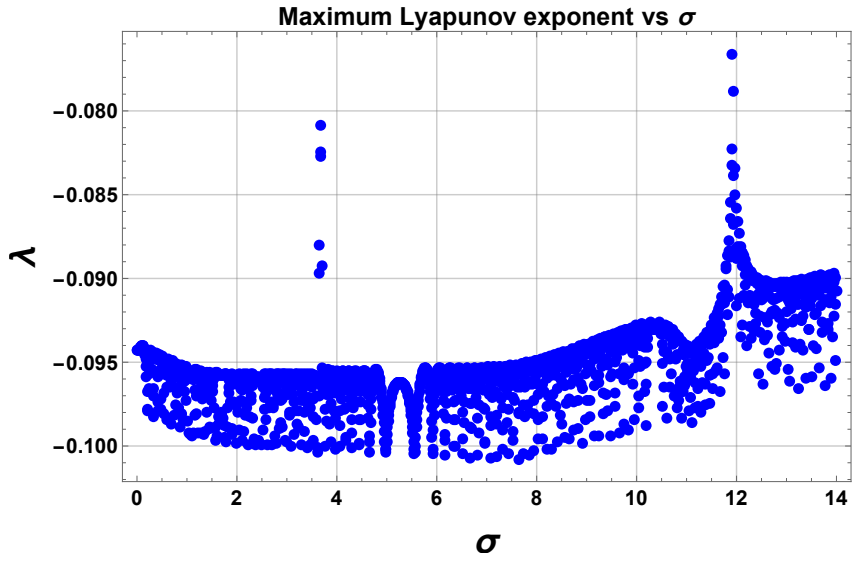


Figure 6.5: Lyapunov diagram of the pendulum system in terms of the external moment σ with a friction number of $\Phi = 0.1$.

For the previous figure, we can see that our system is very stable. The previous result indicates that chaos in the system is completely related to the force (σ) and frequency (Ω) parameters.

Now, we are going to present some bifurcation diagrams. For the first diagram, we use a $\Phi = 0.1$, and we obtain.

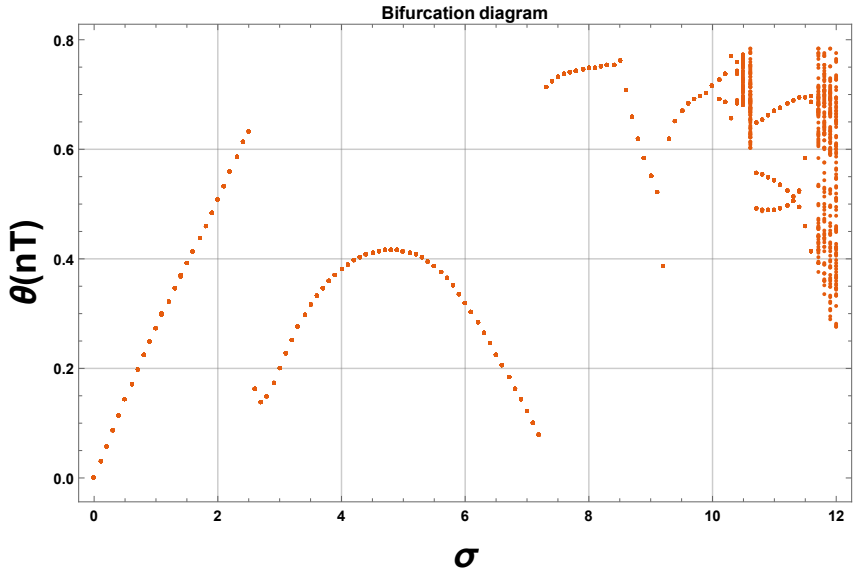


Figure 6.6: Bifurcation diagram for different values of the force with $\Lambda = 1$, $\Phi = 0.1$, $\Theta = \frac{\pi}{4}$ and $\Omega = 1$

Now, we are going to increase the damping number to see how the system reacts to an increase in the damping. We are going to go from 0.1 to 0.25.

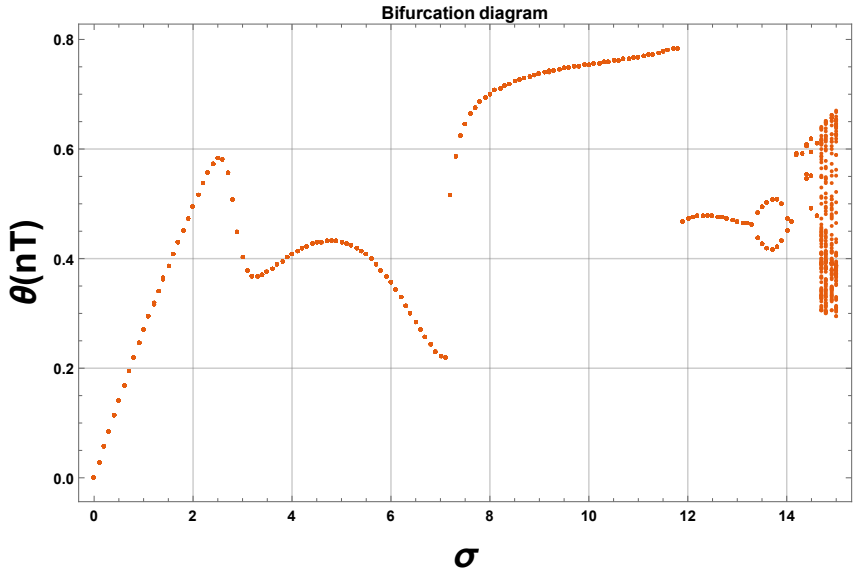


Figure 6.7: Bifurcation diagram for different values of the force with $\Lambda = 1$, $\Phi = 0.1$, $\Theta = \frac{\pi}{4}$ and $\Omega = 1$

As we can see in the previous figure, when the damping increases, the necessary force to generate chaos also increases. Now, we are going to check when the damping is $\Phi = 0.5$

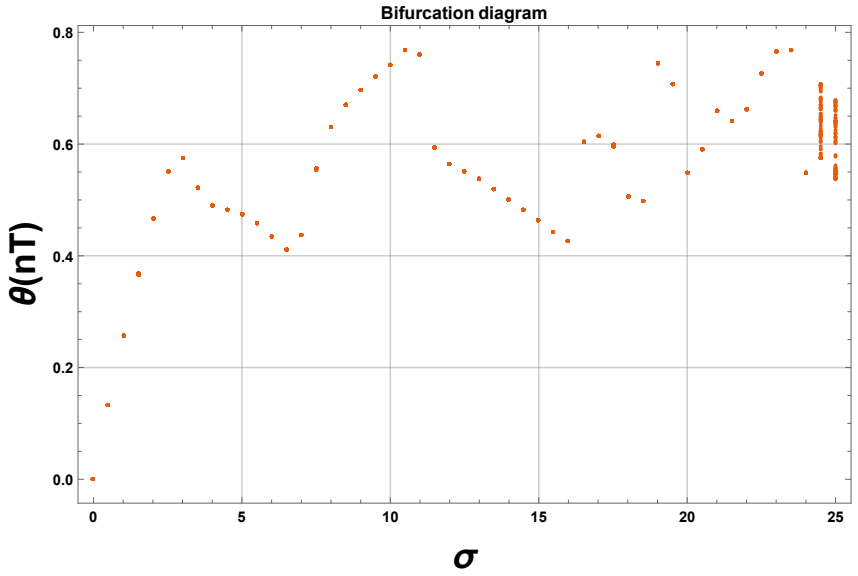


Figure 6.8: Bifurcation diagram for different values of the force with $\Lambda = 1$, $\Phi = 0.1$, $\Theta = \frac{\pi}{4}$ and $\Omega = 1$

From all the previous bifurcation diagrams, we can see that stability needs the damping term. And the increases in the damping term increase the stability of the system.

6.3.5 Numerical

Now, we know that our dynamical system is unsolvable analytically, except for the small-angle approximation. So, to solve a general system, we need to use a numerical method to generate a numerical solution. We use the Runge-Kutta method using the NDSolve function.

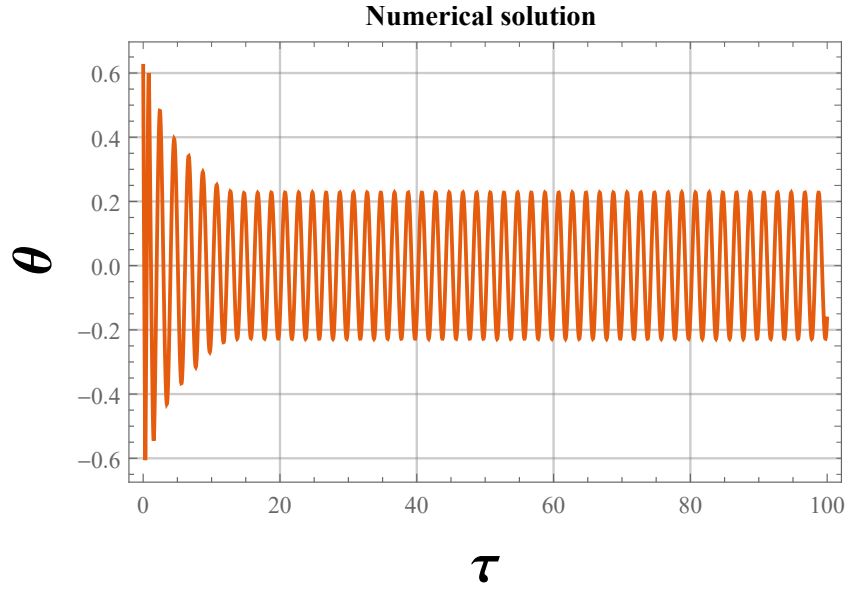


Figure 6.9: Numerical solution of the system with $\Lambda = 1$, $\Phi = 0.5$, $\Theta = \frac{\pi}{5}$, σ=1 and $\Omega = \pi$

With the solution, we are able to find how the angle changes in terms of the dimensionless numbers, for the Energie's number, we obtain.

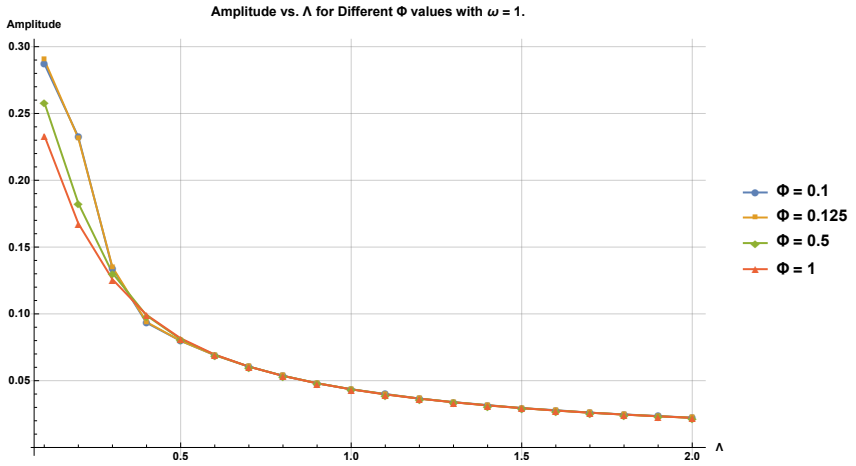


Figure 6.10: Bifurcation diagram for different values of the force with $\Lambda = 1$, $\Phi = 0.1$, $\Theta = \frac{\pi}{4}$ and $\Omega = 1$

6.3.6 Small-angle approximation

As we did in the previous sections, we are going to analyze our system under the small-angle approximation. Using Taylor's series, we obtain:

$$\sin\theta \approx \theta \quad \text{and} \quad \cos(\theta) \approx 1. \tag{6.3.23}$$

Then our differential equation transforms into the following system:

$$\ddot{\theta} + 2\Phi\dot{\theta} + \left[\chi + \frac{\Lambda}{1-\cos(\Theta)}\right]\theta = \sigma\cos(\Omega\tau). \tag{6.3.24}$$

Using the sine square identity and knowing that the operative range of θ is defined by Θ, then we can conclude that for our system, the maximum angle is also less than one. This can be shown in the following equation:

$$\ddot{\theta} + 2\Phi\dot{\theta} + \left[\chi + \frac{\Lambda}{2\sin^2\left(\frac{\Theta}{2}\right)}\right]\theta = \sigma\cos(\Omega\tau) \implies \ddot{\theta} + 2\Phi\dot{\theta} + \left(\chi + \frac{2\Lambda}{\Theta^2}\right)\theta = \sigma\cos(\Omega\tau). \tag{6.3.25}$$

With the previous result, we know the dimensionless natural frequency of our system, which is:

$$\Omega_0^2 = \chi + \frac{2\Lambda}{\Theta^2} \implies \Omega_0 = \sqrt{\chi + \frac{2\Lambda}{\Theta^2}}. \tag{6.3.26}$$

Using the natural frequency, our system for the small-angle approximation is:

$$\ddot{\theta} + 2\Phi\dot{\theta} + \Omega_0^2\theta = \sigma\cos(\Omega\tau). \tag{6.3.27}$$

With our differential system in its simplest form, we can solve it. To solve the system, we need to solve the homogeneous term and the particular term.

To solve the homogeneous term, we need the characteristic equation of the system.

$$s^2 + 2\Phi s + \Omega_0^2 = 0. \tag{6.3.28}$$

The solutions of our characteristic equations are:

$$s = -\Phi \pm \sqrt{\Phi^2 - \Omega_0^2} = -\Phi \pm i\sqrt{\Omega_0^2 - \Phi^2} = -\Phi \pm i\eth, \tag{6.3.29}$$

where $\eth$ is the discriminant of our quadratic solution, and it is defined as $\eth = \sqrt{\Omega_0^2 - \Phi^2}$. With the solutions of our characteristic equation, the solution of our differential equation is:

$$\theta_h(\tau) = Ae^{s_1\tau} + Be^{s_2\tau} \implies \theta_h(\tau) = Ae^{(-\Phi+i\eth)\tau} + Be^{(-\Phi-i\eth)\tau}. \tag{6.3.30}$$

Doing some algebra and using Euler's complex identity:

$$\begin{aligned}
\theta_h(\tau) &= Ae^{(-\Phi+i\eth)\tau} + Be^{(-\Phi-i\eth)\tau} \\
&= e^{-\Phi\tau}\left(Ae^{i\eth\tau} + Be^{-i\eth\tau}\right) \\
&= e^{-\Phi\tau}\left(A\cos(\eth\tau) + iA\sin(\eth\tau) + B\cos(\eth\tau) - iB\sin(\eth\tau)\right) \\
&= e^{-\Phi\tau}\left[(A+B)\cos(\eth\tau) + i(A-B)\sin(\eth\tau)\right] \\
\therefore \boxed{\theta_h(\tau)} &= \boxed{e^{-\Phi\tau}\left(C\cos(\eth\tau) + D\sin(\eth\tau)\right)}.
\end{aligned} \tag{6.3.31}$$

Now, solving the particular term of the system, we obtain:

$$\theta_p(\tau) = Y\cos(\Omega\tau) + X\sin(\Omega\tau), \tag{6.3.32}$$

with

$$Y = \frac{\sigma(\Omega_0^2 - \Omega^2)}{(\Omega_0^2 - \Omega^2)^2 + (2\Phi\Omega)^2}, \qquad X = \frac{\sigma(2\Phi\Omega)}{(\Omega_0^2 - \Omega^2)^2 + (2\Phi\Omega)^2}. \tag{6.3.33}$$

Equivalently,

$$\theta_p(\tau) = A\cos(\Omega\tau - \delta), \quad A = \frac{\sigma}{\sqrt{(\Omega_0^2 - \Omega^2)^2 + (2\Phi\Omega)^2}}, \quad \tan\delta = \frac{2\Phi\Omega}{\Omega_0^2 - \Omega^2},\ \delta \in [0, \pi]. \tag{6.3.34}$$

The complete solution of the system is:

$$\theta(\tau) = \theta_h(\tau) + \theta_p(\tau) \tag{6.3.35}$$

Using the initial conditions:

$$\theta(0) = \Theta = C_1 + Y \quad \Rightarrow \quad C_1 = \Theta - Y, \tag{6.3.36}$$

$$\dot{\theta}(0) = 0 = -\Phi C_1 + \eth C_2 + X\Omega \quad \Rightarrow \quad C_2 = \frac{\Phi C_1 - X\Omega}{\eth}. \tag{6.3.37}$$

Then our system is:

$$\boxed{\theta(\tau) = e^{-\Phi\tau}\left[(\Theta - M)\cos(\eth\tau) + \frac{\Phi(\Theta - M) - N\Omega}{\eth}\sin(\eth\tau)\right] + M\cos(\Omega\tau) + N\sin(\Omega\tau)}. \tag{6.3.38}$$

Now that we know the behavior of the system in small-angle approximation, we can find the behavior of the system when the external force is the force of a fluid.

6.3.7 Fluid force

As we mentioned in the fluid chapter, there are a lot of dimensionless numbers that help us to describe the behavior of the fluid, but for the VIV nature, the most important number is the Strouhal number. Because it describes the vortex shedding behind a body.

As we mentioned, the Gravity-inertial number previously compares the inertial forces to the gravitational forces in a fluid, more specifically, because we have a pendulum.

Now, we are going to rewrite our differential equation using the fluid expressions. A lift force of a fluid, as we have studied previously, has the following form.

$$F_f = \frac{1}{2}\rho U^2 S C_L, \tag{6.3.39}$$

where C_L is the lift coefficient, S is the surface area of the pendulum that interacts with the fluid, and ρ is the density of our fluid. Now, using this force to find the torque of a fluid force, we have.

$$T_f = R \times F_f = \frac{1}{2}\rho U^2 S R C_L, \tag{6.3.40}$$

where R is the distance between the center of mass and the pivot. Using the dimensionless group of the torque and the dimensionless length scale number D, we obtain:

$$\begin{aligned} \sigma = \frac{T_f t_s^2}{I_0} \implies \sigma = \frac{\rho U^2 S C_L R t_s^2}{2I_0} \\ \sigma = \frac{\rho U^2 S R^2 C_L}{2I_0 g} \implies \sigma = \frac{C_L}{2}\left(\frac{U^2}{gD}\right)\left(\frac{\rho S R^2 D}{I_0}\right) \\ \sigma = \frac{C_L \Gamma^2}{2}\left(\frac{\rho S D \chi}{m}\right) \therefore \sigma = \frac{\chi C_L \Gamma^2}{2m^*}. \end{aligned} \tag{6.3.41}$$

We define m^* as the mass ratio between the mass of the structure and the mass of the fluid, that is, $m^* = \left(\frac{m}{\rho D S}\right)$. Something that is relevant to highlight is that the lift coefficient (C_L) could be a function of time, which would depend on the geometry, but in this specific example, we are going to work with a constant lift coefficient.

$$\frac{d^2\theta}{d\tau^2} + 2\Phi\frac{d\theta}{d\tau} + \left[\chi + \frac{\Lambda}{\cos(\theta) - \cos(\Theta)}\right]\sin(\theta) = \frac{\chi C_L \Gamma^2}{2m^*}\cos(\Omega\tau), \tag{6.3.42}$$

as in the general case of a forced system, the amplitude of our oscillation will depend on the Inertia number (χ) and the frequency of the vortices, which in this case is the Strouhal number (St).

Another thing that we need to consider when we are working with a fluid force is the geometry of our structure, cause its shape will define the lift coefficient and the Strouhal number (St). The Stouhal number definition is: $St = \frac{f_s D}{U}$. Then we can find a relation between the Strouhal number and the frequency number. This relation is:

$$\begin{aligned} \omega = 2\pi f_s \implies \frac{\Omega}{t_s} &= 2\pi \left(\frac{StU}{D}\right) \\ \implies \Omega = \frac{2\pi StU t_s}{D} \implies \Omega &= \frac{2\pi StU}{D}\sqrt{\frac{R}{g}} \\ \implies \Omega = \frac{2\pi StU}{\sqrt{gD}}\sqrt{\frac{R}{D}} \implies &\boxed{\Omega = 2\pi St\Gamma\sqrt{\frac{R}{D}}}. \end{aligned} \tag{6.3.43}$$

Then, the frequency of the system is related to the Strouhal and the Gravity-inertial number. If the characteristic length is equal to the distance between the center of mass and the pivot ($D = R$), the frequency number can be simplified as:

$$\Omega = 2\pi St\Gamma. \tag{6.3.44}$$

Then, we can rewrite our dimensionless differential equation as:

$$\frac{d^2\theta}{d\tau^2} + 2\Phi\frac{d\theta}{d\tau} + \left[\chi + \frac{\Lambda}{\cos(\theta) - \cos(\Theta)}\right]\sin(\theta) = \frac{\chi C_L \Gamma^2}{2m^*}\cos(2\pi St\Gamma\tau), \tag{6.3.45}$$

It is important to see that our system now depends on the values of the Strouhal (St), Gravity-inertia (Γ), Energie's (Λ), Inertia (χ), and the friction number (Φ). In this system, the most important numbers are the Strouhal and Gravity-inertial numbers because they define the frequency of the oscillation. The principal problem is that we have a frequency that is a product of two variables. But we can rewrite the system with a new dimensionless time definition to express the frequency of the external force in terms of only one parameter, the Strouhal number.

New dimensionless equation

If we pay attention to the Strouhal number-frequency relation (equation (6.3.43)), we find that there is another possible dimensionless form that could help us to let the dimensionless frequency be only in terms of the Strouhal numbers. This new dimensionless uses fluid and structure properties. The new relation is:

$$\Omega = \frac{2\pi StU t_s}{D} \implies \Omega = 2\pi St \implies \frac{Ut_s}{D} = 1 \quad \therefore \quad \boxed{t_s = \frac{D}{U} = \frac{R}{U}}, \tag{6.3.46}$$

this new time scale lets us rewrite our differential equation. To rewrite it, we are going to use the dimensionless groups that we obtained previously. Starting the new dimensionless analysis in the equation of motion (4.3.5).

Applying the dimensionless groups to our differential equation, we can find the equation of motion

$$\begin{aligned}
I_0\ddot{\theta} + b\dot{\theta} + \left[mgR + \frac{M}{\cos(\theta) - \cos(\Theta)}\right]\sin(\theta) &= T_0\cos(\omega t) \\
\implies \frac{1}{t_s^2}\frac{d^2\theta}{d\tau^2} + \frac{b}{I_0 t_s}\frac{d\theta}{d\tau} + \left[\frac{mgR}{I_0} + \frac{M}{I_0}\frac{1}{\cos(\theta) - \cos(\Theta)}\right]\sin(\theta) &= \frac{T_0}{I_0}\cos(\Omega\tau) \\
\implies \frac{d^2\theta}{d\tau^2} + \frac{bt_s}{I_0}\frac{d\theta}{d\tau} + \left[\frac{mgRt_s^2}{I_0} + \frac{/}{Mt_s^2}I_0\frac{1}{\cos(\theta) - \cos(\Theta)}\right]\sin(\theta) &= \frac{T_0 t_s^2}{I_0}\cos(\Omega\tau) \\
\implies \frac{d^2\theta}{d\tau^2} + \frac{bR}{I_0 U}\frac{d\theta}{d\tau} + \left[\frac{mgRR^2}{I_0U^2} + \frac{MR^2}{I_0U^2}\frac{1}{\cos(\theta) - \cos(\Theta)}\right]\sin(\theta) &= \frac{T_0R^2}{I_0U^2}\cos(\Omega\tau) \\
\implies \frac{d^2\theta}{d\tau^2} + 2\Phi\frac{\sqrt{gR}}{U}\frac{d\theta}{d\tau} + \left[\frac{\chi gR}{U^2} + \frac{\Lambda gR}{U^2}\frac{1}{\cos(\theta) - \cos(\Theta)}\right]\sin(\theta) &= \frac{\sigma gR}{U^2}\cos(\Omega\tau)
\end{aligned} \tag{6.3.47}$$

Using the definition of the Gravity-inertia number ($\Gamma = \frac{U}{\sqrt{gR}}$), we simplify our differential equation as:

$$\begin{aligned}
\frac{d^2\theta}{d\tau^2} + 2\Phi\frac{\sqrt{gR}}{U}\frac{d\theta}{d\tau} + \left[\frac{\chi gR}{U^2} + \frac{\Lambda gR}{U^2}\frac{1}{\cos(\theta) - \cos(\Theta)}\right]\sin(\theta) &= \frac{\sigma gR}{U^2}\cos(\Omega\tau) \\
\implies \frac{d^2\theta}{d\tau^2} + \frac{2\Phi}{\Gamma}\frac{d\theta}{d\tau} + \left[\frac{\chi}{\Gamma^2} + \frac{\Lambda}{\Gamma^2}\frac{1}{\cos(\theta) - \cos(\Theta)}\right]\sin(\theta) &= \frac{\sigma}{\Gamma^2}\cos(\Omega\tau) \\
\implies \frac{d^2\theta}{d\tau^2} + 2\Phi\Gamma\frac{d\theta}{d\tau} + \left[\chi + \frac{\Lambda}{\cos(\theta) - \cos(\Theta)}\right]\sin(\theta) &= \sigma\cos(\Omega\tau)
\end{aligned} \tag{6.3.48}$$

Now, using the fluid form of the dimensionless torque, our differential equation is:

$$\frac{d^2\theta}{d\tau^2} + 2\Phi\Gamma\frac{d\theta}{d\tau} + \left[\chi + \frac{\Lambda}{\cos(\theta) - \cos(\Theta)}\right]\sin(\theta) = \frac{\chi C_L\Gamma^2}{2m^*}\cos(\Omega\tau). \tag{6.3.49}$$

6.3.8 Harnessing energy model

Once we know the way the motion of our system works, we can find an expression to use the motion to generate electricity. To generate it, we are going to use the induction Faraday's law, that is.

$$\varepsilon = -n\frac{d\Phi_B}{dt}, \tag{6.3.50}$$

where ε is the electromotive force induced for the motion of the magnetic flux, n is the number of turns of a coil, and Φ_B is the magnetic flux. We know that the magnetic flux is defined as.

$$\Phi_B = BS\cos(\alpha), \tag{6.3.51}$$

where B is the intensity of the magnetic field, S is the surface area of the coil, and α is the angle between the magnetic field and the normal vector of the surface area of the coil. Now, using the definition of the magnetic flux in Faraday's law.

$$\varepsilon = -n\frac{d}{dt}\left(BS\cos\left(\alpha\right)\right) \implies \varepsilon = nBS\sin\left(\alpha\right)\dot{\alpha}. \tag{6.3.52}$$

Now, since we have the expression for the electromotive force (ε), we need to link it to the motion of the pendulum. To do that, we need to find a relation between α, which is the angle between the magnetic field and the surface normal vector of the coil, and θ, which is the angle of the pendulum.

If we assume that the coil is placed in some way so that its center is aligned with the origin, then it has a fixed angle ν that tells us it position. Then, the angle α is the difference between the pendulum position angle (θ) and the coil position angle (ν).

Then, we can obtain the relation between α and θ and their derivatives.

$$\alpha = \theta - \nu \implies \dot{\alpha} = \dot{\theta}. \tag{6.3.53}$$

Therefore, the induced voltage in the coils is.

$$\varepsilon = nBS\sin\left(\theta - \nu\right)\dot{\theta}. \tag{6.3.54}$$

This last expression is important because we can see the voltage in each coil. However, we know that induction is only generated when the magnet passes over the coil, so the magnet induces a current only in the arc length that the coil uses. To express that we use the Heaviside step function. If each coil has the same width, and that width is w, and its radius is the height of the structure (h). Then, we find that the angle (ϑ) of the length of the arc of each coil is.

$$\vartheta = \frac{w}{h}. \tag{6.3.55}$$

Then, the operative angle of the induction in each coil is:

$$[\nu_i - \frac{\vartheta}{2}, \nu_i + \frac{\vartheta}{2}]. \tag{6.3.56}$$

But, each coil has its unique position angle ν_i. So, it is necessary to find a recursive expression that helps us to express the position of each coil. To do that, we can allocate the space based on the number of coils we want to use. If we distribute the coils geometrically, we obtain that the position angle of each coil is

$$\nu_i = \frac{\Theta(2i - N - 1)}{N + 1}, \quad i = 1, 2, \ldots, N. \tag{6.3.57}$$

where N is the number of coils that we are going to use in our system, and i is the i-th coil that we study. Then, the voltage of each coil can be expressed as.

$$\varepsilon_i = nBS\sin\left(\theta - \frac{\Theta(2i - N - 1)}{N + 1}\right)\dot{\theta}\left[H\left(\theta - \frac{\Theta(2i - N - 1)}{N + 1} + \frac{w}{2h}\right) - H\left(\theta - \frac{\Theta(2i - N - 1)}{N + 1} - \frac{w}{2h}\right)\right]. \tag{6.3.58}$$

or in a simple form.

$$\varepsilon_i = nBS\sin\left(\theta - \nu_i\right)\left(H\left(\theta - \nu_i + \frac{\vartheta}{2}\right) - H\left(\theta - \nu_i - \frac{\vartheta}{2}\right)\right)\dot{\theta}. \tag{6.3.59}$$

Now, to find the voltage generated by the system we sum the voltage generated by each coil. The total voltage is.

$$\varepsilon = \sum_{i=1}^{N} \varepsilon_i. \tag{6.3.60}$$

With the previous information, we can generate a model to see the total voltage generated by our system. We are going to generate a numerical model. The system that we are going to model is a pendulum that has $\Lambda = 1$, $\chi = 1$, $\Phi = 0.5$, $\sigma = 2$, $\Theta = \frac{\pi}{4}$; the induction systema has the following properties, each coil has ten turns ($n = 10$), each coil has an area of one ($S = 1m^2$), the magnet on the pendulum has a magnetic field of one tesla ($B = 1T$), the width of each coils is one meter ($w = 1m$), and the height of the structure is the ten meters ($h = 10m$), and we have five coils ($N = 5$). This generate the following figure.

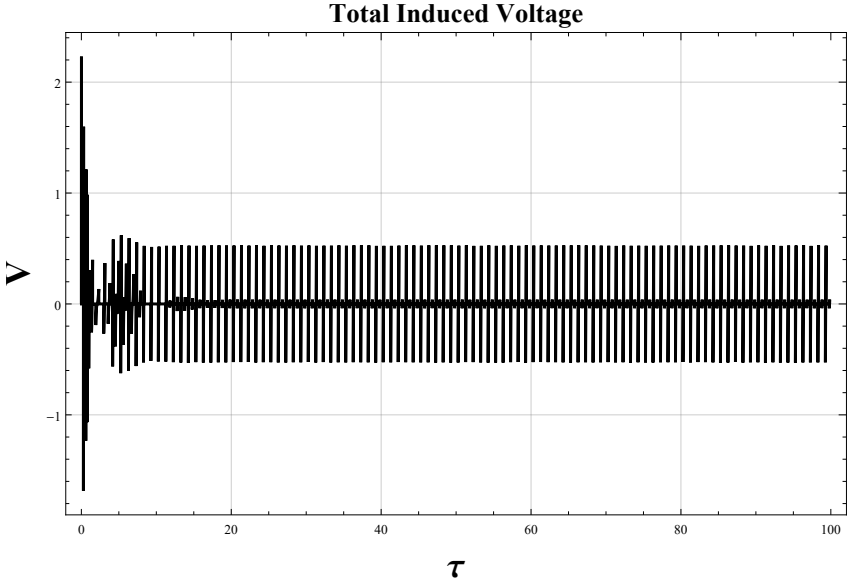


Figure 6.11: Numerical solution of the total voltage in terms of time with $\Lambda = 1$, $\chi = 1$, $\Phi = 0.5$, $\sigma = 2$, $\Theta = \frac{\pi}{4}$, $n = 10$, $S = 1m^2$, $B = 1T$, $w = 1m$, $h = 10m$, and $N = 5$

As we have seen previously, the solution of our dynamical system is extremely dependent on the damping number Φ and the torque number σ. So, we are going to see how the voltage reacts to different values of the damping number Φ. For the following figure that lets us see how the voltage responds to the damping number, we are going to use the same system as the previous figure, with a change in the torque number. For this simulation, we are going to use $\sigma = 1$. The voltage of the system with different values of damping is.

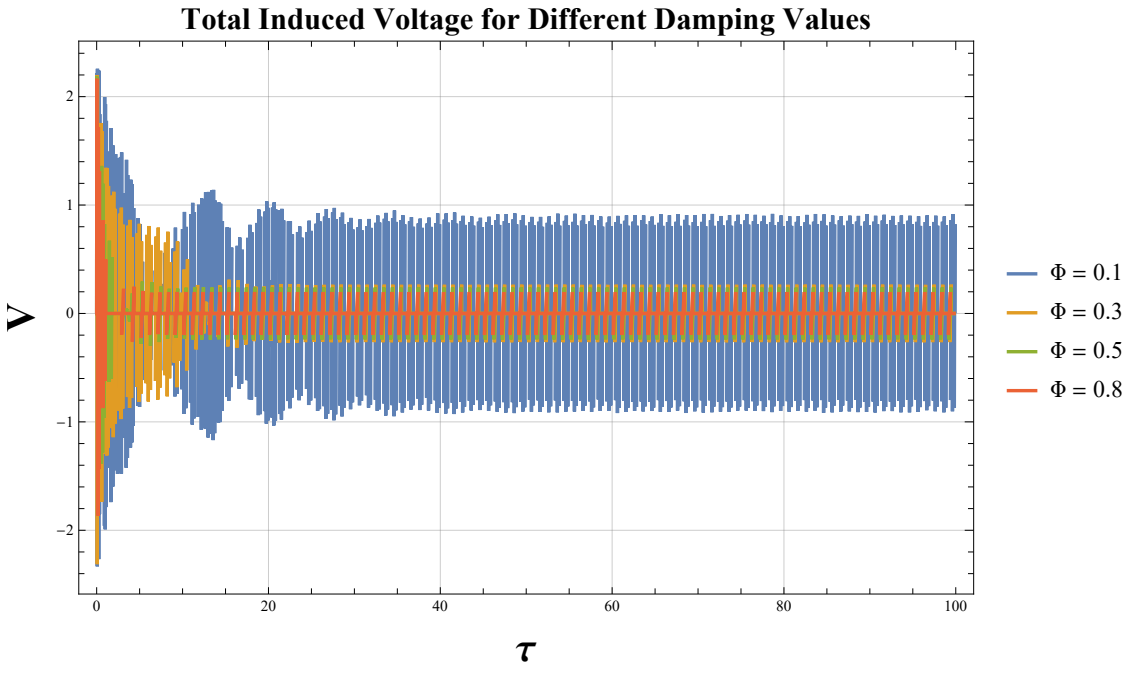


Figure 6.12: Voltage generated with different values of damping

6.4 Conclusions

This dynamical system is a very stable system. The stability is generated by the friction of the environment and the repulsive magnetic force that the magnets generate. But the system could be in chaos if the torque generated by the external force is large enough.

The oscillation frequency of the system depends on the moment of inertia of the structure (I_0), the repulsive magnetic force between the magnets, and the maximum angle of oscillation (Θ). This can be shown easily when the system is dimensionless; we can find three dimensionless numbers that govern the natural frequency of the oscillation; these numbers are the Inertia number (χ), maximum angle of the oscillation (Θ), and the Energie's number (Λ). If the system is in the regime of small oscillations, then the natural frequency is.

$$\Omega_0 = \sqrt{\chi + \frac{2\Lambda}{\Theta^2}}. \tag{6.4.1}$$

This implies that the frequency increases when χ and Λ increase, and when the maximum angle decreases, the frequency also increases. This is logical because, if the intensity of the magnets increases, the pendulum will return faster, and when the maximum angle is short, the distance between the magnets is short. Then, the repulsive force increases faster, causing more oscillations, and that implies a higher frequency.

Something important to highlight is that if the system is undamped and if an external force has enough strength, the system oscillates and decays to a stable state. This stable state keeps the oscillation constant over time. Talking about the external force, this force can be the force of a fluid in motion. Using the lift force of a fluid, we find that the dimensionless torque depends on four dimensionless parameters, the inertia number (χ), the lift coefficient

(C_L), the mass ratio number (m^*), and the Inertial-Gravitational interaction number (Γ). So, these numbers define the magnitude of the torque that the system obtains for the fluid.

Another important parameter to obtain the stable state is the frequency of the external force (Ω). When the frequency of the external force is similar to the natural frequency of the system, the system begins to resonate with the external force. In terms of fluids, the frequency of the external force is defined by the vortex shedding frequency generated for the fluid-structure interaction; then, the vortices that are generated for our system are the source of the oscillation. The frequency of the vortices is defined by the Strouhal number (St). In this system, we can see that the frequency of the external force is always proportional to the Strouhal number, but the definition of the dimensionless number can simplify or complicate the expression of the external frequency. If we only attend to the geometric parameters of the system, the time scale and frequency of the vortices are.

$$t_s = \sqrt{\frac{R}{g}} \quad , \quad \Omega = 2\pi St\Gamma. \tag{6.4.2}$$

And, if we pay attention to the geometric system parameters and the fluid parameters, the time scale and the frequency of the vortices are.

$$t_s = \frac{D}{U} \quad , \quad \Omega = 2\pi St. \tag{6.4.3}$$

These expressions are important because they can define the frequency of the external force. The first expression (6.4.2) lets us control the frequency in terms of the build parameters, and in the second (6.4.3), the frequency is independent of the structure parameters.

Finally, the harnessing energy model lets us see that the system could transform the motion of the pendulum into electricity. Moreover, if we pay attention, the system, when it is in the stable state, generates a voltage that oscillates between a maximum and a minimum voltage, which has the same value if we think in terms of magnitude. So, we can think that this system is analogous to an alternating voltage system.

Conclusions

In this project, we worked with three different systems that used the magnetic force as a restoring force to generate an oscillation. To keep the movement of the systems, we introduced an external force, which was the force of a fluid in motion. So, finally, we built three forced oscillator systems, with the intention of extracting energy from the movement of a structure pushed by a fluid force.

The systems that we studied oscillate, and the external force that is provided for the motion of a fluid generates Vortex-Induced Vibrations (VIV), which create gradients of pressure, causing our systems to start to oscillate in synchrony with the VIV's frequency. So, we can control the frequency of the oscillations with the shape of the structure and the motion of the fluid.

Each system has its own dimensionless numbers that govern and explain the motion of the system. But some dimensionless numbers appear in every system, and those are the dimensionless numbers that are related to the movement of the fluid. That is logical because all of them have the fluid force as the external force that impels the system. The fluid dimensionless numbers that always appear are:

- The Strouhal number (St): The frequency of the VIV's in our system.
- The Inertial-Gravitational interaction number (Γ): The ratio between the movement of the fluid and the movement of the structure driving the gravity.
- The mass ratio (m^*): The ratio of the mass of the fluid that is interacting with the structure, which has its own mass.
- The lift coefficient (C_L): The coefficient that is given by the shape of the body and helps to lift when the fluid passes around it.

Another dimensionless number that always appeared was the friction number (Φ), and it is because of nature that every system always has some friction with the environment. This number is so important because it generates the stability of the system.

From the structural systems, when the movement is locked in a specific region, we found an important dimensionless number that lets us know how the magnetic energy and the gravitational potential energy interact, this number is the Energie's number (Λ). And it is defined as.

$$\Lambda = \frac{M}{mgh}. \tag{6.4.4}$$

Where M is the magnetic energy that magnets have, m is the mass of our magnet that oscillates, g is the gravitational acceleration, and h is the height of the system. This number is important because in closed systems, the system with three magnets (chapter 5) and the pendulum system (chapter 6), the natural frequency has this number inside. So, Energie's number controls the movement of the system.

For the open system, the magneto-gravitational system (chapter 4), we can not find the Energie's number because the forcing increases the height; for that reason, we can not find the Energie's number from the beginning of the movement, and that implies that it does not govern the behavior of the system.

In terms of harnessing energy, we found that we can generate voltage in an open circuit using Faraday's law. In every system, we can induce voltage. The most effective system to generate voltage is the pendulum system, this is because in the length of the arc that it follows, we can put more coils than in the other systems, which can only have one or two coils.

In conclusion, the systems we studied can transform the mechanical energy of a structure into electric energy using Vortex Induced-Vibrations generated by the motion of a fluid. We obtained the theoretical expressions that let us estimate the voltage that we can induce in coils by the force of the vortex-induced vibrations. If we can control the frequency of the VIV with precision, we can build our system with specific dimensions to harness the energy of the fluids effectively. It is important to study the shape of the structures to find the best shape for the movement in each fluid, with the intention of improving the voltage induction.

Appendix

In this appendix, we present the various Wolfram Mathematica codes we wrote throughout the project to simulate our systems and generate our plots.

It is important to note that the codes we present here represent one version, as most of the code can be used across all systems with only minor changes. These changes are the properties or equations of each system.

Phase portrait diagram

This code lets us visualize the phase portrait diagram that a system has. This code needs to know the position and momentum expressions previously and the values of the parameters and conditions that our system has. It is important to note that the code we present at this moment is for the ideal case of a system; for other systems, we need to add the damping and forcing terms.

```
    (*Parameters*)g = 9.81;
m = 1;
M = 10;

(*System of ODEs*)
vx[x_, v_] := v;
vv[x_, v_] := -g + M/(m x);

StreamPlot[{vx[x, v], vv[x, v]}, {x, 0.5, 5}, {v, -5, 5},
 PlotTheme -> "Scientific",(*Visual consistency we agreed on*)
 Frame -> True, Axes -> False, GridLines -> Automatic,
 GridLinesStyle -> Directive[GrayLevel[0.7, 0.35]],
 StreamPoints -> Fine, StreamStyle -> Thick,
 ColorFunction -> ColorData["DeepSeaColors"],
 ColorFunctionScaling -> True,(* You requested this:FRAME LABELS*)
 FrameLabel -> {Style["x", Black, Bold, Large],
   Style["p", Black, Bold, Large]},
 PlotLabel -> Style["Phase Portrait", Black, Bold, Large],
 ImageSize -> 450]
```

Contrast of numerical and asymptotic solutions

This code contrasts the numerical solution of a system with the asymptotic solution that we obtain. This code requires you to know previously the asymptotic solution that you want to contrast.

```
\[CapitalPhi] = 0.01;
\[Delta] = Sqrt[1 - \[CapitalPhi]^2];

glffo = {{, }, {, }, {, }, {, }};

    For[i = 1, i < Length[fel] + 1, i++,
 sf = NDSolve[{x[t] x''[t] + 2 \[CapitalPhi] x[t] x'[t] + x[t] - 1 ==
     0, x[0] == 1 + fel[[i]], x'[0] == 0}, x, {t, 0, 30}];
 ffopf = Plot[{Evaluate[x[t] /. sf],
    1 + E^(-\[CapitalPhi] t)
       fel[[i]] (Cos[\[Delta] t] + \[CapitalPhi]/\[Delta] \
Sin[\[Delta] t])}, {t, 0, 30}, PlotRange -> All,
   PlotLabel ->
    Style["First order solution with \[Epsilon]="@fel[[i]], Black,
     Bold], FrameLabel -> {Style["\[Tau]", Black, Bold, Large],
     Style["\!\(\*OverscriptBox[\(x\), \(~\)]\)", Black, Bold,
      Large]}, PlotTheme -> "Scientific", GridLines -> Automatic,
   PlotStyle -> {Thick, Dashed}];
 ffospf =
  ParametricPlot[{{Evaluate[{x[t], x'[t]} /. sf]}, {1 +
      E^(-\[CapitalPhi] t)
        fel[[i]] (Cos[\[Delta] t] + \[CapitalPhi]/\[Delta] Sin[\
\[Delta] t]),
     fel[[i]] (-((
        E^(-t \[CapitalPhi]) (\[Delta]^2 + \[CapitalPhi]^2) Sin[
          t \[Delta]])/\[Delta])) }}, {t, 0, 20},
   PlotLabel ->
    Style["First order with \[Epsilon]="@fel[[i]], Black, Bold],
   FrameLabel -> {Style["\!\(\*OverscriptBox[\(x\), \(~\)]\)", Black,
      Bold, Large],
     Style["\!\(\*OverscriptBox[OverscriptBox[\(x\), \(~\)], \
\(.\)]\)", Black, Bold, Large]}, PlotTheme -> "Scientific",
   GridLines -> Automatic, PlotStyle -> {Thick, Dashed}];
 glffo[[i, 1]] = ffopf;
 glffo[[i, 2]] = ffospf;
 ]
GraphicsGrid[glffo]
```

Dimensioless groups

This code helps us to find the different dimensionless groups and the scale factors of each dimension of a system. This code needs to introduce the dimensions of each variable that we have in the system and the dimensions of the variables that we use to dimensionless

the system. The code returns us a table that gives us the dimensionless groups of each variable.

```
    vdsys = {{x, 1, 0, 0}, {t, 0, 0, 1}, {A, 1, 0, 0}, {mv, 0, 1, 0}};
kvdsys = {{rv, 1, 0, 0}, {gv, 1, 0, -2}, {Mv, 2, 1, -2}};
TableForm[
 Table[{Solve[{vdsys[[i, 2]] + kvdsys[[1, 2]] \[Alpha] +
       kvdsys[[2, 2]] \[Beta] + kvdsys[[3, 2]] \[Lambda] == 0,
     vdsys[[i, 3]] + kvdsys[[1, 3]] \[Alpha] +
       kvdsys[[2, 3]] \[Beta] + kvdsys[[3, 3]] \[Lambda] == 0,
     vdsys[[i, 4]] + kvdsys[[1, 4]] \[Alpha] +
       kvdsys[[2, 4]] \[Beta] + kvdsys[[3, 4]] \[Lambda] ==
      0}, {\[Alpha], \[Beta], \[Lambda]}],
   vdsys[[i, 1]] kvdsys[[1, 1]]^\[Alpha] kvdsys[[2,
      1]]^\[Beta] kvdsys[[3, 1]]^\[Lambda] /.
    Solve[{vdsys[[i, 2]] + kvdsys[[1, 2]] \[Alpha] +
        kvdsys[[2, 2]] \[Beta] + kvdsys[[3, 2]] \[Lambda] == 0,
      vdsys[[i, 3]] + kvdsys[[1, 3]] \[Alpha] +
        kvdsys[[2, 3]] \[Beta] + kvdsys[[3, 3]] \[Lambda] == 0,
      vdsys[[i, 4]] + kvdsys[[1, 4]] \[Alpha] +
        kvdsys[[2, 4]] \[Beta] + kvdsys[[3, 4]] \[Lambda] ==
       0}, {\[Alpha], \[Beta], \[Lambda]}],
   kvdsys[[1, 1]]^-\[Alpha] kvdsys[[2, 1]]^-\[Beta] kvdsys[[3,
      1]]^-\[Lambda] /.
    Solve[{vdsys[[i, 2]] + kvdsys[[1, 2]] \[Alpha] +
        kvdsys[[2, 2]] \[Beta] + kvdsys[[3, 2]] \[Lambda] == 0,
      vdsys[[i, 3]] + kvdsys[[1, 3]] \[Alpha] +
        kvdsys[[2, 3]] \[Beta] + kvdsys[[3, 3]] \[Lambda] == 0,
      vdsys[[i, 4]] + kvdsys[[1, 4]] \[Alpha] +
        kvdsys[[2, 4]] \[Beta] + kvdsys[[3, 4]] \[Lambda] ==
       0}, {\[Alpha], \[Beta], \[Lambda]}]}, {i, 1, Length[vdsys]}],
 TableHeadings -> {Table[vdsys[[i, 1]], {i, 1, 4}], {"",
    "\[CapitalPi]-group", "scale factor"}}]
 TableForm[{{{{\[Alpha] -> 1, \[Beta] -> 1, \[Lambda] -> -1}}, {((
      g m)/M) x}, {(g^(-1)/m)
      M}}, {{{\[Alpha] -> Rational[
      1, 2], \[Beta] -> 1, \[Lambda] -> Rational[-1, 2]}}, {((
      g m^Rational[1, 2]) M^Rational[-1, 2]) t}, {(
     g^(-1) m^Rational[-1, 2])
      M^Rational[
      1, 2]}}, {{{\[Alpha] -> 1, \[Beta] -> 1, \[Lambda] -> -1}}, {(
     A g) (m/M)}, {(g^(-1)/m) M}}},
 TableHeadings -> {{x, t, A}, {
   "", "\[CapitalPi]-group", "scale factor"}}]
```

Numerical solution

This code generates the numerical solution of a dynamical system, and after that, we plot the solution to see the evolution of the system.

In particular, the code is designed for the pendulum system. To use for the other systems, we need to introduce the correct differential equation and its variables.

```
    \[Chi] = 1;
\[CapitalLambda] = 1;
\[CapitalPhi] = 0.5;
\[Sigma] = 1;
\[CapitalOmega] = \[Pi];
\[CapitalTheta] = \[Pi]/5;

f = NDSolve[{\[Theta]''[\[Tau]] +
     2 \[CapitalPhi] \[Theta]'[\[Tau]] + (\[Chi] + \
\[CapitalLambda]/(-Cos[\[CapitalTheta]] + Cos[\[Theta][\[Tau]]] +
         10^-6)) Sin[\[Theta][\[Tau]]] == \[Sigma] \
Cos[\[CapitalOmega] \[Tau]], \[Theta][
     0] == \[CapitalTheta], \[Theta]'[0] == 0}, \[Theta], {\[Tau], 0,
   500}]

Plot[Evaluate[\[Theta][\[Tau]] /. f], {\[Tau], 0, 100},
 PlotRange -> All,
 PlotLabel -> Style["Numerical solution ", Black, Bold],
 FrameLabel -> {Style["\[Tau]", Black, Bold, Large],
   Style["\[Theta]", Black, Bold, Large]}, PlotTheme -> "Scientific",
 GridLines -> Automatic]
```

Bifurcation

This code is used to generate the bifurcation diagrams. Here, the code solves the system numerically for a specific value of the external force, obtaining the stable states of the system for each value of the external force, and after that, we plot those points.

```
    (*-----USER PARAMETERS-----*)
\[CapitalPhi] = 0.1;(*damping coefficient*)
\[CapitalOmega] = 1;(*forcing frequency*)
T = 2 \[Pi]/\[CapitalOmega];(*forcing period*)
\[CapitalLambda] = 1; (*Energies number*)
\[Chi] = 1;(*Frequency pendulum number*)
\[CapitalTheta] = \[Pi]/4;(*Max angule value*)
(*Integration times (in periods)*)transientPeriods = 200;(*discard \
this many periods at the start*)samplePeriods = 100;(*sample this \
many periods after the transient*)tran =
 transientPeriods T;(*time after which we start sampling*)total = \
(transientPeriods +
    samplePeriods) T;(*total integration time*)(*Parameter range for \
F*)\[Sigma]min = 0;
\[Sigma]max = 12;
\[Sigma]step = 0.1;
\[Sigma]values = Range[\[Sigma]min, \[Sigma]max, \[Sigma]step];

(*-----FUNCTION:compute Poincaré points for a single F-----*)
```

```
getPoincare[\[Sigma]_] :=
  Module[{sol, points},
   sol = NDSolve[{\[Theta]''[\[Tau]] +
        2 \[CapitalPhi] \[Theta]'[\[Tau]] + (\[Chi] + \
\[CapitalLambda]/(-Cos[\[CapitalTheta]] + Cos[\[Theta][\[Tau]]] +
           10^-6)) Sin[\[Theta][\[Tau]]] == \[Sigma] Cos[\
\[CapitalOmega] \[Tau]], \[Theta][0] == \[CapitalTheta], \[Theta]'[
        0] == 0}, \[Theta], {\[Tau], 0, total}, MaxSteps -> Infinity,
     Method ->
      "StiffnessSwitching"    (*helps with potential stiffness*)];
   (*Extract x at times t=tran,tran+T,...,total*)
   points =
    Table[{\[Sigma], \[Theta][\[Tau]] /. sol[[1]]}, {\[Tau], tran,
      total, T}];
   points];

(*-----GENERATE DATA FOR ALL F VALUES-----*)
data = Flatten[
   Table[getPoincare[\[Sigma]], {\[Sigma], \[Sigma]values}], 1];

(*-----PLOT THE BIFURCATION DIAGRAM-----*)
ListPlot[data, PlotRange -> All,
 AxesLabel -> {"Forcing amplitude \[Sigma]", "\[Theta](nT)"},
 PlotStyle -> {PointSize[0.005]},
 LabelStyle -> Directive[Bold, Black],
 PlotLabel -> Style["Bifurcation diagram", Black, Bold],
 PlotRange -> All, ImageSize -> Large, GridLines -> Automatic,
 PlotTheme -> "Scientific",
 FrameLabel -> {Style["\[Sigma]", Black, Bold, Large],
   Style["\[Theta](nT)", Black, Bold, Large]}]
```

Lyapunov exponents

The following code is for generating the diagram of Lyapunov exponents vs the external torque. This code gives us the Lyapunov exponents values for each value of σ.

```
(* ================================================*)(*Maximum \
Lyapunov exponent vs sigma*)(* \
================================================*)ClearAll["Global`*"]

(*----------Fixed parameters----------*)

\[CapitalPhi] = 0.1;(*damping coefficient*)\[CapitalOmega] = \
1.0;(*forcing frequency*)\[CapitalLambda] = 1.0;(*nonlinear \
strength*)\[Chi] = 1.0;(*linear restoring parameter*)\[CapitalTheta] \
= 0.3;(*singular angle parameter*)tmax = 400;(*total integration \
time*)dt = 0.5;(*renormalization interval*)(*----------Initial \
conditions----------*)theta0 = 0.1;
v0 = 0.0;
```

```
\[Delta]0 =
 10^-8;(*initial perturbation norm*)(*----------Range of sigma \
values----------*)sigmaValues = Range[0, 14, 0.01];

(* ================================================*)
(*Main loop over sigma*)
(* ================================================*)

results = Table[(*current state and perturbation*)state = {theta0, v0};
   pert = {\[Delta]0, 0};
   t = 0;
   sum = 0;
   While[t < tmax,(*Solve nonlinear system+variational equations*)
    sol = NDSolve[{(*----------Original system----------*)\[Theta]'[\
\[Tau]] == v[\[Tau]],
        v'[\[Tau]] ==
         sigma Cos[\[CapitalOmega] \[Tau]] -
          2 \[CapitalPhi] v[\[Tau]] - (\[Chi] + \
\[CapitalLambda]/(Cos[\[Theta][\[Tau]]] -
                Cos[\[CapitalTheta]])) \
Sin[\[Theta][\[Tau]]],(*----------Variational equations----------*)\
\[Delta]\[Theta]'[\[Tau]] == \[Delta]v[\[Tau]], \[Delta]v'[\[Tau]] == \
(-\[Chi] Cos[\[Theta][\[Tau]]] - \
\[CapitalLambda]/(Cos[\[Theta][\[Tau]]] - Cos[\[CapitalTheta]])*
             Cos[\[Theta][\[Tau]]] - \[CapitalLambda] Sin[\[Theta][\
\[Tau]]]^2/(Cos[\[Theta][\[Tau]]] -
                  Cos[\[CapitalTheta]])^2) \[Delta]\[Theta][\[Tau]] -
          2 \[CapitalPhi] \[Delta]v[\[Tau]],(*----------Initial \
conditions----------*)\[Theta][0] == state[[1]],
        v[0] == state[[2]], \[Delta]\[Theta][0] ==
         pert[[1]], \[Delta]v[0] == pert[[2]]}, {\[Theta],
        v, \[Delta]\[Theta], \[Delta]v}, {\[Tau], 0, dt},
       MaxSteps -> Infinity, Method -> {"StiffnessSwitching"}][[1]];
    (*----------Update state----------*)
    state = {\[Theta][dt] /. sol, v[dt] /. sol};
    (*----------Update perturbation----------*)
    pert = {\[Delta]\[Theta][dt] /. sol, \[Delta]v[dt] /. sol};
    (*----------Renormalization----------*)norm = Norm[pert];
    sum = sum + Log[norm/\[Delta]0];
    pert = \[Delta]0*pert/norm;
    t = t + dt;];
   (*----------Lyapunov exponent----------*)lyap = sum/tmax;
   {sigma, lyap}, {sigma, sigmaValues}];

(* ================================================*)
(*Plotting*)
(* ================================================*)
```

```
ListPlot[results, PlotRange -> All,
 PlotStyle -> {PointSize[0.015], Blue},
 AxesLabel -> {"\[Sigma]", "\[Lambda]"},
 LabelStyle -> Directive[Bold, Black, 14],
 PlotLabel ->
  Style["Maximum Lyapunov exponent vs \[Sigma]", Black, Bold],
 ImageSize -> Large, GridLines -> Automatic,
 PlotTheme -> "Scientific", Frame -> True,
 FrameLabel -> {Style["\[Sigma]", Black, Bold, Large],
   Style["\[Lambda]", Black, Bold, Large]}, AxesStyle -> Thick]

(*Optional export*)
(*Export["lyapunov_vs_sigma.pdf",%]*)
```

Bibliography

[1] William A. Adkins and Mark G. Davidson. *Ordinary Differential Equations*. United States: Springer, 2012. ISBN: 978-1-4614-3617-1. DOI: 10.1007/978-1-4614-3618-8.

[2] R. Ahamed, I. Howard, and K McKee. "Dynamic analysis of magnetic spring-based nonlinear oscillator system". In: *Nonlinear Dynamics* 111 (2023), pp. 15705–15736. DOI: https://doi.org/10.1007/s11071-023-08669-3.

[3] Peter R. Andronov et al. In: *Aerospace Science and Technology* 93 (2019). DOI: https://doi.org/10.1016/j.ast.2019.105348.

[4] Francois Axisa and Jose Antunes. *Modelling of mechanical systems: Fluid Structure Interaction*. Great Britain: Elsevier Inc., 2007. ISBN: 978-0-750-66847-7.

[5] Sandip Banerjee. *Mathematical modeling: Models, Analysis and Applications*. India: CRC Press, 2014. ISBN: 978-1-4822-2916-5.

[6] Martin Bazant and Joey Gu. *MIT 10.50.CH03x. Analysis of Transport Phenomena*. Massachusetts, United States, 2023. URL: https://courses.mitxonline.mit.edu/learn/course/course-v1:MITxT+10.50.CH03x+1T2023/home.

[7] Maksymilian Bednarek, Donat Lewandowski, and Jan Awrejcewicz. "Magnetic and Electromagnetic Springs Forces: Determination and Usage in Damping Vibrations". In: *Advances in Nonlinear Dynamics*. Ed. by Walter Lacarbonara et al. Cham: Springer International Publishing, 2022, pp. 115–124. ISBN: 978-3-030-81166-2.

[8] Jiri Blazek. *Computational Fluid Dynamics: Principles and Applications*. 3rd. United Kingdom: Elsevier, 2015. ISBN: 978-0-08-099995-1.

[9] M. Brocchini and F. Trivellato. *Vorticity and Turbulence Effects in Fluid Structure Interaction: An Application to Hydraulic Structure Design*. Great Britain: WIT Press, 2006. ISBN: 1-84564-052-7.

[10] Butcher, J. C. *NUMERICAL METHODS FOR ORDINARY DIFFERENTIAL EQUATIONS*. 3rd. Wiley, 2016.

[11] M.G. Calkin. *Lagrangian and Hamiltonian Mechanics*. Singapore: World Scientific Publishing Co., 1996. ISBN: 981-02-2672-1.

[12] Frank S. Crawford Jr. *Ondas: Berkeley physics course-volumen 3 (Juan T. D'alessio, Trad.)* México: Editorial Reverté (Obra original publicada en 1968), 1965. ISBN: 84-291-4023-9.

[13] Aytekin Duranay, Omer Kemal Kinaci, and Michael Bernitsas. "Effect of aspect ratio on hydrokinetic energy harnessing using cylinders in VIV". In: *Journal of Ocean Engineering and Marine Energy* 8 (2022), pp. 217–232. DOI: https://doi.org/10.1007/s40722-022-00226-1.

[14] C. Foias et al. *Navier–Stokes Equations and Turbulence*. United Kingdom: Cambridge University Press, 2004. ISBN: 0-511-03936-0.

[15] S.K. Godunov. *Ecuaciones de la Física Matemática (Juán José Tolosa, Trad.)* México: MIR, 1978.

[16] Herbert Goldstein, Charles P. Poole, and John Safko. *Classical Mechanics*. 3rd. Pearson, 2013. ISBN: 9781292026558.

[17] Peng Han et al. In: *Ocean Engineering* 282 (2023). DOI: https://doi.org/10.1016/j.oceaneng.2023.114869.

[18] Sadri Hassani. *Mathematical Physics: A modern introduction to its Foundations*. 2nd. United States: Springer, 2013. ISBN: 978-3-319-01194-3. DOI: 10.1007/978-3-319-01195-0.

[19] John K. Hunter. *Asymptotic Analysis and Singular Perturbation Theory*. United States: University of California at Davis, 2004. URL: https://www.math.ucdavis.edu/~hunter/notes/asy.pdf.

[20] Hyunjoong Kim and CChee-Han Tan. *Math 6730: Asymptotic and Perturbation Methods*. Universit of Utah, United States, 2018.

[21] L. D. Landau and E. M. Lifshits. *Mechanics*. Russia: Mir, 1960.

[22] Enzo Levi. *Teorías y Métodos de las Matemáticas Aplicadas*. México: Facultad de Ingeniería UNAM, 1965.

[23] J. David Logan. *Applied Partial Differential Equations*. 3th. United States: Springer, 2015. ISBN: 978-3-319-12492-6. DOI: 10.1007/978-3-319-12493-3.

[24] Teruo Matsushita. *Electricity and Magnetism: New Formulation by Introduction of Superconductivity*. Japan: Springer, 2014. ISBN: 978-4-431-54525-5. DOI: 10.1007/978-4-431-54526-2.

[25] Efstathios E (Sthatis) Michaelides. *Alternative Energy Sources*. Germany: Springer, 2012. ISBN: 978-3-642-20950-5. DOI: 10.1007/978-3-642-20951-2.

[26] Yahya Modarres-Sadegh. *Introduction to Fluid-Structure Interactions*. Switzerland: Springer, 2021. ISBN: 978-3-030-85882-7.

[27] Peter J. Olver. *Introduction to Partial Differential Equations*. United States: Springer, 2014. ISBN: 978-3-319-02098-3. DOI: 10.1007/978-3-319-02099-0.

[28] Michael P. Païdoussis, Stuart J. Price, and Emmanuel de Langre. *Fluid-Structure Interactions: CROSS-FLOW-INDUCED INSTABILITIES*. United States of America: Cambrige University Press, 2011. ISBN: 978-0-521-11942-9.

[29] Press, W. H. and Teukolsky, S. A. and Vetterling, W. T. and Flannery, B. P. *NUMERICAL RECIPES: THE ART OF SCIENTIFIC COMPUTING*. 3rd. Cambridge University Press, 2007.

[30] Hazef A. Radi and John O. Rasmussen. *Principles of Physics: For Scientists and Engineers*. Germany: Springer, 2013. ISBN: 978-3-642-23026-4. DOI: 10.1007/978-3-642-23026-4.

[31] José Rogan C. and Víctor Muñoz G. *Apuntes de un curso de FÍSICA MATEMATICA*. 3ra. Chile: Departamento de Física.

[32] Anne Rooney. *La historia de la física: De la filosofía natural al enigma de la materia oscura (Luigi Freda Eslava, Trad.)* México: Grupo Editorial Tomo (Obra original publicada en 2011), 2013. ISBN: 978-607-415-473-3.

[33] Soldovieri C., T. *INTRODUCCION A LA MECANICA DE LAGRAGE Y HAMILTON*. 1era. Venezuela: Preprint, 2019.

[34] Ain A. Sonin. *The Physical Basis of DIMENSIONAL ANALYSIS*. 2nd. Cambridge, Massachusetts, United States: Department of Mechanical Engineering MIT, 2001.

[35] Iain Stewart. *MIT OpenCourseWare 8.09 | Fall 2014 | Undergraduate Classical Mechanics III*. Massachusetts, United States, 2014. URL: https://ocw.mit.edu/courses/8-09-classical-mechanics-iii-fall-2014/.

[36] Jing Tang Xing. *Fluid-Solid Interaction Dynamics: Theory, Variational Principles, Numerical Methods, and Applications*. United Kingdom: Elsevier Inc., 2019. ISBN: 978-0-12-819352-5.

[37] Stephen T. Thornton and Jerry B. Marion. *Classical Dynamics of Particles and Systems*. Cengage Learning, 2003. ISBN: 9780534408961.

[38] Jiyuan Tu, Guan-Heng Yeoh, and Chaoqun Liu. *Computational Fluid Dynamics: A practial approach*. 3rd. Elsevier, 2018. ISBN: 978-0-08-101127-0.

[39] Arnt Inge Vistnes. *Physics of oscillations and waves: With use of Mathlab and Python*. Oslo, Norway: Springer, 2018. ISBN: 978-3-319-72313-6. DOI: https://doi.org/10.1007/978-3-319-72314-3.

[40] Xiaodong (Sheldon) Wang. *Fundamentals of Fluid-Structure Interactions: Analytical and a Computational approaches*. Hungary: Springer, 2008. ISBN: 978-0-444-52807-0.

[41] Frank M. White. *Fluid Mechanics*. 8th. United States of America: McGraw-Hill Education, 2016. ISBN: 978-9-38-596549-4.

[42] C.H.K Williamson and R. Govardhan. "Vortex-Induced Vibrations". In: *Annual Reviews Fluid Mechanics* 36 (2004), pp. 413–455. DOI: 10.1146/annurev.fluid.36.050802.122128.

[43] Thomas Witelski and Mark Bowen. *Methods of Mathematical Modelling: Continuous Systems and Differential Equations*. USA and Japan: Springer, 2015. ISBN: 978-3-319-23041-2. DOI: 10.1007/978-3-319-23042-9.

[44] Thomas Witelski and Mark Bowen. *Physics of oscillations and waves: With use of Mathlab and Python*. Oslo, Norway: Springer, 2018. ISBN: 978-3-319-72313-6. DOI: https://doi.org/10.1007/978-3-319-72314-3.